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A mathematical analysis of motions of viscous incompressible fluid under leak or slip boundary conditions

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Abstract

A mathematical analysis of the boundary value problem for stationary solutions of the Stokes and Navier-Stokes equations under leak or slip boundary conditions of the friction type is made through variational inequalities. As for the Stokes equation, the velocity fields exist uniquely. The accompanying pressure is determined uniquely up to an arbitrary additive constant under the slip boundary condition, while the additive constant in the pressure is restricted in a certain way for the leak boundary condition. In particular, if a leak actually takes place somewhere, then the pressure is completely unique. The results can be extended to the Navier-Stokes flow to a reasonable extent.

1 Introduction

The present paper is concerned with the boundary value problem for steady motions of viscous incompressible fluid under some slip or leak boundary conditions which are to be described below and which we call the slip or leak boundary conditions of friction type.

This work is based on the author's lectures [3] delivered at Collège de France in October 1993, and is an elaborated version of parts of the forthcoming notes [4], [5] on the same topics by the author and collaborators.

In this paper, our method of analysis is that of variational inequalities where the admissible vector functions are taken from solenoidal (divergence-free) functions, and the functional contains a kind of barrier terms to resist against free slip or free leak, while some further study by use of another type of variational inequality with general (not necessarily solenoidal) admissible functions, a saddle-point formulation, and some numerical approaches are treated in [6] and [9].

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1.1 motivations

So far almost exclusively, the Dirichlet boundary condition (adhesion to solid surfaces) has been considered for motions of viscous incompressible fluids in hydrodynamics as well as in mathematics.

However, there exist some flow phenomena, modeling of which might require introduction of slip and/or leak boundary conditions in reality or apparently. As examples, we can refer to the following. (1)flow through a drain or canal with its bottom covered by sherbet of mud and pebbles. (2) flow of melted iron coming out from a smelting furnace. (3) avalanche of water and rocks. (4) blood flow in a vein of an arterial sclerosis patient. (4) polymer-polymer welding and sliding phenomena as studied by P.G.de Gennes.

Furthermore, with some of these examples one observes that some fragile state of the surface or existence of sherbet zone allows for the fluid to slip on the surface, and that as long as the "force of stream" is below a threshold the fluid does not slip. In order to form a mathematical model of such slip phenomena, introduction of slip boundary conditions of friction type, which we describe below, seems to be suitable.

Similarly, leak boundary conditions would be an important concepts when we want to model flow problems involving leak of the fluid through the surface or penetration into the adjacent media. For instance when we deal with the flow problem such as (1) flow through a net or sieve, e.g., a butterfly net. (2) flow through filter, e.g., a vacuum cleaner, diapers, a coffee maker. (3) water flow in a purification plant, filtration of rain to form underground water, (4) oil flow over or beneath sand layers. Indeed, some filters prevent the leak if the "pushing force" is below a threshold, which might suggest to adopt the leak boundary conditions of friction type to be discussed below.

1.2 description of the problem; slip boundary condition

Some specific description of the problem, particularly, the slip and leak boundary conditions of our concern is in order. In this paper we shall consider only stationary motions of viscous incompressible fluid in a bounded domain Ω in R^2 or R^3, while we intend to deal with the time-dependent problem in a forthcoming paper. Thus, inside Ω, the motion of the fluid is governed by the Stokes system (1) or the Navier-Stokes system (2)below.

$$(1) \qquad \begin{cases} -\nu\Delta u + \nabla p &= f, \\ \operatorname{div} u &= 0. \end{cases}$$

$$(2) \qquad \begin{cases} -\nu\Delta u + \nabla p + (u\cdot\nabla)u &= f, \\ \operatorname{div} u &= 0. \end{cases}$$

Here, the vector function u is the velocity, the scalar function p represents the pressure and the given vector function f stands for the external force. The positive constant ν is the kinetic viscosity.

As for the shape of the spatial domain Ω we assume, simply to fix the idea, that Ω is bounded by smooth boundary Γ which is composed of two smooth components (inner wall and outer wall) as below;

$$(3) \qquad \Gamma = \Gamma_0 \cup \Gamma_1$$

Unless otherwise stated, throughout the present paper we impose the usual Dirichlet boundary condition on Γ_0, in order to avoid irrelevant difficulties concerning the solvability of the problem. Namely,

$$u|_{\Gamma_0} = \beta, \tag{4}$$

where the boundary value β is assumed to satisfy the out-flow condition that

$$\int_{\Gamma_0} \beta_n = 0. \tag{5}$$

Here and in what follows, the unit outer normal to the boundary is denoted by n and if b is a vector defined on the boundary, b_n is the normal component of b, while b_t means the tangential component of b. Also, $\dfrac{\partial}{\partial n}$ is the differentiation along n.

Assuming that u is smooth up to the boundary, we set

$$\sigma_t = \sigma_t(u) = \nu \frac{\partial u_t}{\partial n} \tag{6}$$

and can write the slip boundary condition in question, which is to be imposed on Γ_1 ; Namely, we require at each point $s \in \Gamma_1$

$$|\sigma_t(u)| \leq g_t \tag{7}$$

and

$$\begin{cases} |\sigma_t(u)| < g_t \implies u_t = 0, \\ |\sigma_t(u)| = g_t \implies \begin{cases} u_t = 0 \text{ or } u_t \neq 0, \\ \sigma_t(u) \cdot u_t \leq 0. \end{cases} \end{cases} \tag{8}$$

Here g_t is a positive function given on Γ_1. Furthermore, again for simplicity, we impose the following non-leak boundary conditions on Γ_1.

$$u_n|_{\Gamma_1} = 0. \tag{9}$$

Now we can state our boundary value problem for the Stokes equation with the slip boundary condition of the friction type.

Problem 1 (BVP-Sp-S) *Find u and p such that the equations (1) and boundary conditions (4),(7),(8) and (9).*

It is easily seen that under (7), the system of conditions (8) can be equivalently replaced by

$$\sigma_t \cdot u_t + g_t|u_t| = 0, \tag{10}$$

and the whole slip boundary condition can be written as

$$\begin{cases} |\sigma_t(u)| & \leq & g_t, \\ \sigma_t \cdot u_t + g_t|u_t| & = & 0, \end{cases} \tag{11}$$

which is the most convenient for our analysis below.

1.3 description of the problem; leak boundary condition

Quite similarly, we can formulate the leak boundary condition. Actually, putting

$$\sigma_n = \sigma_n(u) = -p + \nu \frac{\partial u_n}{\partial n} \tag{12}$$

we impose on Γ_1

$$|\sigma_n(u)| \le g_n \tag{13}$$

and

$$\begin{cases} |\sigma_n(u)| < g_n & \Longrightarrow \quad u_n = 0, \\ |\sigma_n(u)| = g_n & \Longrightarrow \quad \begin{cases} u_n = 0 \text{ or } u_n \neq 0, \\ \sigma_n(u) u_n \le 0. \end{cases} \end{cases} \tag{14}$$

When we consider this leak condition, let us require the following non-slip boundary condition instead of (9);

$$u_t|_{\Gamma_1} = 0. \tag{15}$$

Then our boundary value problem for the Stokes equation with the leak boundary condition of the friction type reads,

Problem 2 (BVP-Lk-S) *Find u and p such that the equations (1) and boundary conditions (4),(13),(14) and (15).*

At this point, we should note that as is obvious from (13), the pressure p which solves BVP-Lk-S together with u cannot contain a free additive constant. Indeed, this is a remarkable character of BVP-Lk-S which makes the existence proof of the solution more interesting and so more sophisticated. On the other hand, however, we see again that under (13), the condition (14) can equivalently be replaced by

$$\sigma_n \cdot u_n + g_n |u_n| = 0. \tag{16}$$

Actually, for our later analysis, it is convenient to write the whole leak boundary condition as

$$\begin{cases} |\sigma_n(u)| & \le & g_n, \\ \sigma_n u_n + g_n |u_n| & = & 0. \end{cases} \tag{17}$$

1.4 Plan of the paper

The rest of the paper is composed of the following sections.

§2. Slip boundary conditions of friction type for the Stokes equation.
§3. Leak boundary conditions of friction type for the Stokes equation.
§4. Slip and leak boundary conditions of friction type for the Navier-Stokes equation.
§5. Comments and remarks.
References

2 Slip boundary conditions of friction type for the Stokes equation

2.1 Weak formulation of the problem

Now we are going to formulate our problem for the Stokes flow under the slip boundary condition in a somewhat weaker form. As standing assumptions, let us suppose that

$$f \in L^2(\Omega), \quad \beta \in H^{1/2}(\Gamma_1), \quad g_t \in L^2(\Gamma_1). \tag{18}$$

Then we seek the velocity (a vector function) $u \in H^1(\Omega)$, and the pressure (a scalar function) $p \in L^2(\Omega)$. We need the following function spaces and notations. Incidentally, we shall use one and the same symbol to denote a Sobolev space of vector functions and that of scalar functions, when there is no fear of confusion.

$$H^1(\Omega) = \{v \in L^2(\Omega); \quad \nabla v \in L^2(\Omega)\} \tag{19}$$

$$(u, v) = (u, v)_{L^2}, \tag{20}$$

$$a(u, v) = (\nabla u, \nabla v)_{L^2}, \tag{21}$$

$$H_0^1(\Omega) = \{v \in H^1(\Omega); v = 0 \text{ on } \Gamma\}, \tag{22}$$

$$H_{0,\sigma}^1(\Omega) = \{v \in H_0^1(\Omega); \operatorname{div} v = 0\}. \tag{23}$$

Particularly, in dealing with the slip boundary condition, we use

$$K_\beta^S = \{v \in H^1(\Omega); \quad v = \beta \text{ on } \Gamma_0, v_n = 0 \text{ on } \Gamma_1\} \tag{24}$$

and

$$K_0^S = \{v \in H^1(\Omega); \quad v = 0 \text{ on } \Gamma_0, v_n = 0 \text{ on } \Gamma_1\} \tag{25}$$

Obviously, if u, v are in K_β^S, then their difference belongs to K_0^S. The solenoidal subspaces of these are denoted respectively by $K_{\sigma,\beta}^S$and $K_{\sigma,0}^S$. Namely,

$$K_{\sigma,\beta}^S = \{v \in H^1(\Omega); \quad \operatorname{div} v = 0 \text{ in } \Omega, v = \beta \text{ on } \Gamma_0, v_n = 0 \text{ on } \Gamma_1\} \tag{26}$$

and $K_{\sigma,0}^S$.

For $u \in H^1(\Omega)$ and $p \in L^2(\Omega)$, the weak form of the Stokes equation is written as

$$\nu a(u, \phi) - (p, \operatorname{div} \phi) = (f, \phi) \qquad (\forall \phi \in H_0^1(\Omega). \tag{27}$$

As for the slip boundary condition, we must note that if u is sufficiently smooth up to the boundary, σ_t is given by (6). However, under the general circumstances and from the rigorous view point, σ_t is defined as a functional from $H^{1/2}(\Gamma_1)$ through the following identity;

$$\int_{\Gamma_1} \sigma_t \cdot \psi_t \, d\Gamma = \nu a(u, \psi) - (p, \operatorname{div} \psi) - (f, \psi) \quad (\forall \psi \in K_0^S). \tag{28}$$

Let $\eta \in H^{1/2}(\Gamma_1)$. Then there exists $\psi \in K_{\sigma,0}^S$ such that $\psi_t = \eta$ on Γ_1 (e.g., see [12]). The value of the right-hand side of (28) does not depend on the way of extension by virtue of (27).

Now we can state our p.d.e. formulation (PDEF-Sp-S) of our slip problem BVP-Sp-S for the Stokes equation in a more reasonable manner.

Problem 3 (PDEF-Sp-S) .
Find a vector function $u \in K^S_{\sigma,\beta}$ *and* $p \in L^2(\Omega)$ *such that the Stokes equation in the form of (27) holds true and the slip boundary condition, say, (11) is satisfied.*

Remark 2.1 It is a consequence of a well-known fact (e.g., see [11], [12]), that if $u \in K^S_{\sigma,\beta}$ alone satisfies

$$\nu a(u, \phi) = (f, \phi) \qquad (\forall \phi \in H^1_{0,\sigma}(\Omega)), \tag{29}$$

then there exists $p \in L^2(\Omega)$ which satisfies the weak Stokes equation (27) jointly with u. We call such p the pressure associated with u and note that this p contains an arbitrary additive constant.

The solvability of PDEF-Sp-S is obtained through the method of variational inequalities below.

2.2 VI formulation of the problem

First of all we introduce the following barrier term against slip on Γ_1 ;

$$j_t = \int_{\Gamma_1} g_t |v_t| \, d\Gamma. \tag{30}$$

Then we state our variational inequalities which involves u only.

Problem 4 (VIF-Sp-S) .
Find $u \in K^S_{\sigma,\beta}$ *such that the inequality*

$$\nu a(u, v - u) - (f, v - u) + j_t(v) - j_t(u) \geq 0 \quad (\forall v \in K^S_{\sigma,\beta}), \tag{31}$$

holds true.

Let u be a solution of VIF-Sp-S. Then taking a arbitrary $\phi \in K^S_{\sigma,0}$ and substituting $v = u + \phi$ and $v = u - \phi$ into (31), we notice that the weak form of the Stokes equation (29) is satisfied. Therefore, the exists $p \in L^2(\Omega)$ associated with u. Moreover, in terms of σ_t we can equivalently rewrite (31) to

$$\int_{\Gamma_1} \sigma_t \cdot (v_t - u_t) \, d\Gamma + j_t(v) - j_t(u) \geq 0 \qquad (\forall v \in K^S_{\sigma,\beta}). \tag{32}$$

We claim

Theorem 2.1 *The problem VIF-Sp-S has a unique solution.*

Proof. In view of the $H^1(\Omega)-$ellipticity of $a(\ ,\ )$ in $K^S_{\sigma,0}$, and the convexity and lower-simicontinuity (actually, continuity) of the functional j_t from $K^S_{\sigma,\beta}$with the weak topology of $H^1(\Omega)$, the theorem is immediate from a well-known theory (e.g., see ([1]), ([7]), or ([8])). q.e.d.

Theorem 2.2 *The two problems PDEF-Sp-S and VIF-Sp-S are equivalent.*

Proof. PDEF-Sp-S $\Longrightarrow$ VIF-Sp-S.
Let $\{u,p\}$ be a solution of PDEF-Sp-S. Then (32) can be easily verified as

$$
\begin{aligned}
\textstyle\int_{\Gamma_1} \sigma_t \cdot (v-u)_t \, d\Gamma &+ j_t(v) - j_t(u) \\
&= \textstyle\int_{\Gamma_1} (\sigma_t \cdot v_t + g_t|v_t|) \, d\Gamma - \int_{\Gamma_1} (\sigma_t \cdot u_t + g_t|u_t|) \, d\Gamma \\
&= \textstyle\int_{\Gamma_1} (\sigma_t \cdot v_t + g_t|v_t|) \, d\Gamma \geq 0,
\end{aligned} \tag{33}
$$

where on the last line use has been made of (11).
VIF-Sp-S $\Longrightarrow$ PDEF-Sp-S.
Let u be a solution of VIF-Sp-S and p be any pressure associated with u. Then they jointly satisfy (27) and we have (32), from which follows

$$
-\int_{\Gamma_1} \sigma_t \cdot (v_t - u_t) \, d\Gamma \leq \int_{\Gamma_1} g_t |v_t - u_t| \, d\Gamma \quad (\forall v \in K^S_{\sigma,\beta}), \tag{34}
$$

by means of the obvious inequality $|v_t| - |u_t| \leq |v_t - u_t|$. Taking any $\psi \in K^S_{\sigma,0}$ and substituting $v = u + \psi$ and $v = u - \psi$ into (34), we get

$$
\left| \int_{\Gamma_1} \sigma_t \cdot \psi_t \, d\Gamma \right| \leq \int_{\Gamma_1} g_t |\psi_t| \, d\Gamma \quad (\forall \psi \in K^S_{\sigma,0}). \tag{35}
$$

From (35) follows (7) by means of a duality argument and in view of the arbitrariness of ψ_t on Γ_1. In fact, we can apply the following lemma, which is a slight modificatin of the well-known theorem that $(L^1(\Omega))^* = L^\infty(\Omega)$.

Lemma 2.1 *Suppose that $g = g(s)$ is a positive function on Γ_1. Then the dual space of the Banach space*

$$
L^1_g(\Gamma_1) = \{\eta \, ; \, \|\eta\| = \int_{\Gamma_1} g(s)|\eta(s)| \, d\Gamma < +\infty\} \tag{36}
$$

is the Banach space

$$
L^\infty_g(\Gamma_1) = \{\zeta \, ; \, \|\zeta\| = \text{ess.sup}_{s \in \Gamma_1} \frac{|\zeta(s)|}{g(s)} < +\infty\}. \tag{37}
$$

Having obtained $|\sigma_t| \leq g_t$, we take $v \in K^S_{\sigma,\beta}$ with $v_t = 0$ on Γ_1 and substitute it into (32). Then we have

$$
-\int_{\Gamma_1} \sigma_t \cdot u_t \, d\Gamma - \int_{\Gamma_1} g_t |u_t| \, d\Gamma \geq 0, \tag{38}
$$

which yields firstly

$$
\int_{\Gamma_1} (\sigma_t \cdot u_t + \int_{\Gamma_1} g_t |u_t|) \, d\Gamma = 0,
$$

with the aid of (7), and hence (10). Thus we have shown that the solution $\{u,p\}$ solves PDEF-Sp-S. q.e.d.

Combining the preceding theorems we have

Theorem 2.3 *The problem PDEF-Sp-S has a solution $\{u,p\}$. The velocity u is unique, while the pressure p is unique except for an arbitrary additive constant.*

3 Leak boundary conditions of friction type for the Stokes equation

3.1 Weak formulation of the problem

In parallel to the study of the slip boundary condition, we are going to formulate the problem BVP-Lk-S in a weaker form. Here, concerning the positive function g_n we assume

(39) $$g_n \in L^2(\Gamma_1).$$

In dealing with the leak boundary conditions, we use

(40) $$K_\beta^L = \{v \in H^1(\Omega); \quad v = \beta \text{ on } \Gamma_0 \,, v_t = 0 \text{ on } \Gamma_1 \}$$

and

(41) $$K_0^L = \{v \in H^1(\Omega); \quad v = 0 \text{ on } \Gamma_0 \,, v_t = 0 \text{ on } \Gamma_1 \}$$

If u, v are in K_β^L, their difference belongs to K_0^L. Furthermore, we put

(42) $$K_{\sigma,\beta}^L = \{v \in H^1(\Omega); \quad \operatorname{div} v = 0 \text{ in } \Omega, v = \beta \text{ on } \Gamma_0 \,, v_t = 0 \text{ on } \Gamma_1 \}$$

and

(43) $$K_{\sigma,0}^L = \{v \in H^1(\Omega); \quad \operatorname{div} v = 0 \text{ in } \Omega, v = 0 \text{ on } \Gamma_0 \,, v_t = 0 \text{ on } \Gamma_1 \}$$

As for σ_n, we note that if u, p are sufficiently smooth up to the boundary, σ_n is by (12). However, under the general circumstances and from the rigorous view point, σ_n should be regarded to be defined as a functional from a subspace of $H^{1/2}(\Gamma_1)$ through the following identity;

(44) $$\int_{\Gamma_1} \sigma_n \psi_n \, d\Gamma = \nu a(u, \psi) - (p, \operatorname{div} \psi) - (f, \psi) \quad (\forall \psi \in K_0^L).$$

Remark 3.1 Let $\eta \in H^{1/2}(\Gamma_1)$. Then there exists $\psi \in K_{\sigma,0}^L$ with $\psi_n = \eta$ on Γ_1, if and only if

(45) $$\int_{\Gamma_1} \eta \, d\Gamma = 0.$$

Provided (45), the value of the right-hand side of (44) does not depend on the way of extension by virtue of (27).

Now we can state properly the p.d.e. formulation (PDEF-Lk-S) of our leak problem for the Stokes equation.

Problem 5 (PDEF-Lk-S) .
Find a vector function $u \in K_{\sigma,\beta}^L$ and $p \in L^2(\Omega)$ such that the weak Stokes equation (27) holds true and the leak boundary condition, say, (17) is satisfied.

Again, the solvability of PDEF-Lk-S is shown with resort to the variational inequality below.

3.2 VI formulation of the problem

Now we introduce the following barrier term against leak on Γ_1 ;

(46) $$j_n = \int_{\Gamma_1} g_n |v_n| \, d\Gamma.$$

Then we state a version of our problem in terms of variational inequalities and complementary conditions.

Problem 6 (VIF-Lk-S) .
Find $u \in K^L_{\sigma,\beta}$ and $p \in L^2(\Omega)$ with the following properties;

1. *u satisfies the inequality*

(47) $$\nu a(u, v-u) - (f, v-u) + j_t(v) - j_t(u) \geq 0 \quad (\forall v \in K^L_{\sigma,\beta}).$$

2. *p is a pressure associated with u such that (13) holds true.*

In the same way as before, we can verify that any solution of (47) and any of its associated pressure jointly satisfy the weak Stokes equation (27). Therefore, as to σ_n, the identity (44) can be used. Moreover, if ψ is restricted to the solenoidal space $K^L_{\sigma,0}$, the identity (44)is reduced to

(48) $$\int_{\Gamma_1} \sigma_n \psi_n = \nu a(u, \psi) - (f, \psi) \quad (\forall \psi \in K^L_{\sigma,0}).$$

Consequently, the variational inequality (47) can be rewritten as

(49) $$\int_{\Gamma_1} \sigma_n (v_n - u_n) \, d\Gamma + j_n(v) - j_n(u) \geq 0 \quad (\forall v \in K^L_{\sigma,\beta}).$$

We claim

Theorem 3.1 *The problem VIF-Lk-S has a solution $\{u, p\}$.*

Proof. By means of standard arguments, it is easy to see that u which solves the variational inequality (47) does exist uniquely. Also it is easy to verify that u satisfies (29) and hence admits of an associated pressure. For the time being, let us take any associated pressure p and fix it. Then we note (49), whence follows by the same argument as before

(50) $$\left| \int_{\Gamma_1} \sigma_n (v-u)_n \, d\Gamma \right| \leq \int_{\Gamma_1} |(v-u)_n| \, d\Gamma, \quad (\forall v \in K^L_{\sigma,\beta}).$$

This implies

(51) $$\left| \int_{\Gamma_1} \sigma_n \eta \, d\Gamma \right| \leq \int_{\Gamma_1} g_n |\eta| \, d\Gamma \qquad (\forall \eta \in Y_0),$$

where

(52) $$Y_0 = \{\eta \in H^{1/2}(\Gamma_1) \; ; \quad \int_{\Gamma_1} \eta \, d\Gamma = 0\}.$$

It should be noted that since Y_0 is not dense in $L_g^1(\Gamma_1)$, we cannot directly apply Lemma 2.1. However, by virtue of the Hahn-Banach theorem and of Lemma 2.1 , we can show that there exists a function $\lambda \in L_g^\infty(\Gamma_1)$ with the property that

$$|\lambda| \leq g_n \quad \text{a. e. in } \Gamma_1 , \tag{53}$$

and

$$\int_{\Gamma_1} \sigma_n \eta \, d\Gamma = \int_{\Gamma_1} \lambda \eta \, d\Gamma, \quad (\forall \eta \in Y_0). \tag{54}$$

The last identity implies that with some constant c we have

$$\lambda = \sigma_n + c. \tag{55}$$

Therefore, if we put

$$p^* = p - c, \text{and} \quad {\sigma_n}^* = \sigma_n \text{with } p \text{ replaced by } p^*, \tag{56}$$

then we have ${\sigma_n}^* = \lambda$ and hence the first inequality of the leak boundary condition, namely, $|{\sigma_n}^*| \leq g_n$. Also, noting (54) and $(v_n - u_n) \in Y_0$, we see that (49) holds true with σ_n replaced by ${\sigma_n}^*$. Thus we have shown that $\{u, p^*\}$ solves VIF-Lk-S. q.e.d.

Theorem 3.2 *The two problems PDEF-Lk-S and VIF-Lk-S are equivalent.*

Proof. PDEF-Lk-S $\Longrightarrow$ VIF-Lk-S.
Let $\{u, p\}$ be a solution of PDEF-Lk-S. Then (49) can be easily verified as

$$\begin{aligned} \textstyle\int_{\Gamma_1} \sigma_n (v-u)_n \, d\Gamma \ &+ \ j_n(v) - j_n(u) \\ &= \textstyle\int_{\Gamma_1} (\sigma_n v_n + g_t |v_n|) \, d\Gamma - \int_{\Gamma_1} (\sigma_n u_n + g_n |u_n|) \, d\Gamma \\ &= \textstyle\int_{\Gamma_1} (\sigma_n v_n + g_t |v_n|) \, d\Gamma \geq 0, \end{aligned} \tag{57}$$

where on the last line use has been made of (17).
VIF-Lk-S $\Longrightarrow$ PDEF-Lk-S.
Let u, p be a solution of VIF-Lk-S. In view of the proof of the preceding theorem, it remains only to prove (16). To this end, we take $v \in K_{\sigma,\beta}^L$ with $v_n = 0$ on Γ_1 and substitute it into (49). Then we have

$$-\int_{\Gamma_1} \sigma_n u_n \, d\Gamma - \int_{\Gamma_1} g_n |u_n| \, d\Gamma \geq 0, \tag{58}$$

which yields (10) with the aid of (13). q.e.d.

As a consequence we have

Theorem 3.3 *The problem PDEF-Lk-S is solvable. The velocity is unique, while the additive constant in the pressure is restricted through (13).*

Remark 3.2 *On the choice of the associated pressure;* In the proof of Theorem 3.1, it was necessary to choose the additive constant in the original pressure p so that the resulting p^* satisfies (13), and in order to show this re-choice is possible, we have employed the Hahn-Banach theorem concerning extension of bounded linear functionals defined on a subspace. However, a more intuitive argument is possible as we describe below. Suppose we have reached (51). Then we claim that from (51) follows

(59) $$|\sigma_n(t) - \sigma_n(s)| \leq g(t) + g(s) \quad (\forall t, s \in \Gamma_1).$$

Proof of (59). Preferring clearness of the essential idea, let us here assume that functions are sufficiently smooth, for justification can be carried out in a standard way. For $s = t$, (59) is trivial. For different t, s, we introduce non-negative functions $e_{\varepsilon,t}$ and $e_{\varepsilon,s}$ which approximate delta functions on Γ_1 with singularities at t, s, respectively and which are normalized as

$$\int_{\Gamma_1} e_{\varepsilon,t}\, d\Gamma = 1 \text{ and } \int_{\Gamma_1} e_{\varepsilon,s}\, d\Gamma = 1.$$

Then we form $\zeta = e_{\varepsilon,t} - e_{\varepsilon,s}$ and note that $\zeta \in Y_0$. Thus we can substitute $\eta = \zeta$ into (51) and go to the limit. Then we have (59). (59) means

$$\sigma_n(t) - \sigma_n(s) \leq g(t) + g(s),$$

and hence

$$\sigma_n(t) - g(t) \leq \sigma_n(s) + g(s),$$

where t and s are separated to each side. Consequently,

(60) $$k_1 \equiv \sup_{t \in \Gamma_1} (\sigma_n(t) - g(t)) \leq \inf_{s \in \Gamma_1} (\sigma_n(s) + g(s)) \equiv k_2.$$

Therefore, we can choose a constant k^* such that

(61) $$k_1 \leq k^* \leq k_2.$$

Then

$$\sigma_n(t) - g(t) \leq k^* \leq \sigma_n(s) + g(s) \quad (\forall t, s \in \Gamma_1),$$

and we have $\sigma_n(s) - k^* \geq -g(s)$ $(\forall s)$ as well as $\sigma_n(t) - k^* \leq g(t)$ $(\forall t)$. Namely, putting

(62) $$\sigma_n^* \equiv \sigma_n - k^*, \quad \text{or equivalently } p* = p + k^*$$

we have

(63) $$-g(t) \leq \sigma_n^*(t) \leq g(t).$$

Thus we have derived (13) with σ_n replaced by ${\sigma_n}^*$.

Incidentally, the admissible range of k^* is given by (61). From physical view point, it is interesting to note that if some leak actually takes place, then the σ_n and so the the admissible associated pressure p is uniquely determined by the leak boundary condition.

4 Slip and leak boundary conditions of friction type for the Navier-Stokes equation

4.1 Slip boundary conditions

formulations and preliminary considerations
The weak p.d.e. formulation (PDEF-Sp-NS) of the problem with the slip boundary condition for the Navier-Stokes equation is now stated.

Problem 7 (PDEF-Sp-NS) .
Find a vector function $u \in K^S_{\sigma,\beta}$ *and* $p \in L^2(\Omega)$ *such that*

1. *the weak form of the Navier-Stokes equation*

$$(64) \qquad \nu a(u,\phi) - (p, \operatorname{div}\phi) + ((u\cdot\nabla)u,\phi) - (f,\phi) = 0 \quad (\forall\phi \in H^1_0(\Omega))$$

   holds true,

2. *the slip boundary condition (11) is satisfied.*

Obviously, (64) is reduced to the following identity which involves u only;

$$(65) \qquad \nu a(u,\phi) + ((u\cdot\nabla)u,\phi) - (f,\phi) = 0 \quad (\forall\phi \in H^1_{0,\sigma}(\Omega))$$

We give some preliminary remarks which are well-known or, at least, essentially well-known.

We put

$$(66) \qquad b(u,v,w) = ((u\cdot\nabla)v,w)$$

for $u, v, w \in K^S_{\sigma,\beta}$. Since our domain Ω is bounded in R^2 or R^3, the tri-linear form $b(u,v,w)$ is continuous from $H^1(\Omega)\times H^1(\Omega)\times H^1(\Omega)$. Furthermore, if $u_n \times v\cdot w$ vanishes on the boundary Γ , then we have

$$(67) \qquad b(u,v,w) = -b(u,w,v), \text{ in particular, } b(u,v,v) = 0.$$

This is the case, for instance, if $u \in K^S_{\sigma,\beta}$ and $v \in K^S_{\sigma,0}$. We take a solenoidal extension $\tilde\beta \in K^S_{\sigma,\beta}$ whose support is confined in a sufficiently narrow boundary strip adjacent to Γ_0 . Such an extension does exists by virtue of the outflow condition

$$(68) \qquad \int_{\Gamma_0} \beta_n \, d\Gamma = 0.$$

Furthermore, the following lemma which might be traced back to an early paper by J. Leray is true (e.g., see [2], [11]).

Lemma 4.1 *For any* $\varepsilon > 0$, *there exists a solenoidal extension of* $\tilde\beta = \tilde\beta_\varepsilon$ *such that*

$$(69) \qquad |b(\psi,\tilde\beta,\psi)| \le \varepsilon\|\nabla\psi\|^2 \quad (\forall\psi \in K^S_{\sigma,0}).$$

Furthermore, we may require that $\tilde\beta$ *is supported by a narrow strip adjacent to* Γ_0 . *Particularly,* $\tilde\beta$ *vanishes on* Γ_1 .

Now we state the corresponding variational inequality which involves only u.

Problem 8 (VIF-Sp-NS) .
Find $u \in K^S_{\sigma,\beta}$ *such that the inequality*

$$(70)\qquad \nu a(u, v-u) + b(u, u, v-u) - (f, v-u) + j_t(v) - j_t(u) \geq 0 \quad (\forall v \in K^S_{\sigma,\beta}),$$

is satisfied.

Again in terms of σ_t, the variational inequality above can be written as

$$(71)\qquad \int_{\Gamma_1} \sigma_t \cdot (v_t - u_t)\, d\Gamma + j_t(v) - j_t(u) \geq 0 \qquad (\forall v \in K^S_{\sigma,\beta}).$$

Consequently, we can apply the same argument as for the Stokes equation and obtain

Theorem 4.1 *The two problems PDEF-Sp-NS and VIF-Sp-NS are equivalent.*

existence of solution

We proceed to the existence proof of the solution of VIF-Sp-NS, for which we make use of the Galerkin method and the Leray-Schauder fixed point theorem. We begin with derivation of an *a priori* estimate for the solution of VIF-Sp-NS. To this end, we firstly choose and fix the solenoidal extension $\tilde{\beta}$in Lemma 4.1 so that

$$(72)\qquad |b(\psi, \tilde{\beta}, \psi)| \leq \frac{\nu}{2} \|\nabla \psi\|^2,$$

and seek the solution in the form

$$(73)\qquad u = \tilde{\beta} + U \quad (U \in K^S_{\sigma,0}).$$

Lemma 4.2 (*a priori estimate) There exists a positive constant* $C^* = C^*(\nu, \Omega, \tilde{\beta}, \|f\|_{L^2(\Omega)})$ *such that any solution* u *of VIF-Sp-NS is bounded as*

$$(74)\qquad \|u - \tilde{\beta}\|_{H^1} \leq C^*, \text{ and so } \|u\|_{H^1} \leq \|\tilde{\beta}\|_{H^1} + C^*.$$

Proof of the lemma. Substitution of $u = \tilde{\beta} + U, \quad (U \in K^S_{\sigma,0})$ into (70) yields

$$(75)\quad \nu a(\tilde{\beta} + U, \psi) + b(\tilde{\beta} + U, \tilde{\beta} + U, \psi) - (f, \psi) + j_t(\tilde{\beta} + U + \psi) - j_t(\tilde{\beta} + U) \geq 0,$$

where we have put $\psi = v - u$. Since $\tilde{\beta} = 0$ on Γ_1 , we note

$$j_t(\tilde{\beta} + U + \psi) = j_t(U + \psi), \; j_t(\tilde{\beta} + U) = j_t(U).$$

Then we put $\psi = -U$ in (75), obtaining

$$(76)\qquad -\nu a(U, U) - \nu a(\tilde{\beta}, U) + (f, U) - b(U, \tilde{\beta}, U) - b(\tilde{\beta}, \tilde{\beta}, U) \geq 0,$$

in consideration of

$$(77)\qquad b(U + \tilde{\beta}, U, U) = 0, \quad j_t(0) = 0, \quad j_t(U) \geq 0.$$

Hence

$$\text{(78)} \qquad \begin{aligned} \nu a(U,U) - |b(U,\tilde\beta,U)| &\le \nu a(U,U) + b(U,\tilde\beta,U) \\ &\le -\nu a(\tilde\beta,U) + (f,U) - b(\tilde\beta,\tilde\beta,U), \end{aligned}$$

whence follows

$$\text{(79)} \qquad \frac{\nu}{2}\|\nabla U\|^2 \le C_1\|\nabla U\|$$

for a positive constant C_1 depending on $\Omega, \tilde\beta$ and $\|f\|$. Thus we have $\|\nabla U\| \le \frac{2}{\nu}C_1$, and hence

$$\text{(80)} \qquad \|U\|_{H^1} \le C_2$$

for another positive constant C_2 depending on $\Omega, \tilde\beta$ and $\|f\|$. With $C^* = C_2$, we obtain the lemma. q.e.d.

In order to construct a sequence of approximate solutions, we introduce a series of finite-dimensional subspace $\mathcal{M}_N$ of $K^S_{\sigma,0}(N = 1,2,3,\ldots)$ in the following way. Let $\{e_k \in K^S_{\sigma,0}\}_{k=1}^{\infty}$ be a basis of $K^S_{\sigma,0}$ in the sense that they are linear independent and their linear hull is dense in $K^S_{\sigma,0}$ under the $H^1(\Omega)$-topology. We may assume that each e_k is smooth. Then we set

$$\text{(81)} \qquad \mathcal{M}_N = \text{ linear span of } \{e_1, e_2, \ldots, e_N\}.$$

We want to obtain the N-th approximate solution u_N in $\mathcal{M}_N$ as the solution of the following variational inequality;

Problem; VIF$_N$-Sp-NS

Find $u_N = \tilde\beta + U_N$ with $U_N \in \mathcal{M}_N$, such that

$$\text{(82)} \qquad \begin{aligned} \nu a(u_N, v-u_N) + b(u_N,u_N,v-u_N) &- (f, v-u_N) \\ &+ j_t(v) - j_t(u_N) \ge 0 \quad (\forall v \in \tilde\beta + \mathcal{M}_N). \end{aligned}$$

Since the nature of this approximate problem is the same as (or simpler that) the VIF-Sp-NS, we have the following

Lemma 4.3 (*a priori estimate* of approximate solutions) *The solution u_N of VIF$_N$-Sp-NS admits of the estimate*

$$\text{(83)} \qquad \|u_N - \tilde\beta\|_{H^1} \le C^*, \text{ and so } \|u_N\|_{H^1} \le \|\tilde\beta\|_{H^1} + C^*.$$

with the same constant C^* as in (74).

As to the existence of the approximate solution u_N, we have

Lemma 4.4 *For each N, the solution $u_N = \tilde\beta + U_N$ of VIF$_N$-Sp-NS exists.*

Proof. With intention to apply the Leray-Schauder fixed-point theorem, we introduce a real parameter $\tau \in [0,1]$, and for any given $U_N \in \mathcal{M}_N$ we define $W_N \in \mathcal{M}_N$ through the following variational inequality in $\mathcal{M}_N$ stated in terms of $u_N = \tilde\beta + U_N$ and $w_N = \tilde\beta + W_N$;

$$\text{(84)} \qquad \begin{aligned} \nu a(w_N, \psi_N) + \tau b(u_N,u_N,\psi_N) &- (f, \psi_N) \\ &+ j_t(w_N + \psi_N) - j_t(w_N) \ge 0 \quad (\forall \psi_N \in \mathcal{M}_N). \end{aligned}$$

Since for the given u_N the form $b(u_N,u_N,\psi_N)$ defines a bounded linear functional on $\mathcal{M}_N$, the existence and uniqueness of $W_N = w_N - \tilde\beta$ follows by a standard argument (which is actually the same as for VIF-Sp-S). We denote the mapping which carries U_N to W_N by $\Phi = \Phi(\tau, \cdot) : \mathcal{M}_N \to \mathcal{M}_N$. It is easily shown that this finite dimensional mapping $\Phi(\tau, U_N)$ is continuous in τ and U_N.

Suppose now that $U_N = \Phi U_N$, namely, U_N is a fixed-point of Φ. Then in the same way as in the derivation of (74),(83), we obtain following *a priori* estimate;

$$\|U_N\|_{H^1} \leq C^*, \tag{85}$$

where C^* is the same positive constant as in (74) and (83).

Finally, we take any positive number $R > C^*$ and fix it. Then we define a closed ball K_N in $\mathcal{M}_N$ by

$$K_N = \{V_N \in \mathcal{M}_N \; ; \; \|V_N\|_{H^1} \leq R\}. \tag{86}$$

For $\tau = 0$, we see that the fixed point U_N of Φ exists in K_N and is unique. For $0 < \tau \leq 1$, any possible fixed point U_N of Φ is subject to the estimate (85) and, therefore, can not reach the boundary of K_N. Thus we can apply the Leray-Schauder theorem and obtain the lemma. q.e.d.

Now we claim

Theorem 4.2 *The problem VIF-Sp-NS has a solution.*

Proof. Let $u_N = \tilde{\beta} + U_N$ be the solution of VIF$_N$-Sp-NS. Since they are bounded as

$$\|u_N\|_{H^1} \leq \|\tilde{\beta}\|_{H^1} + C^*,$$

we can select a subsequence $u_n = u_{n(N)}$ which converges to $u^* \in K^S_{\sigma,\beta}$ weakly in $H^1(\Omega)$. Verification that u^* is the required solution is as follows. Take any $\psi \in \mathcal{M}_N$ and fix it. Then for sufficiently large n, it holds that

$$\nu a(u_n, \psi) + b(u_n, u_n, \psi) - (f, \psi) + j_t(u_n + \psi) - j_t(u_n) \geq 0. \tag{87}$$

Noting that u_n converges strongly in $L^2(\Omega)$ as well as in $L^4(\Omega)$, and that $(u_n)_t =$ tangential component of u_n on Γ_1 converges strongly in $L^2(\ \Gamma_1\ )$, we can go to the limit in (87 and get

$$\nu a(u^*, \psi) + b(u^*, u^*, \psi) - (f, \psi) + j_t(u^* + \psi) - j_t(u^*) \geq 0 \tag{88}$$

Since $\cup_{N=1}^{\infty} \mathcal{M}_N$ is dense in $K^S_{\sigma,0}$, and since each term of (88) is continuous in ψ with $H^1(\Omega)$-strong topology, we have

$$\nu a(u^*, \psi) + b(u^*, u^*, \psi) - (f, \psi) + j_t(u^* + \psi) - j_t(u^*) \geq 0, \quad (\forall \psi \in K^S_{\sigma,0}), \tag{89}$$

which means that u^* is a solution of VIF-Sp-NS. q.e.d.

As a corollary we have

Theorem 4.3 *The problem PDEF-Sp-NS has a solution.*

Remark 4.1 Uniqueness of solutions of PDEF-Sp-NS and VIF-Sp-NS can be shown if the Reynolds number is sufficiently large.

4.2 Leak boundary conditions

As for the leak boundary conditions for the Navier-Stokes equation, the formulation of PDEF-Lk-NS and VIF-Lk-NS is quite parallel to that for the slip boundary condition. Their equivalence is obtained also similarly. However, existence of solutions becomes more difficult. This is because we now only have

$$b(u,v,v) = \frac{1}{2}\int_{\Gamma_1} u_n|v|^2\,d\Gamma \quad (\forall u,v \in K^L_{\sigma,0}) \tag{90}$$

instead of $b(u,v,v) = 0$ which is the case with $K^S_{\sigma,0}$. Still we can show, as will be given elsewhere in detail,

Theorem 4.4 *If the Reynolds number is sufficiently small, then the problem VIF-Lk-NS has a solution.*

5 Comments and remarks

I. slip and leak boundary conditions and sub-differentials

The slip boundary condition, say, in the form of (11) can be compactly written by use of the sub-differential of $|\cdot|$. In fact, the multi-valued sub-differential $\partial|\cdot|$ of $|\cdot| : \lambda \to |\lambda| \quad (\lambda \in R^1)$ is given by

$$\partial|\lambda| = \begin{cases} 1 & (\lambda > 0) \\ \text{the interval } [-1,1] & (\lambda = 0) \\ -1 & (\lambda < 0) \end{cases} \tag{91}$$

We can easily verify that the slip boundary condition (11) is equivalent to

$$-\sigma_t(u) \in \partial(g_t(s)|u_t|) \quad (\text{a.e. on } \Gamma_1). \tag{92}$$

Therefore if δ is a small positive number, the boundary condition

$$-\sigma_t(u) = g_t(s)\tanh\left(\frac{u_t}{\delta}\right) \quad (\text{a.e. on } \Gamma_1) \tag{93}$$

would be a good approximation of the slip boundary condition. Similarly, the leak boundary condition (17) can be written as

$$-\sigma_n(u,p) \in \partial(g_t(s)|u_n|) \quad (\text{a.e. on } \Gamma_1). \tag{94}$$

co-existence of slip and leak

We can deal with the boundary conditions under which leak and slip may take place at at the same time, by taking the admissible set

$$K_{\sigma,\beta} = \{v \in H^1(\Omega)\ ;\ \operatorname{div} v = 0 \text{ in } \Omega, v = \beta \text{ on } \Gamma_0\} \tag{95}$$

and considering variational inequalities involving the barrier term

$$j(v) = j_t(v) + j_n(v) = \int_{\Gamma_1} (g_t(s)|v_t| + g_n(s)|v_n|)\,d\Gamma. \tag{96}$$

The analysis goes through without any increased difficulty.

alternative choice of $a(\ ,\ )$

From the view point of hydrodynamics, we might have to use as the $H^1(\Omega)$-ellipticity form $a(u,v)$ the following 'deformation integral form' $E(u,v)$ instead of the Dirichlet form $(\nabla u, \nabla v)_{L^2(\Omega)}$;

(97)
$$E(u,v) = \frac{1}{2}\int_\Omega \sum_{i,j} e_{i,j}(u)e_{i,j}(v)dx$$

where

(98)
$$e_{i,j}(u) = \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}.$$

This replacement does not affect the form of the equations in Ω, namely, the Stokes or the Navier-Stokes equations remain the same, while σ_t and σ_n in the slip and leak boundary conditions must be changed duly, since these boundary conditions are a kind of the natural boundary conditions. Details will be given in [6] and elsewhere.

alternative formulations with non-solenoidal admissible set

As mentioned in Introduction, we can formulate variational inequality with admissible functions which are not necessarily solenoidal. For instance, in order to deal with VIF-Sp-S, we adopt as the admissible set

(99)
$$K_\beta^L = \{v \in H^1(\Omega)\ ;\ v = \beta \text{ on } \Gamma_0\ ,\ v_t = 0 \text{ on } \Gamma_1\ \}$$

and pose the following problem

Problem 9 *Find $u \in K_\beta^L$ and $p \in L^2(\Omega)$ such that*

(100)
$$\nu a(u, v-u) - (p, \operatorname{div}(v-u)) - (f, v-u) + j_t(v) - j_t(u) \geq 0 \quad (\forall v \in K_\beta^L),.$$

and

(101)
$$(\operatorname{div} u, q) = 0 \quad (\forall q \in L^2(\Omega)).$$

It is interesting to note that any solution u, p of this new variational inequality solves PDEF-Sp-S automatically. However, its solvability is obtained by means of the previous argument in Section 3. Furthermore, such formulation with the non-solenoidal admissible set enable us to formulate a kind of saddle-point search method, and a certain numerical approaches. Again, details will be given in [6] and elsewhere.

References

[1] G. Duvaut and J. L. Lions. *Les Inéquations en Mećhanique et en Physique*, Dunod, 1972; (English version), Springer, 1976.

[2] H. Fujita. On the existence and regularity of the steady-state solutions of the Navier-Stokes equations, J. Fac. Sci.Univ.Tokyo Sect.IA, 9(1991), pp.59-102.

[3] H. Fujita. Flow problems with unilateral boundary conditions, Leçons, Collège de France, October 1993.

[4] H. Fujita and H. Kawarada. Notes on stationary motions of viscous incompressible fluid under leak boundary conditions, preprint, 1994

[5] H. Fujita, H. Kawarada and H. Morimoto. Notes on stationary motions of viscous incompressible fluid under slip boundary conditions, preprint, 1994

[6] H. Fujita, H. Kawarada, and A. Sasamoto. Analytical and numerical approaches to stationary flow problems with leak and slip boundary conditions, to appear in Proc. of 2nd Japan-China Joint Seminar on Numerical Mathematics held on August 22 – 26, 1994, Chofu, Japan.

[7] R. Glowinski, J. L. Lions and R. Trémolières. *Numerical Analysis of Variational Inequalities*, (Studies in mathematics and its applications), North-Holland, 1981.

[8] R. Glowinski. *Numerical Methods for Nonlinear Variational Problems.* (Springer series in computational physics), Springer-Verlag, 1984.

[9] H. Kawarada, H. Kawahara and H. Fujita. Fictitious domain method via singular perturbation and its application to flow problems with unilateral boundary conditions, to appear in East–West J. of Numer. Math.

[10] N. K. Kikuchi and J. J. Oden. *Contact Problems in Elasticity*, (SIAM studies in applied mathematics), SIAM, 1988.

[11] O. A. Ladyshenskaya. *The Mathematical Theory of Viscous Incompressible Flow*, Revised English edition translated by R. A. Silverman, Gordon and Breach, 1963, (Russian original,1961).

[12] R. Temam. *Navier-Stokes Equations*, (Studies in mathematics and its applications), North-Holland, 1977.