数理解析研究所講究録
第 890 巻 1994 年 63-69

Singular difference integrals, hypersingular integrals and their applications

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§1. Singular difference integrals and hypersingular integrals

For a function u on the $n-$dimensional Euclidean space R^n, the difference $\Delta_t^\ell u$ and the remainder R_t^ℓ of order ℓ with increment t are defined by

$$\Delta_t^\ell u(x)=\sum_{j=0}^{\ell}(-1)^j C_j^\ell u(x+(\ell-j)t),$$

$$R_t^\ell u(x)=u(x+t)-\sum_{|\gamma|\leq\ell-1}\frac{D^\gamma u(x)}{\gamma!}t^\gamma$$

where $\gamma=(\gamma_1,\cdots,\gamma_n)$ is a multi-index, $D^\gamma=D_1^{\gamma_1}\cdots D_n^{\gamma_n}, t^\gamma=t_1^{\gamma_1}\cdots t_n^{\gamma_n}$ and $|\gamma|=\gamma_1+\cdots+\gamma_n$. The following integral transforms $D^{\alpha,\ell}u$ and $H^{\alpha,\ell}u(\alpha>0,\ell$ a positive integer)

$$D^{\alpha,\ell}u(x)=\lim_{\epsilon\to 0}\int_{|t|\geq\epsilon}\frac{\Delta_t^\ell u(x)}{|t|^{n+\alpha}}dt,$$

$$H^{\alpha,\ell}u(x)=\lim_{\epsilon\to 0}\int_{|t|\geq\epsilon}\frac{R_t^\ell u(x)}{|t|^{n+\alpha}}dt$$

are called singular difference integral and hypersingular integral,respectively. The Schwartz space S is the set of infinitely differentiable functions which decrease at infinity faster than any power. For $u\in S, D^{\alpha,\ell}u(x)$ exists for $\alpha<2[(\ell+1)/2]$, and $H^{\alpha,\ell}u(x)$ exists for $\ell-1<\alpha<2[(\ell+1)/2]$ where $[r]$ denotes the integral part of r. For $u\in S'$(the dual of S), the Fourier transform of u is denoted by Fu. If u is an integrable function, then the Fourier transform of u is defined by

$$Fu(\xi)=\int u(x)e^{-x\cdot\xi}dx$$

with $x\cdot\xi=\sum_{j=1}^n x_j\xi_j$. S.G.Samko calculated the Fourier transform of $D^{\alpha,\ell}u$ for $u\in S$.

PROPOSITION 1.1.([6]) *Let $u\in S$ and $0<\alpha<2[(\ell+1)/2]$. Then*

$$F(D^{\alpha,\ell}u)(\xi)=d_{\alpha,\ell}|\xi|^\alpha Fu(\xi)$$

with

$$d_{\alpha,\ell}=(-1)^\ell\frac{\pi^{(n/2)+1}\sum_{j=0}^{\ell-1}(-1)^{j+1}C_j^\ell(\ell-j)^\alpha}{2^{\alpha+1}\Gamma(1+(\alpha/2))\Gamma((n+\alpha)/2)\sin\frac{\pi}{2}\alpha}.$$

We calculate the Fourier transform of $H^{\alpha,\ell}u$.

LEMMA 1.2.([4]) *If* $2[(\ell-1)/2] < \alpha < 2[(\ell+1)/2]$, *then*

$$\psi(\xi) = \lim_{\epsilon\to 0,\delta\to\infty} \int_{\epsilon\le|t|\le\delta} \frac{e^{it\cdot\xi} - \sum_{|\gamma|\le\ell-1} \frac{t^\gamma}{\gamma!}(i\xi)^\gamma}{|t|^{n+\alpha}} dt$$

exists and

$$\psi(\xi) = e_{\alpha,\ell}|\xi|^\alpha$$

with

$$e_{\alpha,\ell} = \frac{2^{1-\alpha}\pi^{(n/2)+1}}{\alpha\Gamma(\alpha/2)\Gamma((n+\alpha)/2)\sin\frac{\pi}{2}\alpha}.$$

PROPOSITION 1.3. *Let* $u \in S$ *and* $\ell - 1 < \alpha < 2[(\ell+1)/2]$. *Then*

$$F(H^{\alpha,\ell}u)(\xi) = e_{\alpha,\ell}|\xi|^\alpha Fu(\xi).$$

In fact, we have

$$\begin{aligned}
&F(H_\epsilon^{\alpha,\ell}u)(\xi) \\
&= \int\Big(\int_{|t|\ge\epsilon} \frac{u(x+t) - \sum_{|\gamma|\le\ell-1}\frac{D^\gamma u(x)}{\gamma!}t^\gamma}{|t|^{n+\alpha}} dt\Big)e^{-ix\cdot\xi}dt \\
&= \int_{|t|\ge\epsilon} \frac{1}{|t|^{n+\alpha}} \int u(x+t)e^{-ix\cdot\xi}dxdt - \sum_{|\gamma|\le\ell-1} \int_{|t|\ge\epsilon} \frac{t^\gamma}{\gamma!|t|^{n+\alpha}}dt \int D^\gamma u(x)e^{-ix\cdot\xi}dx \\
&= Fu(\xi)\int_{|t|\ge\epsilon} \frac{e^{it\xi} - \sum_{|\gamma|\le\ell-1}\frac{t^\gamma}{\gamma!}(i\xi)^\gamma}{|t|^{n+\alpha}} dt.
\end{aligned}$$

Hence the proposition follows from Lemma 1.2.

§2. The truncated integrals of the Riesz kernels

For $\alpha > 0$, the Riesz kernel of order α is given by

$$\kappa_\alpha(x) = \frac{1}{\gamma_{\alpha,n}} \begin{cases} |x|^{\alpha-n}, & \alpha < n, \text{or } \alpha \ge n, \alpha - n \ne \text{even}, \\ (\delta_{\alpha,n} - \log|x|)|x|^{\alpha-n}, & \alpha \ge n, \alpha - n = \text{even} \end{cases}$$

with

$$\delta_{\alpha,n} = \frac{\Gamma'(\alpha/2)}{2\Gamma(\alpha)} + \frac{1}{2}(1 + \frac{1}{2} + \cdots + \frac{1}{(\alpha-n)/2} - C) - \log\pi$$

where C is Euler' constant. With the above normalizing constants $\gamma_{\alpha,n}$ and $\delta_{\alpha,n}$ we have

$$F\kappa_\alpha(\xi) = \text{Pf.}|\xi|^{-\alpha} \tag{2.1}$$

where Pf. stands for the pseudo function[7:section 3 on Chap II]. Let $\alpha > 0$ and ℓ be a positive integer. We consider the truncated integrals of the Riesz kernels:

$$\rho_\epsilon^{\alpha,\ell}(x) = \int_{|t|\geq\epsilon} \frac{\Delta_t^\ell \kappa_\alpha(x)}{|t|^{n+\alpha}} dt,$$

$$\mu_\epsilon^{\alpha,\ell}(x) = \int_{|t|\geq\epsilon} \frac{R_t^\ell \kappa_\alpha(x)}{|t|^{n+\alpha}} dt.$$

We set $\rho^{\alpha,\ell}(x) = \rho_1^{\alpha,\ell}(x)$ and $\mu^{\alpha,\ell}(x) = \mu_1^{\alpha,\ell}(x)$. We note that $\rho^{\alpha,\ell}(x)$ is finite for every x, and $\mu^{\alpha,\ell}(x)$ is finite for $\alpha > \ell - 1$ and $x \neq 0$. Properties of $\rho^{\alpha,\ell}$ and $\mu^{\alpha,\ell}$ are investigated in [2],[3],[4] and [6].

LEMMA 2.1.(i) *Let ℓ be a positive integer,and moreover assume that $\ell > \alpha - n$ in case $\alpha - n$ is a nonnegative even number. Then*

$$\rho_\epsilon^{\alpha,\ell}(x) = \frac{1}{\epsilon^n}\rho^{\alpha,\ell}(\frac{x}{\epsilon}).$$

(ii) *Let $\alpha > \ell - 1$, and moreover assume that $\ell > \alpha - n$ in case $\alpha - n$ is a nonnegative even number. Then*

$$\mu_\epsilon^{\alpha,\ell}(x) = \frac{1}{\epsilon^n}\mu^{\alpha,\ell}(\frac{x}{\epsilon}).$$

LEMMA 2.2. (i) *Let $2[(\ell+1)/2] > \alpha$. Then for $|x| \geq 1$*

$$|\rho^{\alpha,\ell}(x)| \leq C|x|^{\alpha-2[2(\ell+1)/2]-n}$$

and for $|x| < 1$

$$|\rho^{\alpha,\ell}(x)| \leq C \begin{cases} |x|^{\alpha-n}, & \alpha < n, \\ (1 - \log|x|), & \alpha = n, \\ 1, & \alpha > n. \end{cases}$$

(ii) *Let $\ell - 1 < \alpha < 2[(\ell+1)/2]$. Then*

$$|\mu^{\alpha,\ell}(x)| \leq C \begin{cases} |x|^{\alpha-[\alpha]-1-n}, & |x| \geq 1, \\ |x|^{\alpha-2[(\ell-1)/2]-n}, & |x| < 1. \end{cases}$$

By Lemma 2.2, if $2[(\ell+1)/2] > \alpha$, then $\rho^{\alpha,\ell}$ is integrable, and if $\ell - 1 < \alpha < 2[(\ell+1)/2]$, then $\mu^{\alpha,\ell}$ is integrable. We denote

$$d_{\alpha,\ell}^1 = \int \rho^{\alpha,\ell}(x)dx, \qquad e_{\alpha,\ell}^1 = \int \mu^{\alpha,\ell}(x)dx.$$

S.G.Samko[6] remarked

PROPOSITION 2.3. *For* $2[(\ell+1)/2] > \alpha, d_{\alpha,\ell} = d^1_{\alpha,\ell}$ *and hence* $d^1_{\alpha,\ell} \neq 0$ *for* $2[(\ell+1)/2] > \alpha$ *and* $\alpha \neq$ *odd.*

We note

PROPOSITION 2.4.([4]) *For* $\ell-1 < \alpha < 2[(\ell+1)/2], e_{\alpha,\ell} = e^1_{\alpha,\ell}$ *and hence* $e^1_{\alpha,\ell} \neq 0$ *for* $\ell-1 < \alpha < 2[(\ell+1)/2]$.

§3. The spaces of Riesz potentials

For $f \in S$, the Riesz potential of order α of f is defined by

$$U^f_\alpha(x) = \int \kappa_\alpha(x-y)f(y)dy.$$

By (2.1) for $f \in S$ we have

$$F(U^f_\alpha)(\xi) = \mathrm{Pf.}|\xi|^{-\alpha}Ff(\xi). \tag{3.1}$$

In order to define the Riesz potential of an L^p-function, for an integer $k < \alpha$ we introduce

$$\kappa_{\alpha,k}(x,y) = \begin{cases} \kappa_\alpha(x-y) - \sum_{|\gamma|\le k} \frac{D^\gamma\kappa_\alpha(-y)}{\gamma!}x^\gamma, & 0 \le k < \alpha, \\ \kappa_\alpha(x-y), \quad k \le -1 & \end{cases}$$

We have

PROPOSITION 3.1.([1]) *Let* $f \in L^p$ *and* $k = [\alpha - (n/p)]$.
(i) *If* $\alpha - (n/p)$ *is not a nonnegative integer, then*

$$U^f_{\alpha,k}(x) = \int \kappa_{\alpha,k}(x,y)f(y)dy$$

exists and satisfies

$$(\int |U^f_{\alpha,k}(x,y)|^p|x|^{-\alpha p}dx)^{1/p} \le C||f||_p.$$

(ii) *If* $\alpha - (n/p)$ *is a nonnegative integer, then* $U^{f_1}_{\alpha,k-1}$ *and* $U^{f_2}_{\alpha,k}$ *exist and satisfy*

$$(\int |U^{f_1}_{\alpha,k-1}(x)|^p|x|^{-\alpha p}(1+|\log|x||)^{-p}dx)^{1/p} \le C||f_1||_p,$$

$$(\int |U^{f_2}_{\alpha,k}(x)|^p|x|^{-\alpha p}(1+|\log|x||)^{-p}dx)^{1/p} \le C||f_2||_p$$

where $f_1 = f|_{B_1}$ *is the restriction of* f *to* $B_1 = \{|x| < 1\}$ *and* $f_2 = f - f_1$.

By Propositions 1.1,1.3 and (3.1) it seems that the integral transforms $\frac{1}{d_{\alpha,\ell}} D^{\alpha,\ell}$ and $\frac{1}{e_{\alpha,\ell}} H^{\alpha,\ell}$ are the inverse operators of the Riesz potential operator. Precisely speaking

PROPOSITION 3.2. (I)([3]) *Let* $f \in L^p, k = [\alpha - (n/p)]$ *and* $\ell > \alpha - (n/p)$.
(i) *If* $\alpha - (n/p)$ *is not a nonnegative integer, then*

$$D_\epsilon^{\alpha,\ell} U_{\alpha,k}^f = \rho_\epsilon^{\alpha,\ell} * f$$

and hence

$$D^{\alpha,\ell} U_{\alpha,k}^f = d_{\alpha,\ell} f$$

where the symbol $*$ *stands for the convolution.*
(ii) *If* $\alpha - (n/p)$ *is a nonnegative integer, then*

$$D_\epsilon^{\alpha,\ell}(U_{\alpha,k-1}^{f_1} + U_{\alpha,k}^{f_2}) = \rho_\epsilon^{\alpha,\ell} * f$$

and hence

$$D^{\alpha,\ell}(U_{\alpha,k-1}^{f_1} + U_{\alpha,k}^{f_2}) = d_{\alpha,\ell} f$$

with $f_1 = f|_{B_1}$ *and* $f_2 = f - f_1$.
(II)([2]) *Let* $f \in L^p, k = [\alpha - (n/p)]$ *and* $\alpha - (n/p) < \ell < \alpha + 1$.
(i) *If* $\alpha - (n/p)$ *is not a nonnegative integer, then*

$$H_\epsilon^{\alpha,\ell} U_{\alpha,k}^f = \mu_\epsilon^{\alpha,\ell} * f$$

and hence

$$H^{\alpha,\ell} U_{\alpha,k}^f = e_{\alpha,\ell} f.$$

(ii) *If* $\alpha - (n/p)$ *is a nonnegative integer, then*

$$H_\epsilon^{\alpha,\ell}(U_{\alpha,k-1}^{f_1} + U_{\alpha,k}^{f_2}) = \mu_\epsilon^{\alpha,\ell} * f$$

and hence

$$H^{\alpha,\ell}(U_{\alpha,k-1}^{f_1} + U_{\alpha,k}^{f_2}) = e_{\alpha,\ell} f.$$

Taking Proposition 3.1 into account, we define the Riesz potential spaces of L^p-functions as follows:

$$R_\alpha^p = \begin{cases} \{U_{\alpha,k}^f; f \in L^p\}, & \alpha - (n/p) \neq \text{ a nonnegative integer}, \\ \{U_{\alpha,k-1}^{f_1} + U_{\alpha,k}^{f_2}; f \in L^p, f_1 = f|_{B_1}, f_2 = f - f_1\}, & \\ \qquad \alpha - (n/p) = \text{ a nonnegative integer} & \end{cases}$$

with $k = [\alpha - (n/p)]$.

We give characterizations of the Riesz potential spaces using the singular difference integrals and hypersingular integrals.

THEOREM 3.3.([3]) *Let* $[(\ell+1)/2] > \alpha$ *and* $\alpha =$ *an odd number. Then* $u \in R^p_\alpha + P_k$ *if and only if*

$$\text{(i)} \qquad \int |u(x)|^p (1+|x|)^{-\alpha p} (\log(e+|x|)^{-p} dx < \infty,$$

$$\text{(ii)} \qquad \lim_{\epsilon \to 0} \int_{|t| \geq \epsilon} \frac{\Delta^\ell_t u(x)}{|t|^{n+\alpha}} dt \quad \text{exists in} \quad L^p$$

where P_k is the set of polynomials of degree k.

For $1 < r_0, r_1, \cdots, r_{\ell-1} < \infty$, we denote

$$W^{r_0, r_1, \cdots, r_{\ell-1}}_{\ell-1} = \{u; D^\gamma u \in L^{r_j} \text{ for } |\gamma| = j, j = 0, 1, \cdots, \ell-1\}.$$

COROLLARY 3.4. *Let* $[(\ell+1)/2] > \alpha$ *and* $\alpha \neq$ *an odd number. Then* $u \in (R^p_\alpha + P_k) \cap W^{r_0, r_1, \cdots, r_{\ell-1}}_{\ell-1}$ *if and only if*

$$\text{(i)} \qquad u \in W^{r_0, r_1, \cdots, r_{\ell-1}}_{\ell-1},$$

$$\text{(ii)} \quad \lim_{\epsilon \to 0} \int_{|t| \geq \epsilon} \frac{\Delta^\ell_t u(x)}{|t|^{n+\alpha}} dt \text{ exists in } L^p$$

for $r_0 \geq p$ in case of $\alpha - (n/p) \geq 0, p \leq r_0 \leq p_\alpha$ in case of $\alpha - (n/p) < 0$ where $1/p_\alpha = (1/p) - (\alpha/n)$.

THEOREM 3.5.([4]) *Let* $\ell - 1 < \alpha < \min(2[(\ell+1)/2], \ell + (n/p))$. *Then* $u \in (R^p_\alpha + P_k) \cap W^{r_0, r_1, \cdots, r_{\ell-1}}_{\ell-1}$ *if and only if*

$$\text{(i)} \qquad u \in W^{r_0, r_1, \cdots, r_{\ell-1}}_{\ell-1},$$

$$\text{(ii)} \quad \lim_{\epsilon \to 0} \int_{|t| \geq \epsilon} \frac{R^\ell_t u(x)}{|t|^{n+\alpha}} dt \quad \text{exists in } L^P$$

for $r_0 \geq p$ in case of $\alpha - (n/p) \geq 0, p \leq r_0 \leq p_\alpha$ in case of $\alpha - (n/p) < 0$.

THEOREM 3.6.([5]) *Let* $\alpha - (n/p) < 0$ *and* $\ell - 1 < \alpha < \min(2[(\ell+1)/2], \frac{1}{2}(\ell + (n/p))$. *Then* $u \in R^p_\alpha$ *if and only if*

$$\text{(i)} \qquad u \in W^{p_\alpha, p_{\alpha-1}, \cdots, p_{\alpha-(\ell-1)}}_{\ell-1},$$

$$\text{(ii)} \quad \lim_{\epsilon \to 0} \int_{|t| \geq \epsilon} \frac{R^\ell_t u(x)}{|t|^{n+\alpha}} dt \text{ exists in } L^p.$$

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