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Generalized Poisson integrals on unbounded domains and their applications

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PART 1. Introduction and the case of half-spaces

Let $\mathbb{R}$ and $\mathbb{R}_+$ be the set of all real numbers and all positive real numbers, respectively. The boundary and the closure of a set S in the n-dimensional Euclidean space $\mathbb{R}^n$ $(n\geq 2)$ are denoted by ∂S and $\overline{S}$, respectively. We also introduce the spherical coordinates (r,Θ), $\Theta=(\theta_1,\theta_2,\ldots,\theta_{n-1})$, in $\mathbb{R}^n$ which are related to the cartesian coordinates (X,y), $X=(x_1,x_2,\ldots,x_{n-1})$ by the formulas

$$x_1 = r\left(\prod_{j=1}^{n-1} \sin\theta_j\right) \quad (n\geq 2), \qquad y = r\cos\theta_1,$$

and if $n\geq 3$,

$$x_{n+1-k} = r\left(\prod_{j=1}^{k-1} \sin\theta_j\right)\cos\theta_k \quad (2\leq k\leq n-1),$$

where

$$0\leq r<+\infty, \quad 0\leq\theta_j\leq\pi \ (1\leq j\leq n-2;\ n\geq 3), \quad -2^{-1}\pi<\theta_{n-1}\leq 2^{-1}3\pi \quad (n\geq 2).$$

The unit sphere (the unit circle, if n=2) and the upper half unit sphere $\{(1,\theta_1,\theta_2,\ldots,\theta_{n-1})\in\mathbb{R}^n;\ 0\leq\theta_1<\frac{\pi}{2}\}$ (the upper half unit circle $\{(1,\theta_1)\in\mathbb{R}^2;\ -\frac{\pi}{2}<\theta_1<\frac{\pi}{2}\}$, if n=2) in $\mathbb{R}^n$ are denoted by S^{n-1} and S_+^{n-1}, respectively. The half-space

$$\{(X,y)\in\mathbb{R}^n;\ X\in\mathbb{R}^{n-1},\ y>0\} = \{(r,\Theta)\in\mathbb{R}^n;\ \Theta\in S_+^{n-1},\ 0<r<+\infty\}$$

is denoted by T_n.

Given a domain $D\subset\mathbb{R}^n$ and a continuous function g on a subset $S\subset\partial D$, we say that h is a solution of the (classical) Dirichlet problem on D with g, if h is harmonic in D and

$$\lim_{P\in D,\ P\to Q} h(P) = g(Q)$$

for every $Q\in S$. If D is a bounded domain and g is a bounded function on ∂D, then the existence of a solution of the Dirichlet problem and its uniqueness is completely known (e.g. see [10, Theorem 5.21]). Otherwise we may suppose that D is always an unbounded domain by using the Kelvin transformation.

When D is the typical unbounded domain $\mathbb{T}_n$, the following results are known. Let $g(X)$ be a continuous function on $\partial\mathbb{T}_n=\mathbb{R}^{n-1}$ satisfying (1) with a non-negative integer ℓ:

$$\int_{\mathbb{R}^{n-1}} \frac{|g(X)|}{1+|X|^{n+\ell}}\, dX < +\infty. \tag{1}$$

Then Armitage [1, Theorem 2] gave the explicit form of a solution of the Dirichlet problem on $\mathbb{T}_n$ with g (also see Siegel [15, P.1 and p.7]). Further, for any continuous function $g(X)$ on $\partial\mathbb{T}_n$ Finkelstein and Scheinberg [7] showed the existence of a solution of the Dirichlet problem on $\mathbb{T}_n$ with g and Gardiner [8] gave the solution explicitely.

About the uniqueness of solutions of the Dirichlet problem on $\mathbb{T}_n$, Helms [11, p.42 and p.158] states that even if $g(X)$ is a bounded continuous function on $\partial\mathbb{T}_n$, the solution of the Dirichlet problem on $\mathbb{T}_n$ with g is not unique and to obtain the unique solution H(P) ($P=(X,y)\in\mathbb{T}_n$) we must specify the behavior of H(P) as $y\to+\infty$. In connection with this remark, Siegel [15, Theorems 1 and 3] gave the following result to more restricted boundary function g than (1).

Let ℓ *be a non-negative integer. If* $g(X)$ $(X\in\partial\mathbb{T}_n=\mathbb{R}^{n-1})$ *is a continuous function on* $\partial\mathbb{T}_n$ *such that*

$$|g(X)| \le G(x) \qquad (X\in\mathbb{R}^{n-1},\ |X|=x>0)$$

for a continuous function $G(x)$ $(x\in\mathbb{R})$, $G(x)=G(-x)$,

$$\int_{-\infty}^{+\infty} \frac{|G(x)|}{1+|x|^{n+\ell}}\,dx < +\infty,$$

then there exists a solution $H(\mathbb{T}_n,\ell;g)(P)$ *of the Dirichlet problem on* $\mathbb{T}_n$ *with g satisfying*

$$H(\mathbb{T}_n,\ell;g)(P)=o(r^{\ell+1}/\cos\theta_1) \qquad (r\to+\infty)$$

$$(P=(r,\Theta)\in\mathbb{T}_n,\ \Theta=(\theta_1,\theta_2,\ldots\theta_{n-1})).$$

If $h(P)$ *is a solution of the Dirichlet problem on* $\mathbb{T}_n$ *with g such that*

$$h(P) = o(r^{\ell+1}/\cos\theta_1) \qquad (P=(r,\Theta)\in\mathbb{T}_n),$$

then

$$h(P) = H(\mathbb{T}_n,\ell;g)(P) + \Gamma(h)(P) \qquad (P\in\mathbb{T}_n),$$

where $\Gamma(h)(P)$ *is a harmonic polynomial (of* $P=(x_1,x_2,\ldots,x_{n-1},y)\in\mathbb{R}^n$) *of degree* ℓ *vanishing on*

$$\partial\mathbb{T}_n=\{(x_1,x_2,\ldots,x_{n-1},0)\in\mathbb{R}^n;\ (x_1,x_2,\ldots,x_{n-1})\in\mathbb{R}^{n-1}\}.$$

To answer the question of Siegel [15, p.8] Yoshida [19] proved

THEOREM Y_1 [19, Theorems 1 and 2]. *Let* $g(Q)$ *be a continuous function on* $\partial\mathbb{T}_n$ $(n\ge 2)$ *satisfying* (1) *with a non-negative integer* ℓ. *Then there exists a solution* $H(\mathbb{T}_n,g;\ell)(P)$ *of the Dirichlet problem with g satisfying*

$$\lim_{r\to\infty} r^{-\ell-1}\int_{S_+^{n-1}} H(\mathbb{T}_n,\ell;g)(r,\Theta)\cos\theta_1\,d\sigma_\Theta = 0$$

$$(P=(r,\Theta)\in\mathbb{T}_n,\ \Theta=(\theta_1,\theta_2,\ldots\theta_{n-1})),$$

where $d\sigma_\Theta$ *is the surface element of* S^{n-1}. *If* $h(P)$ *is a solution of* *the Dirichlet problem on* $\mathbb{T}_n$ *with* g *satisfying*

$$\lim_{r\to\infty} r^{-\ell-1}\int_{\mathbb{S}_+^{n-1}} h^+(r,\theta)\cos\theta_1\, d\sigma_\theta = 0,$$

then

$$h(P) = H(\mathbb{T}_n,\ell;g)(P) + \Pi(h)(P), \quad \Pi(h)(P) = \begin{cases} y\Pi^*(h)(P) & (\ell\geq 1) \\ 0 & (\ell=0) \end{cases}$$

for every $P=(X,y)\in\mathbb{T}_n$, *where* $\Pi^*(h)(P)$ *is a polynomial of* $P=(x_1,x_2,\ldots,x_{n-1},y)\in\mathbb{R}^n$ *of degree at most* $\ell-1$ *and even with respect to the variable* y.

PART 2. The conical case

1. Introduction

A half-space is a special one of more general unbounded domains cones. To generalize Theorem Y_1 to results about cones, we shall first pay attention to Yoshida's result [18, Theorem 3] concerning the Dirichlet problem on a cone. To state it, we need some preliminaries.

Let Δ_n ($n\geq 2$) be the Laplace operator and Λ_n be the spherical part of the spherical coordinates of Δ_n:

$$\Delta_n = \frac{n-1}{r}\frac{\partial}{\partial r} + \frac{\partial^2}{\partial r^2} + \frac{1}{r^2}\Lambda_n.$$

Given a domain Ω on S^{n-1} ($n\geq 2$), consider the Dirichlet problem for

(2) $$(\Lambda_n + \lambda)F = 0 \quad \text{on } \Omega$$
$$F = 0 \quad \text{on } \partial\Omega.$$

we denote the least positive eigenvalue of (2) by $\lambda(\Omega,1)$ and the normalized positive eigenfunction corresponding $\lambda(\Omega,1)$ by $f_1^{\Omega}(\theta)$. We shall denote two solutions of the equation

$$t^2 + (n-2)t - \lambda(\Omega,1) = 0$$

by $\alpha(\Omega,1)$, $-\beta(\Omega,1)$ ($\alpha(\Omega,1)$, $\beta(\Omega,1)>0$). Given a domain Ω on S^{n-1}, the set

$$\{(r,\theta)\in\mathbb{R}^n;\ (1,\theta)\in\Omega,\ r\in\mathbb{R}_+\} \quad\text{and}\quad \{(r,\theta)\in\mathbb{R}^n;\ (1,\theta)\in\partial\Omega,\ r\in\mathbb{R}_+\}$$

in $\mathbb{R}^n$ are denoted by $C_n(\Omega)$ and $S_n(\Omega)$, respectively. If $n=2$, then $C_2(\Omega)$ is an angular domain.

In the following, we put the strong assumption relative to Ω on S^{n-1}: if $n\geq 3$, Ω is a $C^{2,\alpha}$-domain ($0<\alpha<1$) on S^{n-1} surrounded by a finite number of mutually disjoint closed hypersurfaces (e.g. see [10, pp.88-89] for the definition of $C^{2,\alpha}$-domain).

Let $G_{C_n(\Omega)}((r_1,\theta_1),(r_2,\theta_2))$ ((r_1,θ_1), $(r_2,\theta_2)\in C_n(\Omega)$) be the Green function of a cone $C_n(\Omega)$ and let s_n denote the surface area $2\pi^{n/2}\{\Gamma(n/2)\}^{-1}$ of S^{n-1}. The function

$$c_n^{-1}\frac{\partial}{\partial\nu}G_{C_n(\Omega)}(P,Q), \qquad c_n = \begin{cases} 2\pi & (n=2) \\ (n-2)s_n & (n\geq 3) \end{cases}$$

of $Q\in\partial C_n(\Omega)$ for any fixed $P\in C_n(\Omega)$ is an ordinary Poisson kernel, where $\frac{\partial}{\partial\nu}$ denotes the differentiatin at Q along the inward normal into $C_n(\Omega)$. Let $F(r,\theta)$ be a function on $C_n(\Omega)$. We put

(3) $$\mu_0(F) = \lim_{r\to\infty} r^{-\alpha(\Omega,1)}\int_\Omega F(r,\theta)f_1^\Omega(\theta)d\sigma_\theta$$

$$\text{and} \qquad \eta_0(F) = \lim_{r\to\infty} r^{\beta(\Omega,1)}\int_\Omega F(r,\theta)f_1^\Omega(\theta)d\sigma_\theta,$$

if they exist, where $d\sigma_\theta$ is the surface element on S^{n-1}.

THEOREM Y_2 [18, Theorem 3 and Lemma 3]. *Let $g(Q)=g(t,\Xi)$ be a continuous function on $S_n(\Omega)$ satisfying*

(4) $$\int^{+\infty} t^{-\alpha(\Omega,1)-1}\left(\int_{\partial\Omega}|g(t,\Xi)|d\sigma_\Xi\right)dt < +\infty$$

$$\text{and} \qquad \int_0 t^{\beta(\Omega,1)-1}\left(\int_{\partial\Omega}|g(t,\Xi)|d\sigma_\Xi\right)dt < +\infty$$

(if $n=2$ and $\Omega=(\gamma,\delta)$, then

$$\int_{\partial\Omega}|g(t,\Xi)|d\sigma_\Xi = |g(t,\gamma)| + |g(t,\delta)|).$$

Then the Poisson integral

$$H(C_n(\Omega);g)(P) = c_n^{-1}\int_{S_n(\Omega)} g(Q)\frac{\partial}{\partial\nu}G_{C_n(\Omega)}(P,Q)d\sigma_Q$$

is a solution of the classical Dirichlet problem on $C_n(\Omega)$ with g such that

$$\mu_0(|H(C_n(\Omega);g)|) = 0 \quad \text{and} \quad \eta_0(|H(C_n(\Omega);g)|) = 0,$$

where $d\sigma_\theta$ *is the surface element on* S^{n-1}. *If* $h(P)$ *is any solution of the classical Dirichlet problem on* $C_n(\Omega)$ *with* g, *then all of the limits* $\mu_0(h)$, $\eta_0(h)$ $(-\infty<\mu_0(h), \eta_0(h)\leq+\infty)$, $\mu_0(|h|)$ *and* $\eta_0(|h|)$ $(0\leq\mu_0(|h|), \eta_0(|h|)\leq+\infty)$ *exist, and if*

$$\mu_0(|h|) < +\infty \quad \text{and} \quad \eta_0(|h|) < +\infty,$$

then

$$h(P) = H(C_n(\Omega),g)(P) + (\mu_0(h)r^{-\alpha(\Omega,1)} + \eta_0(h)r^{\beta(\Omega,1)})f_1^\Omega(\theta)$$

for any $P=(r,\theta)\in C_n(\Omega)$.

In this paper we shall show the existence of solutions of the Dirichlet problem on a cone (Theorems 1 and 2) and a type of uniqueness of them (Theorems 7 and 8) by introducing the conical generalized Poisson kernels and Poisson integrals, the special cases of which are $H(\mathbb{T}_n,\ell;g)$ and $H(C_n(\Omega);g)(P)$. They generalize Theorem Y_1 to the conical case and Theorem Y_2 to more unrestricted boundary function than (4). To prove the uniqueness, we shall give two results (Theorems 5 and 6) which are the conical version of Kuran's result [12, Theorem 10]. We also generalize the results of Finkelstein and Scheinberg [7] and Gardinar [8] to the conical case (Theorems 3 and 4). Finally a result of Yoshida [19, Theorem 3] will be generalized in the conical form (Theorem 9).

2. Results about the existence of solutions.

We denote the non-decreasing sequence of positive eigenvalues of (2) by $\{\lambda(\Omega,k)\}_{k=1}^{\infty}$. In this expression we write $\lambda(\Omega,k)$ the same number of times as the dimension of the corresponding eigenspace. When the normalized eigenfunction corresponding $\lambda(\Omega,k)$ is denoted by $f_k^{\Omega}(\theta)$, the set of sequential eigenfunctions corresponding to the same value of $\lambda(\Omega,k)$ in the sequence $\{f_k^{\Omega}(\theta)\}_{k=1}^{\infty}$ makes an orthonormal basis for the eigenspace of the eigenvalue $\lambda(\Omega,k)$. We can also say that for each $\Omega\subset\mathbb{S}^{n-1}$ there is a sequence $\{k_i\}$ of positive integers such that $k_1=1$, $\lambda(\Omega,k_i)<\lambda(\Omega,k_{i+1})$,

$$\lambda(\Omega,k_i)=\lambda(\Omega,k_i+1)=\lambda(\Omega,k_i+2)=\ldots=\lambda(\Omega,k_{i+1}-1) \qquad (i=1,2,3,\ldots)$$

and $\{f_{k_i}^{\Omega}, f_{k_i+1}^{\Omega}, \ldots, f_{k_{i+1}-1}^{\Omega}\}$ is an orthonormal basis for the eigenspace of the eigenvalue $\lambda(\Omega,k_i)$ $(i=1,2,3,\ldots)$. It is well known that $k_2=2$ and $f_1^{\Omega}(\theta)>0$ for any $\theta\in\Omega$ (see Courant and Hilbert [5, p.451 and p.458]). With respect to $\{k_i\}$, the following Remark 1 shows that even in the case $\Omega=\mathbb{S}_+^{n-1}$ $(n=2,3,4,\ldots)$, not only the simplest case $k_i=i$ $(i=1,2,3,\ldots)$ but also other complex cases can appear.

If we note that Ω is an $(n-1)$-dimensional compact Riemannian manifold with its boundary to be sufficiently regular, we know that

$$\lambda(\Omega,k) \sim A(\Omega,n)k^{2/(n-1)} \qquad (k\to+\infty)$$

(e.g. see Cheng and Li [4]) and

$$\sum_{\lambda(\Omega,k)\le x} \{f_k^{\Omega}(\theta)\}^2 \sim B(\Omega,n)x^{(n-1)/2} \qquad (x\to+\infty)$$

uniformly with respect to θ (e.g. Minakshisundaram and Pleijel [13], and also Essén and Lewis [6, p.120 and pp.126-128]), where $A(\Omega,n)$ and $B(\Omega,n)$ are both constants depending Ω and n, respectively. Hence there exist two positive constants M_1, M_2 such that

$$(5) \qquad M k_1{}^{2/(n-1)} \le \lambda(\Omega,k) \qquad (k=1,2,\ldots)$$

and

$$|f_k^{\Omega}(\theta)| \le M_2 k^{1/2} \qquad (\theta\in\Omega,\ k=1,2,\dots)$$

If we denote two solutions of the equation

$$t^2 + (n-2)t - \lambda(\Omega,k) = 0$$

by $\alpha(\Omega,k)$, $-\beta(\Omega,k)$ ($\alpha(\Omega,k)$, $\beta(\Omega,k)>0$), then we also have

$$\alpha(\Omega,k),\ \beta(\Omega,k) \ge M_3 k^{1/(n-1)} \qquad (k=1,2,\dots),$$

from (5), where M_3 is a positive constant independent of k. We remark that both

$$r^{\alpha(\Omega,k)} f_k^{\Omega}(\theta) \quad \text{and} \quad r^{-\beta(\Omega,k)} f_k^{\Omega}(\theta) \qquad (k=1,2,\dots.)$$

are harmonic on $C_n(\Omega)$ and vanish continuously on $S_n(\Omega)$. For a domain Ω and the sequence $\{k_i\}$ mentioned above, by $I(\Omega,k_\ell)$ we denote the set of all positive integers less than k_ℓ ($\ell=1,2,3,\dots$). In spite of the fact $I(\Omega,k_1)=\phi$, the summation over $I(\Omega,k_1)$ of a function S(k) of a variable k will be used by promising

$$\sum_{k\in I(\Omega,k_1)} S(k) = 0.$$

REMARK 1. Suppose $\Omega=S_+^{n-1}$ ($n\ge2$). Then

(6) $$c_n^{-1} \frac{\partial}{\partial\nu} G_{\mathbb{T}_n}((r,\theta),(t,\Xi)) = 2s_n^{-1}\sum_{k=0}^{\infty} c_{k,n+2} r^{k+1} t^{-k-n} \cos\theta_1 L_{k,n+2}(\cos\gamma)$$

for any $(X,y)=(r,\theta)\in\mathbb{T}_n$ and $(Z,0)=(t,\Xi)\in\partial\mathbb{T}_n$ satisfying $r<t$, where $c_{k,n+2}=\binom{k+n-1}{k}$, $L_{k,n+2}$ is the (n+2)-dimensional Legendre polynomial of degree k and γ is the angle between M=(X,0) and N=(Z,0) defined by

$$\cos\gamma = \frac{(M,\ N)}{|M|\,|N|}$$

(see Armitage [1, Theorem E]). On the other hand, Remark 4 in Section 4 applied to $\Omega=S_+^{n-1}$ gives the Fourier series expansion of the function

$$c_n^{-1}\frac{\partial}{\partial\nu}G_{\mathbb{T}_n}((r,\theta),(t,\Xi))\qquad (r<t)$$

of θ with respect to the sequence of eigenfunctions of (2). Hence, in comparison with (2-4) we obtain

(7) $$\alpha(S_+^{n-1},k_i) = i,$$

$$\beta(S_+^{n-1},k_i) = n+i-2 \qquad (i=1,2,3,...;\ n=2,3,4,...).$$

Consider the simplest case n=2 i.e. $\Omega=S_+^1$. For $(r,\theta_1)\in\mathbb{T}_2$ and $(t,\Xi)=t\in\mathbb{R}$, we see $\cos\gamma=\frac{t}{|t|}\sin\theta_1$ and hence

$$k_i = i \qquad (i=1,2,3,...)$$

$$f_k^{\Omega}(\theta_1) = \rho_k\cos\theta_1 L_{k-1,4}(\sin\theta_1) \qquad (k=1,2,...),$$

where

$$\rho_k = \{\int_{-2^{-1}\pi}^{2^{-1}\pi}(\cos\theta_1 L_{k-1,4}(\sin\theta_1))^2 d\theta_1\}^{-1/2}.$$

Next, suppose n=3 i.e. $\Omega=S_+^2$. Then for $(r,\theta)=(X,y)\in\mathbb{T}_3$, $\theta=(\theta_1,\theta_2)$ and $(t,\Xi)\in\partial\mathbb{T}_3=\mathbb{R}^2$, $\Xi=(\frac{\pi}{2},\xi_2)$, we see

$$\cos\gamma = \sin\theta_1\sin\theta_2\sin\xi_2 + \sin\theta_1\cos\theta_2\cos\xi_2.$$

If we put

$$L_{0,5} = \Phi_{0,0} = 1$$

and

$$\begin{aligned} L_{k,5}(\sin\theta_1\sin\theta_2\sin\xi_2 + \sin\theta_1\cos\theta_2\cos\xi_2) &= \Phi_{k,0}(\theta_1,\theta_2)\cos^k\xi_2 + \Phi_{k,1}(\theta_1,\theta_2)\cos^{k-2}\xi_2 + \cdots \\ &\quad + \Phi_{k,[2^{-1}k]}(\theta_1,\theta_2)\cos^{k-2[2^{-1}k]}\xi_2 \\ &\quad + \Psi_{k,0}(\theta_1,\theta_2)\cos^{k-1}\xi_2\sin\xi_2 + \Psi_{k,1}(\theta_1,\theta_2)\cos^{k-3}\xi_2\sin\xi_2 + \cdots \\ &\quad + \Psi_{k,[(k-1)/2]}(\theta_1,\theta_2)\cos^{k-1-2[(k-1)/2]}\xi_2\sin\xi_2 \end{aligned}$$

$$(k=1,2,3...),$$

then

$$k_i = 1 + \frac{(i-1)i}{2} \qquad (i=1,2,3,...)$$

and $$f_1^{\Omega}(\theta) = (2ns_n^{-1})^{1/2}\cos\theta_1,$$

$$f_{k_i+j}^{\Omega}(\theta) = \begin{cases} \rho_{k_i+j}\Phi_{i-1,j}(\theta_1,\theta_2)\cos\theta_1, \\ \qquad (j=0,1,\dots,[(i-1)/2]) \\ \rho_{k_i+j}\Psi_{i-1,j-[(i-1)/2]-1}(\theta_1,\theta_2)\cos\theta_1 \\ \qquad (j=[(i-1)/2]+1,\dots,[(i-1)/2]+[(i-2)/2]+1) \end{cases}$$
$$(i=2,3,4,\dots),$$

where

$$\rho_{k_i+j} = \begin{cases} \{\int_{S_+^2}\Phi_{i-1,j}^2(\theta_1,\theta_2)\cos^2\theta_1\, d\sigma_\theta\}^{-1/2} \\ \qquad (j=0,1,\dots,[(i-1)/2]) \\ \{\int_{S_+^2}\Psi_{i-1,j-[(i-1)/2]-1}^2(\theta_1,\theta_2)\cos^2\theta_1\, d\sigma_\theta\}^{-1/2} \\ \qquad (j=[(i-1)/2]+1,\dots,[(i-1)/2]+[(i-2)/2]+1) \end{cases}$$
$$(i=2,3,4,\dots).$$

The Fourier coefficient

$$\int_{\Omega} F(\theta) f_k^{\Omega}(\theta)d\sigma_\theta$$

of a function $F(\theta)$ on Ω with respect to the orthnormal sequence $\{f_k^{\Omega}(\theta)\}$ is denoted by $c(F,k)$, if it exists. Now we shall define generalized Poisson kernels of the conical type. For two non-negative integers ℓ, m and two points $P=(r,\theta)\in C_n(\Omega)$, $Q=(t,\Xi)\in S_n(\Omega)$, we put

(8) $$\overline{V}(C_n(\Omega),\ell)(P,Q) = \sum_{k\in I(\Omega,k_{\ell+1})} 2^{\alpha(\Omega,k)+n-1}c((H_\Xi)_1,k)t^{-\alpha(\Omega,k)-n+1}r^{\alpha(\Omega,k)}f_k^{\Omega}(\theta)$$

and

$$\underline{V}(C_n(\Omega),m)(P,Q) = \sum_{k\in I(\Omega,k_{m+1})} (\tfrac{3}{2})^{\beta(\Omega,k)}2^{n-1}c((H_\Xi)_3,k)t^{\beta(\Omega,k)-n+1}r^{-\beta(\Omega,k)}f_k^{\Omega}(\theta),$$

where

$$(H_{\Xi})_r(\theta) = c_n^{-1}\frac{\partial}{\partial\nu}G_{C_n(\Omega)}((r,\theta),(2,\Xi)) \qquad (r=1,3).$$

We introduce two functions of $P\in C_n(\Omega)$ and $Q=(t,\Xi)\in S_n(\Omega)$

$$\overline{W}(C_n(\Omega),\ell)(P,Q) = \begin{cases} \overline{V}(C_n(\Omega),\ell)(P,Q) & (1\le t<+\infty) \\ 0 & (0<t<1) \end{cases}$$

and

$$\underline{W}(C_n(\Omega),m)(P,Q) = \begin{cases} \underline{V}(C_n(\Omega),m)(P,Q) & (0<t<1) \\ 0 & (1\le t<+\infty). \end{cases}$$

The generalized Poisson kernel $K(C_n(\Omega),\ell,m)(P,Q)$ with respect to $C_n(\Omega)$ is defined by

$$K(C_n(\Omega),\ell,m)(P,Q) = c_n^{-1}\frac{\partial}{\partial\nu}G_{C_n(\Omega)}(P,Q)-\overline{W}(C_n(\Omega),\ell)(P,Q)-\underline{W}(C_n(\Omega),m)(P,Q).$$

In fact

$$K(C_n(\Omega),\ell,0)(P,Q) = c_n^{-1}\frac{\partial}{\partial\nu}G_{C_n(\Omega)}(P,Q)-\overline{W}(C_n(\Omega),\ell)(P,Q)) \qquad (\ell\ge 1)$$

and

$$K(C_n(\Omega),0,0)(P,Q) = c_n^{-1}\frac{\partial}{\partial\nu}G_{C_n(\Omega)}(P,Q).$$

REMARK 2. Put $\Omega=S_+^{n-1}$ and $r_2=1$ in Remark 3 of Section 4. Then from (7) we have

$$c_n^{-1}\frac{\partial}{\partial\nu}G_{T_n}((r,\theta),(t,\Xi)) = \sum_{i=0}^{\infty} 2^{n-1+i}r^i t^{1-n-i}\Big(\sum_{k=k_i}^{k_{i+1}-1} c((H_{\Xi})_1,k)f_k^{\Omega}(\theta)\Big)$$

for any $(r,\theta)\in T_n$ and any $(t,\Xi)\in\partial T_n$ $(r<t)$, which is (6). Hence we obtain

$$2^{n+i}\Big(\sum_{k=k_{i+1}}^{k_{i+2}-1} c((H_{\Xi})_1,k)f_k^{\Omega}(\theta)\Big) = 2s_n^{-1}c_{i,n+2}\cos\theta_1 L_{i,n+2}(\cos\gamma)$$

$$(i=0,1,2,\ldots).$$

Since

$$\overline{V}(\mathbb{T}_n,\ell)(P,Q) = \sum_{i=0}^{\ell-1} 2^{n+i} r^{i+1} t^{-n-i} \left(\sum_{k=k_{i+1}}^{k_{i+2}-1} c((H_\Xi)_1,k) f_k^\Omega(\theta) \right)$$

from (7), we finally have

$$\overline{V}(\mathbb{T}_n,\ell)(P,Q) = 2s_n^{-1} \sum_{i=0}^{\ell-1} c_{i,n+2} r^{i+1} t^{-i-n} \cos\theta_1 L_{i,n+2}(\cos\gamma).$$

This shows that our kernel $K(\mathbb{T}_n,\ell,0)(P,Q)$ $(\ell\geq 1)$ concides with ones in Armitage [1], Siegel [15] and Yoshida [19].

Let $F(P)=F(r,\theta)$ be a function on $C_n(\Omega)$ and put

$$N(F)(r) = \int_\Omega F(r,\theta) f_1^\Omega(\theta) d\sigma_\theta.$$

For two non-negative integers p and q we write

$$\mu_p(F) = \lim_{r\to\infty} r^{-\alpha(\Omega,k_{p+1})} N(F)(r) \quad \text{and} \quad \eta_q(F) = \lim_{r\to 0} r^{\beta(\Omega,k_{q+1})} N(F)(r),$$

if they exist. Since $k_1=1$, we know that these with $p=q=0$ are consistant with (3).

The following theorem is a generalization of the first part of Theorem Y_2 which is the case $\ell=m=0$ of Theorem 1.

THEOREM 1. *Let* ℓ, m *be two non-negative integers and* $g(Q)=g(t,\Xi)$ *be a continuous function on* $S_n(\Omega)$ *satisfying* (9) *with* ℓ *and* (10) *with* m:

$$(9) \qquad \int^{+\infty} t^{-\alpha(\Omega,k_{\ell+1})-1} \left(\int_{\partial\Omega} |g(t,\Xi)| d\sigma_\Xi\right) dt < +\infty$$

and

$$(10) \qquad \int_0 t^{\beta(\Omega,k_{m+1})-1} \left(\int_{\partial\Omega} |g(t,\Xi)| d\sigma_\Xi\right) dt < +\infty.$$

Then

$$H(C_n(\Omega),\ell,m;g)(P) = \int_{S_n(\Omega)} g(Q) K(C_n(\Omega),\ell,m)(P,Q) d\sigma_Q$$

is a solution of the classical Dirichlet problem on $C_n(\Omega)$ *with* g *satisfying*

$$\mu_\ell(|H(C_n(\Omega),\ell,m;g)|) = \mu_m(|H(C_n(\Omega),\ell,m;g)|) = 0.$$

To emphasize that Theorem 1 is also a natural generalization of the first part of Theorem Y_1, the following Theorem 2 is more desirable than Theorem 1.

THEOREM 2. *Let* $g(Q)=g(t,\Xi)$ *be a continuous function on* $\partial C_n(\Omega)$ *satisfying* (9) *with a non-negative integer* ℓ. *Then*

$$H(C_n(\Omega),\ell,0;g)(P) = \int_{S_n(\Omega)} g(Q)K(C_n(\Omega),\ell,0)(P,Q)d\sigma_Q$$

is a solution of the classical Dirichlet problem on $C_n(\Omega)$ *with* g *satisfying*

$$\mu_\ell(|H(C_n(\Omega),\ell,0;g)|) = 0.$$

By taking $\Omega=S_+^{n-1}$, we obtain from (7)

COROLLARY 1 (Yoshida [19, Theorem 1]). *Let* $g(X)$ *be a continuous function on* $\partial\mathbb{T}_n=\mathbb{R}^{n-1}$ *satisfying* (1) *with a non-negative integer* ℓ. *Then* $H(\mathbb{T}_n,\ell,0;g)(P)$ *is a solution of the Dirichlet problem on* $\mathbb{T}_n$ *with* g *such that*

$$\mu_\ell(|H(\mathbb{T}_n,\ell,0;g)|) = 0.$$

To solve the Dirichlet problem on $C_n(\Omega)$ with any function $g(Q)$, we shall define other Poisson kernels. Let $\varphi(t)$ (resp. $\psi(t)$) be a positive continuous function of $t\geq1$ (resp. $0<t\leq1$) satisfying

$$\varphi(1) = 2^{-\alpha(\Omega,1)} \quad (\text{resp. } \psi(1) = (\tfrac{3}{2})^{-\beta(\Omega,1)}).$$

For a domain $\Omega\subset\mathbb{S}^{n-1}$ and the sequence $\{\alpha(\Omega,k_i)\}_{i=1}^{\infty}$ (resp. $\{\beta(\Omega,k_i)\}_{i=1}^{\infty}$), denote the set

$$\{t\geq 1;\ -\alpha(\Omega,k_i)= (\log 2)^{-1}(\log(t^{n-1}\varphi(t))\}$$

$$(\text{resp. } \{0<t\leq 1;\ -\beta(\Omega,k_i)= (\log \tfrac{3}{2})^{-1}\log(t^{n-1}\psi(t))\})$$

by $\overline{S}(\Omega,\varphi,i)$ (resp. $\underline{S}(\Omega,\psi,i)$). Then $1\in\overline{S}(\Omega,\varphi,1)$ (resp. $1\in\underline{S}(\Omega,\psi,1)$). When there is an integer N such that $\overline{S}(\Omega,\varphi,N)\neq\phi$ and $\overline{S}(\Omega,\varphi,N+1)=\phi$ (resp. $\underline{S}(\Omega,\psi,N)\neq\phi$ and $\underline{S}(\Omega,\psi,N+1)=\phi$), denote the set $\{i;\ 1\leq i\leq N\}$ of integers by $\overline{J}(\Omega,\varphi)$ (resp. $\underline{J}(\Omega,\psi)$). Otherwise, denote the set of all positive integers by $\overline{J}(\Omega,\varphi)$ (resp. $\underline{J}(\Omega,\psi)$). Let $\overline{t}(i)= \overline{t}(\Omega,\varphi,i)$ (resp. $\underline{t}(i)= \underline{t}(\Omega,\psi,i)$) be the minimum (resp. maximum) of elements t in $\overline{S}(\Omega,\varphi,i)$ (resp. $\overline{S}(\Omega,\psi,i)$) for each $i\in\overline{J}(\Omega,\varphi)$ (resp. $\underline{J}(\Omega,\psi)$). In the former case, we put $\overline{t}(N+1)=+\infty$ (resp. $\underline{t}(N+1=0)$. Then $\overline{t}(1)=1$ (resp. $\underline{t}(1)=1$).

We define $\overline{W}(C_n(\Omega),\varphi)(P,Q)$ $(P\in C_n(\Omega),\ Q=(t,\Xi)\in S_n(\Omega))$ by

$$\overline{W}(C_n(\Omega),\varphi)(P,Q) = \begin{cases} 0 & (0<t<1) \\ \overline{V}(C_n(\Omega),i)(P,Q) & (\overline{t}(i)\leq t<\overline{t}(i+1);\ i\in\overline{J}(\Omega,\varphi)). \end{cases}$$

We also define $\underline{W}(C_n(\Omega),\varphi)(P,Q)$ $(P\in C_n(\Omega),\ Q=(t,\Xi)\in S_n(\Omega))$ by

$$\underline{W}(C_n(\Omega),\psi)(P,Q) = \begin{cases} 0 & (1<t<+\infty) \\ \underline{V}(C_n(\Omega),i)(P,Q) & (\underline{t}(i+1)<t\leq\underline{t}(i);\ i\in\underline{J}(\Omega,\psi)). \end{cases}$$

The Poisson kernel $K(C_n(\Omega),\varphi,\psi)(P,Q)$ and $K(C_n(\Omega),\varphi)(P,Q)$ $(P\in C_n(\Omega),\ Q\in S_n(\Omega))$ are defined by

$$K(C_n(\Omega),\varphi,\psi)(P,Q)$$
$$= c_n^{-1}\frac{\partial}{\partial\nu}G_{C_n(\Omega)}(P,Q) - \overline{W}(C_n(\Omega),\varphi)(P,Q) - \underline{W}(C_n(\Omega),\psi)(P,Q)$$

and

$$K(C_n(\Omega),\varphi)(P,Q) = c_n^{-1}\frac{\partial}{\partial\nu}G_{C_n(\Omega)}(P,Q) - \overline{W}(C_n(\Omega),\varphi)(P,Q)).$$

Now we have

THEOREM 3. *Let $g(Q)$ be a continuous function on $S_n(\Omega)$. Then there are two positive continuous functions $\varphi(t)$ of $t\geq 1$ and $\psi(t)$ of $0<t\leq 1$ such that*

$$H(C_n(\Omega),\varphi,\psi;g)(P) = \int_{S_n(\Omega)} g(Q)K(C_n(\Omega),\varphi,\psi)(P,Q)d\sigma_Q$$

is a solution of the Dirichlet problem on $C_n(\Omega)$ with g.

We can also obtain

THEOREM 4. *Let $g(Q)$ be a continuous function on $\partial C_n(\Omega)$. Then there is a positive continuous function $\varphi(t)$ of $t\geq 1$ such that*

$$H(C_n(\Omega),\varphi;g)(P) = \int_{S_n(\Omega)} g(Q)K(C_n(\Omega),\varphi)(P,Q)d\sigma_Q$$

is a solution of the Dirichlet problem on $C_n(\Omega)$ with g.

If we take $\Omega=S_+^{n-1}$ in Theorem 4, then we have

COROLLARY 2 (Finkelstein and Scheinberg [7] and Gardinar [8]). *Let $g(Q)$ be a continuous function on ∂T_n. Then there is a positive continuous function $\varphi(t)$ of $t\geq 1$ such that*

$$H(T_n,\varphi;g)(P) = \int_{\partial T_n} g(Q)K(T_n,\varphi)(P,Q)d\sigma_Q$$

is a solution of the Dirichlet problem on T_n with g.

3. Results about a type of uniqueness of solutions.

The following result is just a generalization of Picard's theorem stating that if H is a positive harmonic function in the Euclidean space then H is a constant.

Let $h(r,\theta)$ *be harmonic on* $\mathbb{R}^d$ $(d\geq 2)$. *If, for some positive* t,

$$r^{-t-1}\mathfrak{M}(h^+)(r) \to 0 \quad (r \to +\infty), \qquad \mathfrak{M}(h^+)(r) = \int_{\mathbb{S}^{d-1}} h^+(r,\theta)d\sigma_\theta,$$

then for some positive integer ℓ *less than* $t+1$

$$h(r,\theta) = C + \sum_{k=1}^{\ell} P_k(r,\theta) \qquad ((r,\theta)\in\mathbb{R}^d),$$

where C *is a constant and* $P_k(r,\theta)=r^kY_k(\theta)$ *is a homogeneous harmonic polynomial of order* k ($Y_k(\theta)$ *is a spherical harmonic function of Laplace*) (see e.g. Brelot [2, Appendix, §26]).

It is well known that the potential theory in $\mathbb{R}^n$ is intimately related to the potential theory in $\mathbb{R}^{n+2}$ and many results on harmonic functions in $\mathbb{R}^n$ can easily obtained by a passage to $\mathbb{R}^{n+2}$. By using this fact, Kuran proved the following theorem.

THEOREM K (Kuran [12, Theorem 10]). *Let* $h(X,y)$ $(=h(r,\theta))$ *be a harmonic function on* $\mathbb{T}_n$ *such that* h *vanishes continuously on* $\partial\mathbb{T}_n$.

If, for some positive t,

$$\lim_{r\to\infty} r^{-t-2}\mathcal{D}(yh^+,r) = 0, \qquad \mathcal{D}(yh^+,r)=(s_r^+)^{-1}\int_{S_r^+} yh^+(r,\theta)\, dS_r^+,$$

where $S_r^+=\{(r,\theta)\in\mathbb{T}_n;\ \theta\in\mathbb{S}_+^{n-1}\}$, s_r^+ *is the surface area of the spherical part of* S_r^+ *and* dS_r^+ *is the surface element of* S_r^+, *then*

$$h = y\Pi$$

in $\mathbb{T}_n$, *where* Π *is a polynomial of* $(x_1,x_2,\ldots,x_{n-1},y)$ *in* $\mathbb{R}^n$ *of degree* *less than* t *and even with respect to the variable* y.

Though his method is not applicable, we shall try to extend these results for functions defined on cones, one of which is the half-space. And we can obtain

THEOREM 5. *Let* p, q *be two positive integers and* $h(r,\theta)$ *be a harmonic function in* $C_n(\Omega)$ *vanishing continuously on* $S_n(\Omega)$. *If* h *satisfies* (11) *with* p *and* (12) *with* q:

(11) $$\mu_p(h^+) = 0$$

and

(12) $$\eta_q(h^+) = 0,$$

then

$$h(r,\theta)= \sum_{k\in I(\Omega,k_{p+1})} A_k(h)r^{\alpha(\Omega,k)}f_k^{\Omega}(\theta) + \sum_{k\in I(\Omega,k_{q+1})} B_k(h)r^{-\beta(\Omega,k)}f_k^{\Omega}(\theta)),$$

for every $(r,\theta)\in C_n(\Omega)$, *where* $A_k(h)$ $(k=1,2,\dots,k_{p+1}-1)$ *and* $B_k(h)$ $(k=1,2,\dots,k_{q+1}-1)$ *are all constants.*

In comparison with Theorem K, the following type is prefered.

THEOREM 6. *Let* $h(r,\theta)$ *be a harmonic function in* $C_n(\Omega)$ *vanishing continuously on* $\partial C_n(\Omega)$. *If* h *satisfies* (11) *with a positive integer* p, *then*

$$h(r,\theta)= \sum_{k\in I(\Omega,k_{p+1})} A_k(h)r^{\alpha(\Omega,k)}f_k^{\Omega}(\theta)$$

for every $(r,\theta)\in C_n(\Omega)$, *where* $A_k(h)$ $(k=1,2,\dots,k_{p+1}-1)$ *are all constants.*

When we specialize $\Omega=S_{+}^{n-1}$ i.e. $C_n(\Omega)=\mathbb{T}_n$, from Theorem 6 and (7), we obtain the following corollary 3. The equality

$$\mathcal{D}(yh^{+},r) = 2s_n^{-1}rN(h^{+})(r)$$

shows that this is equal to Theorem K

COROLLARY 3. *Let* $h(r,\theta)$ *be a harmonic function in* $\mathbb{T}_n$ *vanishing continuously on* $\partial\mathbb{T}_n$. *If, for some positive* t,

$$\lim_{r\to\infty} r^{-(t+1)}N(h^{+})(r) = 0,$$

then

$$h(r,\theta) = y\Pi,$$

where Π *is a polynomial in* R^n *with degree less than* t *and even respect to the variable* y.

By using Theorem 5, we can prove the following Theorem 7.

THEOREM 7. *Let* ℓ, m *be two non-negative integers and* p, q *be two positive integers satisfying* $p\geq\ell$, $q\geq m$. *Let* $g(t,\Xi)$ *be a continuous function on* $S_n(\Omega)$ *satisfying* (9) *with* ℓ *and* (10) *with* m. *If* $h(r,\theta)$ *is a solution of the Dirichlet problem on* $C_n(\Omega)$ *with* g *satisfying*

(13) $$\mu_p(h^{+}) = 0$$

and

$$\eta_q(h^{+}) = 0,$$

then

$$h(r,\theta) = H(C_n(\Omega),\ell,m;g)(P) + \sum_{k\in I(\Omega,k_{p+1})} A_k(h)r^{\alpha(\Omega,k)}f_k^{\Omega}(\theta) + \sum_{k\in I(\Omega,k_{q+1})} B_k(h)r^{-\beta(\Omega,k)}f_k^{\Omega}(\theta)$$

for every $P=(r,\theta)\in C_n(\Omega)$, *where* $A_k(h)$ $(k=1,2,\dots,k_{p+1}-1)$ *and* $B_k(h)$ $(k=1,2,\dots,k_{q+1}-1)$ *are all constants.*

If we take $\ell=m=0$ and $p=q=1$ in Theorem 7, then we have the following result containing the second part of Theorem Y_2.

COROLLARY 4. *Let* $g(Q)$ *be a continuous function on* $S_n(\Omega)$ *satisfying* (4). *If* $h(r,\theta)$ *is a solution of the Dirichlet problem on* $C_n(\Omega)$ *with* g *satisfying*

$$\mu_1(h^+) = \eta_1(h^+) = 0,$$

then

$$h(r,\theta) = H(C_n(\Omega);g)(P) + A_1(h)r^{\alpha(\Omega,1)}f_1^{\Omega}(\theta) + B_1(h)r^{-\beta(\Omega,1)}f_1^{\Omega}(\theta)$$

for every $P=(r,\theta)\in C_n(\Omega)$, *where* $A_1(h)=\mu_0(h)$ *and* $B_1(h)=\eta_0(h)$.

By Theorem 6, we also have

THEOREM 8. *Let* ℓ *be a non-negative integer and* p *be a positive integer satisfying* $\ell\le p$. *Let* $g(t,\Xi)$ *be a continuous function on* $\partial C_n(\Omega)$ *satisfying* (9) *with* ℓ. *If* $h(r,\theta)$ *is a solution of the Dirichlet problem on* $C_n(\Omega)$ *with* g *satisfying* (13) *with* p, *then*

$$h(r,\theta) = H(C_n(\Omega),\ell,0;g)(P) + \sum_{k\in I(\Omega,k_{p+1})} A_k(h)r^{\alpha(\Omega,k)}f_k^{\Omega}(\theta)$$

for every $P=(r,\theta)\in C_n(\Omega)$, *where* $A_k(h)$ $(k=1,2,\dots,k_{p+1}-1)$ *are all constants.*

If we put $\Omega=S_+^{n-1}$, $\ell=\rho$ and $p=\rho$ (resp. $\ell=\rho-1$ and $p=\rho$) (ρ is a positive integer) in Theorem 8, we obtain from Corollary 3

COROLLARY 5 (Yoshida [19, Theorem 2 (resp. Corollary 2)]). *Let ρ be a positive integer and $g(X)$ be a continuous function on $\partial\mathbb{T}_n=\mathbb{R}^{n-1}$ satisfying (1) with ρ (resp. (1) with $\rho-1$). If $h(P)$ is a solution of the Dirichlet problem on $\mathbb{T}_n$ with g such that*

$$\lim_{r\to\infty} r^{-(\rho+1)}N(h^{+})(r) = 0,$$

then

$$h(P) = H(\mathbb{T}_n,\rho,0;g)(P) + yF(h)(P)$$

$$(\textit{resp. } h(P) = H(\mathbb{T}_n,\rho-1,0;g)(P) + yF(h)(P)),$$

where $F(h)(P)$ is a harmonic polynomial (of $P=(x_1,x_2,\dots,x_{n-1},y)\in\mathbb{R}^n$) of at most degree $\rho-1$ vanishing on $\partial\mathbb{T}_n$ and even with respect to the variable y.

The following Theorem 9 also generalizes a result of Yoshida [19, Theorem 3].

THEOREM 9. *If $h(r,\theta)$ is a harmonic function on $C_n(\Omega)$ and is continuous on $\overline{C_n(\Omega)}$ such that the restriction $h=h|_{\partial C_n(\Omega)}$ of h to $\partial C_n(\Omega)$ satisfies*

$$\int^{+\infty} t^{-\alpha(\Omega,k_{\ell+1})-1}\left(\int_{\partial\Omega}|h(t,\Xi)|d\sigma_{\Xi}\right)dt < +\infty$$

for some non-negative integer ℓ and

$$\overline{\lim_{r\to\infty}}\frac{\log N(h^{+})(r)}{\log r} < +\infty,$$

then for some positive integer p

$$h(r,\theta) = H(C_n(\Omega),\ell,0;h)(P) + \sum_{k\in I(\Omega,k_{p+1})} A_k(h)r^{\alpha(\Omega,k)}f_k^{\Omega}(\theta)$$

at every $P=(r,\theta)\in C_n(\Omega)$, *where* $A_k(h)$ $(k=1,2,\ldots,k_{p+1}-1)$ *are all constants.*

The Proof of all results in this part will be found in Yoshida and Miyamoto [20].

PART 3. The cylindrical case

1. Introduction

There is another typical unbounded domain which is a cylinder

$$\Gamma_n(D) = D \times R$$

with a bounded domain $D \subset R^{n-1}$. The existence and the uniqueness of solutions of the Dirichlet problem on $\Gamma_n(D)$ with a continuous function on $\partial\Gamma_n(D)$ are worth being inquired. In this direction, Yoshida [**18**] proved the following Theorem A. To state it we need some preliminaries.

Consider the Dirichlet problem

$$(14) \qquad \begin{aligned} (\Delta_{n-1} + \lambda)f &= 0 \quad \textit{in } D \\ f &= 0 \quad \textit{on } \partial D \end{aligned}$$

for a bounded domain $D \subset R^{n-1} (n \geq 2)$, where $\Delta_1 = d^2/dx^2$. Let $\lambda(D,1)$ be the least positive eigenvalue of (14) and $f_1^D(X)$ be the normalized eigenfunction corresponding to $\lambda(D,1)$. In order to make the subsequent consideration simpler, we put a strong assumption on D throughout the whole this paper: If $n \geq 3$, then D is a $C^{2,\alpha}$-domain ($0 < \alpha < 1$) in R^{n-1} surrouded by a finite number of mutually disjoint closed hypersurfaces (for example, see Gilberg and Trudinger [**9**,pp.88-89] for the definition of $C^{2,\alpha}$-domain). Let $G_{\Gamma_n(D)}(P_1, P_2)$ be the Green function of $\Gamma_n(D)$ ($P_1, P_2 \in \Gamma_n(D)$) and $\partial G_{\Gamma_n(D)}(P,Q)/\partial\nu$ be the differentiation at $Q \in \partial\Gamma_n(D)$ along the inward normal into $\Gamma_n(D)$ ($P \in \Gamma_n(D)$).

Given a function $F(X,y)$ on $\Gamma_n(D)$,we denote the function of y defined by the integral

$$\int_D F(X,y) f_1^D(X) dX$$

by $N(F)(y)$, where dX denotes the $(n-1)$-dimensional volume element. We write

$$\mu_0(N(F)) = \lim_{y\to\infty} \exp(-\sqrt{\lambda(D,1)}y) N(F)(y)$$

and

$$\eta_0(N(F)) = \lim_{y\to-\infty} \exp(\sqrt{\lambda(D,1)}y) N(F)(y),$$

if they exist.

Theorem A (Yoshida [**18**, Theorem 6]). *Let $g(Q)$ be a continuous function on $\partial\Gamma_n(D)$ satisfying*

$$(15) \qquad \int_{-\infty}^{\infty} \exp(-\sqrt{\lambda(D,1)}|y|)\left(\int_{\partial D} |g(X,y)| d\sigma_X\right) dy < \infty,$$

where $d\sigma_X$ *is the surface area element of* ∂D *at* X *and if* $n = 2$ *and* $D = (\gamma, \delta)$, *then*

$$\int_{\partial D} |g(X,y)| d\sigma_X = |g(\gamma, y)| + |g(\delta, y)|.$$

Then the Poisson integral

$$PI_g(P) = c_n^{-1} \int_{\partial \Gamma_n(D)} g(Q) \frac{\partial}{\partial \nu} G_{\Gamma_n(D)}(P,Q) d\sigma_Q$$

is a solution of the Dirichlet problem on $\Gamma_n(D)$ *with* g, *where*

$$c_n = \begin{cases} 2\pi & (n = 2) \\ (n-2)s_n & (n \geq 3) \end{cases} \text{ (s_n is the surface area of the unit sphere S^{n-1})}$$

and $d\sigma_Q$ *is the surface area element on* $\partial\Gamma_n(D)$ *at* Q. *Let* $h(P)$ *be any solution of the Dirichlet problem on* $\Gamma_n(D)$ *with* g. *Then all of the limits* $\mu_0(N(h))$, $\eta_0(N(h))$ $(-\infty < \mu_0(N(h)), \eta_0(N(h)) \leq \infty)$, $\mu_0(N(|h|))$ *and* $\eta_0(N(|h|))$ $(0 \leq \mu_0(N(|h|)), \eta_0(N(|h|)) \leq \infty)$ *exist, and if*

$$\mu_0(N(|h|)) < \infty \text{ and } \eta_0(N(|h|)) < \infty, \tag{16}$$

then

$$h(P) = PI_g(P) + (\mu_0(N(h)) \exp(\sqrt{\lambda(D,1)}y) + \eta_0(N(h)) \exp(-\sqrt{\lambda(D,1)}y)) f_1^D(X)$$

for any $P = (X, y) \in \Gamma_n(D)$.

This Theorem A shows that under the conditions (15)and (16) the existence and a type of uniqueness of solutions for the Dirichlet problem on $\Gamma_n(D)$ can be proved,respectively.

If $n = 2$, then $\Gamma_n(D)$ is a strip.The strip $\Gamma_2((0,\pi))$ with $D = (0,\pi)$ is simply denoted by Γ_2. With respect to the Dirichlet problem on Γ_2, Widder obtained

Theorem B (Widder [13, Theorems 1 and 3]). *If* $g_i(t)$ $(i = 1, 2)$ *is a continuous function on* R *satisfying*

$$\int_{-\infty}^{\infty} |g_i(t)| \exp(-|t|) dt < \infty,$$

then

$$H(\Gamma_2; g_1, g_2)(x,y) = \frac{1}{2\pi} \int_{-\infty}^{\infty} P(x, t-y) g_1(t) dt + \frac{1}{2\pi} \int_{-\infty}^{\infty} P(\pi - x, t-y) g_2(t) dt$$

$$(P(x,y) = \frac{\sin x}{\cosh y - \cos x})$$

is a harmonic function in Γ_2 *and a continuous function on* $\overline{\Gamma_2}$ *such that*

$$H(\Gamma_2; g_1, g_2)(0, y) = g_1(y) \text{ and } H(\Gamma_2; g_1, g_2)(\pi, y) = g_2(y) \quad (-\infty < y < \infty).$$

If $h(x,y)$ is a harmonic function in Γ_2 and a continuous function on $\overline{\Gamma_2}$ such that

$$h(0,y) = g_1(y), \quad h(\pi,y) = g_2(y) \quad (-\infty < y < \infty)$$

and

$$\int_0^{\pi} |h(x,y)|dx = o(e^{|y|}) \quad (|y| \to \infty),$$

then

$$h(x,y) = H(\Gamma_2; g_1, g_2)(x,y)$$

on $\overline{\Gamma_2}$.

Though by a conformal mapping a strip is reduced to T_2 which was treated in [**20**] as a special case, it may be interested to be indipendently treated as a special case of cylinders.

In this paper, the first parts of Theorems A and B will be extended by defining generalized Poisson integrals with continuous functions under more unrestricted conditions than (15) and (16) (Theorem 1 and Corollary 1). We shall also prove that for any continuous function g on $\Gamma_n(D)$ there is a solution of the Dirichlet problem on $\Gamma_n(D)$ with g (Theorem 2 and Corollary 2). The results (Theorem 3 and Corollary 3) which generalize the second parts of Theorems A and B will be connected with a type of uniqueness of solutions for the Dirichlet problem on $\Gamma_n(D)$.

2. Statements of our results

We denote the non-decreasing sequence of positive eigenvalues of (15) by $\{\lambda(D,k)\}_{k=1}^{\infty}$. In this expression we write $\lambda(D,k)$ the same number of times as the dimension of the corresponding eigenspace. When the normalized eigenfunction corresponding $\lambda(D,k)$ is denoted by f_k^D, the set of sequential eigenfunctions corresponding to the same value of $\lambda(D,k)$ in the sequence $\{f_k^D\}_{k=1}^{\infty}$ makes an orthonormal basis for the eigenspace of the eigenvalue $\lambda(D,k)$. We can also say that for each $D \subset R^{n-1}$ there is a sequence $\{k_i\}$ of positive integers such that $k_1 = 1, \lambda(D,k_i) < \lambda(D,k_{i+1})$

$$\lambda(D,k_i) = \lambda(D,k_i+1) = \lambda(D,k_i+2) = \ldots = \lambda(D,k_{i+1}-1)$$

and $\{f_{k_i}^D, f_{k_i+1}^D, \ldots, f_{k_{i+1}-1}^D\}$ is an orthnormal basis for the eigenspace of the eigenvalue $\lambda(D,k_i)$ $(i = 1,2,3,\ldots)$. It is well known that $k_2 = 2$ and $f_1^D(X) > 0$ for any $X \in D$ (See Courant and Hilbert [**5**, p.451 and p.458]). With respect to $\{k_i\}$, the following Example (2) shows that even in the case where D is an open disk in R^2 , not the simplest case $k_i = i$ $(i = 1,2,3,\ldots)$, but complex case can appear.

When D has sufficiently smooth boundary, we know that

$$\lambda(D,k) \sim A(D,n)k^{2/(n-1)} \quad (k \to \infty)$$

and

$$\sum_{\lambda(D,k)\leq x} \{f_k^D(X)\}^2 \sim B(D,n)x^{(n-1)/2} \quad (x \to \infty)$$

uniformly with respect to $X \in D$, where $A(D,n)$ and $B(D,n)$ are both constants depending on D and n (e.g. see Weyl [**7**] and Carleman [**3**]). Hence there exist two positive constants M_1, M_2 such that

$$M_1 k^{2/(n-1)} \le \lambda(D,k) \quad (k=1,2,\ldots)$$

and

$$|f_k^D(X)| \le M_2 k^{1/2} \quad (X \in D, k=1,2,\ldots).$$

We remark that both

$$\exp(\sqrt{\lambda(D,k)}y) f_k^D(X) \ \ \textit{and} \ \ \exp(-\sqrt{\lambda(D,k)}y) f_k^D(X) \quad (k=1,2,\ldots)$$

are harmonic on $\Gamma_n(D)$ and vanish continuously on $\partial\Gamma_n(D)$.
For a domain D and the sequence $\{k_i\}$ mentioned above, by $I(D,k_j)$ we denote the set of all positive integers less than k_j $(j=1,2,3\ldots)$. In spite of the fact $I(D,k_1)=o$. the summation over $I(D,k_1)$ of a function $S(j)$ of a variable j will be used by promising $\sum_{k\in I(D,k_1)} S(k)=0$.

Examples. (1) Let $D=(0,\pi)$. Then (14) is reduced to find solutions $f(x)$ $(0 \le x \le \pi)$ such that

$$\frac{d^2 f(x)}{dx^2} + \lambda f(x) = 0 \quad (0 < x < \pi)$$

and

$$f(0) = f(\pi) = 0$$

It is easy to see that $k_i = i$, $\lambda(D,k)=k^2$ and $f_k^D(x)=\sqrt{\frac{2}{\pi}}\sin kx \quad (k=1,2,3,\ldots)$.

(2) Let $D=\{(x,y)\in R^2; x^2+y^2<1\}$. Let $\{\alpha_{n,m}\}_{m=1}^{\infty}$ be the increasing sequence of positive real numbers $\alpha_{n,m}$ such that

$$J_n(\alpha_{n,m}) = 0 \quad (n=0,1,2,\ldots),$$

where $J_n(z)$ is the Bessel function of order n. If the spherical coordinates $x=r\cos\theta, y=r\sin\theta$ $(0\le r<1,\ 0\le\theta<2\pi)$ are introduced, then $J_n(\alpha_{n,m}r)\cos n\theta$ and $J_n(\alpha_{n,m}r)\sin n\theta$ $(n\ne 0, m=1,2,3,\ldots)$ are two eigenfunctions coresponding to the eigenvalue $\lambda=\alpha_{n,m}^2$ (see Courant and Hilbert [**5**]). Since we do not know how the zeros of Bessel functions distribute, we cannot explicitly determine the sequence $\{k_i\}$ with respect this D.

The Fourier coefficient

$$\int_D F(X) f_k^D(X) dX$$

of a function $F(X)$ on D with respect to the orthonormal sequence $\{f_k^D(X)\}$ is denoted by $c(F,k)$, if it exists. Now we shall define generalized Poisson kernels. Let l and m be two non-negative integers. For two points $P=(X,y)\in\Gamma_n(D)$, $Q=(X^*,y^*)\in\partial\Gamma_n(D)$, we put

$$\overline{V}(\Gamma_n(D),l)(P,Q)$$

$$= \sum_{k \in I(D,k_{l+1})} \exp(\sqrt{\lambda(D,k)})c((H_{X^*})_1, k) f_k^D(X) \exp(\sqrt{\lambda(D,k)}y) \exp(-\sqrt{\lambda(D,k)}y^*),$$

and

$$\underline{V}(\Gamma_n(D),m)(P,Q)$$
$$= \sum_{k \in I(D,k_{m+1})} \exp(\sqrt{\lambda(D,k)})c((H_{X^*})_1, k) f_k^D(X) \exp(-\sqrt{\lambda(D,k)}y) \exp(\sqrt{\lambda(D,k)}y^*),$$

where

$$(H_{X^*})_1(X) = c_n^{-1} \frac{\partial}{\partial \nu} G_{\Gamma_n(D)}((X,1),(X^*,0)).$$

We remark that $\overline{V}(\Gamma_n(D),l)(P,Q)$ and $\underline{V}(\Gamma_n(D),m)(P,Q)$ are two harmonic functions of $P \in \Gamma_n(D)$ for any fixed $Q \in \partial\Gamma_n(D)$. We introduce two functions of $P \in \Gamma_n(D)$ and $Q = (X^*, y^*) \in \partial\Gamma_n(D)$

$$\overline{W}(\Gamma_n(D),l)(P,Q) = \begin{cases} \overline{V}(\Gamma_n(D),l)(P,Q) & (y^* \geq 0) \\ 0 & (y^* < 0) \end{cases}$$

and

$$\underline{W}(\Gamma_n(D),m)(P,Q) = \begin{cases} \underline{V}(\Gamma_n(D),m)(P,Q) & (y^* \leq 0) \\ 0 & (y^* > 0). \end{cases}$$

The Poisson kernel $K(\Gamma_n(D),l,m)(P,Q)$ with respect to $\Gamma_n(D)$ is defined by

$$K(\Gamma_n(D),l,m)(P,Q) = c_n^{-1} \frac{\partial}{\partial \nu} G_{\Gamma_n(D)}(P,Q) - \overline{W}(\Gamma_n(D),l)(P,Q) - \underline{W}(\Gamma_n(D),m)(P,Q).$$

We note

$$K(\Gamma_n(D),0,0)(P,Q) = c_n^{-1} \frac{\partial}{\partial \nu} G_{\Gamma_n(D)}(P,Q).$$

Let p , q be two non-negative integers and $I(y)$ be a function on R. The finite or infinite limits

$$\lim_{y \to \infty} \exp(-\sqrt{\lambda(D,k_{p+1})}y) I(y) \quad \text{and} \quad \lim_{y \to -\infty} \exp(\sqrt{\lambda(D,k_{q+1})}y) I(y)$$

are denoted by $\mu_p(I)$ and $\eta_q(I)$, respectively, when they exist.

Theorem 10. *Let l, m be two non-negative integers and $g(Q) = g(X^*, y^*)$ be a continuous function on $S_n(D)$ satisfying*

(17) $$\int^{\infty} \exp(-\sqrt{\lambda(D,k_{l+1})}y^*)\left(\int_{\partial D} |g(X^*,y^*)| d\sigma_{X^*}\right) dy^* < \infty$$

and

$$\int_{-\infty} \exp(\sqrt{\lambda(D,k_{m+1})}y^*)\left(\int_{\partial D} |g(X^*,y^*)| d\sigma_{X^*}\right) dy^* < \infty.$$

Then

$$H(\Gamma_n(D),l,m,;g)(P) = \int_{S_n(D)} g(Q) K(\Gamma_n(D),l,m)(P,Q) d\sigma_Q$$

is a solution of the Dirichlet problem on $\Gamma_n(D)$ *with* g *satisfying*

$$\mu_l(N(|H(\Gamma_n(D),l,m;g)|)) = \eta_m(N(|H(\Gamma_n(D),l,m;g)|)) = 0.$$

If $n = 2$ and $D = (0,\pi)$, then we immediately obtain the following Corollary 6 which generalizes Theorem B.

Corollary 6. *Let* l,m *be two non-negative integers and* $g_1(y^*), g_2(y^*)$ *be two continuous functions on* R *satisfying*

$$\int^{\infty} |g_i(y^*)| \exp(-(l+1)y^*)dy^* < \infty \tag{18}$$

and

$$\int_{-\infty} |g_i(y^*)| \exp(-(m+1)y^*)dy^* < \infty \quad (i=1,2).$$

Then

$$H(\Gamma_2,l,m;g_1,g_2)(x,y) =$$
$$\int_{-\infty}^{\infty} g_1(y^*)K(\Gamma_2,l,m)((x,y),(0,y^*))dy^* + \int_{-\infty}^{\infty} g_2(y^*)K(\Gamma_2,l,m)((x,y),(\pi,y^*))dy^*$$

is a harmonic function in Γ_2 *and a continuous function on* $\overline{\Gamma_2}$ *such that*

$$H(\Gamma_2,l,m;g_1,g_2)(0,y^*) = g_1(y^*)$$

and

$$H(\Gamma_2,l,m;g_1,g_2)(\pi,y^*) = g_2(y^*) \quad (-\infty < y^* < \infty)$$

To solve the Dirichlet problem on $\Gamma_n(D)$ with any function $g(Q)$ on $\partial\Gamma_n(D)$, we shall define another Poisson kernel. Let $\varphi(t)$ be any positive continuous function of $t \geq 0$ satisfying

$$\varphi(0) = \exp(-\sqrt{\lambda(D,1)}).$$

For a domain $D \subset R^{n-1}$ and the sequence $\{\lambda(D,k_i)\}$, denote the set

$$\{t \geq 0; \exp(-\sqrt{\lambda(D,k_i)}) = \varphi(t)\}$$

by $S(D,\varphi,i)$. Then $0 \in S(D,\varphi,1)$. When there is an integer N such that $S(D,\varphi,N) \neq \phi$ and $S(D,\varphi,N+1) = \phi$, denote the set $\{i; 1 \leq i \leq N\}$ of integers by $J(D,\varphi)$. Otherwise, denote the set of all positive integers by $J(D,\varphi)$. Let $t(i) = t(D,\varphi,i)$ be the minimum of elements t in $S(D,\varphi,i)$ for each $i \in J(D,\varphi)$. In the former case, we put $t(N+1) = \infty$. Then $t(1) = 0$.

We define $\overline{W}(\Gamma_n(D),\varphi)(P,Q)$ $(P \in \Gamma_n(D),\ Q = (X^*,y^*) \in S_n(D))$ by

$$\overline{W}(\Gamma_n(D),\varphi)(P,Q) = \begin{cases} 0 & (y^* < 0) \\ \overline{V}(\Gamma_n(D),i)(P,Q) & (t(i) \leq y^* < t(i+1); i \in J(D,\varphi)). \end{cases}$$

We also define $\underline{W}(\Gamma_n(D),\varphi)(P,Q)$ $(P \in \Gamma_n(D), Q=(X^*,y^*) \in S_n(D))$ by

$$\underline{W}(\Gamma_n(D),\varphi)(P,Q) = \begin{cases} 0 & (y^*>0) \\ \underline{V}(\Gamma_n(D),i)(P,Q) & (-t(i+1)<y^* \le -t(i); i \in J(D,\varphi)). \end{cases}$$

The Poisson kernel $K(\Gamma_n(D),\varphi)(P,Q)$ $(P \in \Gamma_n(D), Q \in S_n(D))$ is defined by

$$K(\Gamma_n(D),\varphi)(P,Q) = c_n^{-1}\frac{\partial}{\partial \nu}G_{\Gamma_n(D)}(P,Q) - \overline{W}(\Gamma_n(D),\varphi)(P,Q) - \underline{W}(\Gamma_n(D),\varphi)(P,Q).$$

Now we have

Theorem 11. *Let $g(Q)$ be a continuous function on $\partial\Gamma_n(D)$. Then there is a positive continuous function $\varphi(t)$ of $t \ge 0$ connected with g such that*

$$H(\Gamma_n(D),\varphi;g)(P) = \int_{\partial\Gamma_n(D)} g(Q)K(\Gamma_n(D),\varphi)(P,Q)d\sigma_Q$$

is a solution of the Dirichlet problem on $\Gamma_n(D)$ with g.

If we take $n=2$ and $D=(0,\pi)$ in Theorem 11, we obtain

Corollary 7. *Let $g_1(y^*)$ and $g_2(y^*)$ be two continuous functions on R. Then there is a positive continuous functions $\varphi(t)$ of $t \ge 0$ such that*

$$H(\Gamma_2,\varphi;g_1,g_2)(x,y)$$
$$= \int_{-\infty}^{\infty} g_1(y^*)K(\Gamma_2,\varphi)((x,y),(0,y^*)dy^* + \int_{-\infty}^{\infty} g_2(y^*)K(\Gamma_2,\varphi)((x,y),(\pi,y^*)dy^*$$

is a harmonic function in Γ_2 and a continuous function on $\overline{\Gamma_2}$ such that

$$H(\Gamma_2,\varphi;g_1,g_2)(0,y^*) = g_1(y^*), \quad H(\Gamma_2,\varphi;g_1,g_2)(\pi,y^*) = g_2(y^*) \quad (-\infty < y^* < \infty).$$

Theorem 12. *Let l,m be two non-negative integers and p,q be two positive integers satisfying $p \ge l, q \ge m$. Let $g(X^*,y^*)$ be a continuous function on $\partial\Gamma_n(D)$ satisfying (17). If $h(X,y)$ is a solution of the Dirichlet problem on $\Gamma_n(D)$ with g satisfying*

$$\mu_p(N(h^+)) = 0 \quad and \quad \eta_q(N(h^+)) = 0$$

then

$$h(X,y) = H(\Gamma_n(D),l,m;g)(P)$$
$$+ \sum_{k \in I(D,k_{p+1})} A_k(h)exp(\sqrt{\lambda(D,k)}y)f_k^D(X) + \sum_{k \in I(D,k_{q+1})} B_k(h)exp(-\sqrt{\lambda(D,k)}y)f_k^D(X)$$

for every $P=(X,y) \in \Gamma_n(D)$, where $A_k(h)(k=1,2,\dots,k_{p+1}-1)$ and $B_k(h)(k=1,2,\dots,k_{q+1}-1)$ are all constants.

If we take $n = 2$ and $D = (0, \pi)$ in Theorem 12, then we have

Corollary 8. *Let l, m be two non-negative integers and p, q be two positive integers satisfying $p \geq l$, $q \geq m$. Let $g_1(y^*)$, $g_2(y^*)$ be two continuous function on R satisfying (18). If $h(x,y)$ is a harmonic function in Γ_2 and a continuous function on $\overline{\Gamma_2}$ such that*

$$h(0, y^*) = g_1(y^*) \quad \text{and} \quad h(\pi, y^*) = g_2(y^*) \quad (-\infty < y^* < \infty),$$

and

$$\lim_{y\to\infty} \exp(-(p+1)y) \int_0^\pi h(x,y) \sin x dx = \lim_{y\to-\infty} \exp((q+1)y) \int_0^\pi h(x,y) \sin x dx = 0,$$

then

$$h(x,y) = H(\Gamma_2, l, m; g_1, g_2)(x,y) + \sum_{k=1}^{p} A_k(h) \exp(ky) \sin kx + \sum_{k=1}^{q} B_k(h) \exp(-ky) \sin kx$$

for every $(x,y) \in \Gamma_2$, where $A_k(h)$ $(k = 1, 2 \ldots, p)$ and $B_k(h)$ $(k = 1, 2 \ldots, q)$ are all constants.

The proof of all results in this part will be found in Miyamoto [**14**].

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