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The Kontorovich-Lebedev Transform and Its Convolution

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Abstract

The present paper deals with modern results for the known Kontorovich-Lebedev transform by index of the Macdonald function and related convolution in Lebesgue $L_{\nu,p}$-spaces of functions. We study the mapping and factorization properties of these operators and demonstrate applications to corresponding class of integral equations of first and second kind.

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1. Introduction

In this paper we consider the known Kontorovich-Lebedev (K-L) transform first introduced in [6] by the formula

$$(\mathfrak{KL}f)(\tau) = \int_0^\infty K_{i\tau}(y)f(y)dy, \tag{1.1}$$

where $K_{i\tau}(y)$ is the Macdonald function [1] of the argument $y > 0$ and the pure imaginary index $i\tau$ ($\tau \in \mathbf{R}$). The Macdonald function in (1.1) is defined by the integral [16]

$$K_{i\tau}(x) = \frac{1}{2}\int_{i\delta-\infty}^{i\delta+\infty} e^{-x\cosh\beta}e^{i\tau\beta}d\beta \quad (x > 0), \tag{1.2}$$

where the parameter δ is taken from the interval $[0, \pi/2)$, coordinating with the known one from [1] according to analytic properties of the integrand (1.2). Recently many results and wide list of references for K-L transform (1.1) have been collected by the first author in the monograph [16].

Here we attract our attention to the Lebesgue weighted space $L_p(\rho) \equiv L_p(\mathbf{R}_+; \rho)$ ($p \geqq 1$) with the weight function $\rho(x) > 0$ and the finite norm

$$||f||_{L_p(\rho)} = \left(\int_0^\infty \rho(t)|f(t)|^p dt\right)^{1/p}. \tag{1.3}$$

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In particular, when $\rho(x) = x^{\nu p-1}$ $(x > 0, \nu \in \mathbf{R})$ we will denote such a space as $L_{\nu,p}(\mathbf{R}_+)$. Let us mention here useful inequalities for our considerations in the paper which are the weighted Hölder inequality

$$\int_0^\infty |f(t)g(t)|dt \leqq ||f||_{L_p(\rho)}||g||_{L_q(\rho^{1-q})}, \quad q = \frac{p}{p-1} \tag{1.4}$$

and the generalized Minkowski inequality

$$\left(\int_0^\infty dx \left|\int_0^\infty f(x,y)dy\right|^p\right)^{1/p} \leqq \int_0^\infty \left(\int_0^\infty |f(x,y)|^p dx\right)^{1/p} dy. \tag{1.5}$$

Another purpose of our investigation is to study a so-called convolution operator related to K-L transform (1.1) defined by the double integral

$$(f * g)(x) = \frac{1}{2x}\int_0^\infty \int_0^\infty \exp\left(-\frac{1}{2}\left[\frac{xu}{y} + \frac{xy}{u} + \frac{yu}{x}\right]\right) f(u)g(y)du\,dy \quad (x > 0). \tag{1.6}$$

This operator was first introduced in [5] formally as an example of the integral nonstandard convolution. Later this operator was considered in detail by the first author in [12] - [15] in a slightly different form. Moreover this convolution was generalized for other index transforms and applications to various types of integral equations were obtained. In [2] - [4] the convolution like (1.6) and its analogue for the Mehler-Fock transform in the space of generalized functions have been considered. In [16] great attention to the convolution operator (1.6) and its applications is attracted and this class of convolutions essentially is completed the double integral type convolutions studied in [9].

This paper is intended to draw a parallel between known results for K-L transform (1.1) and its convolution (1.6) in the spaces L_1 and L_2 (see [16]) and new ones in the weighted space $L_{\nu,p}$. As conclusion we extend our understanding of these objects and their applications to integral equations. Some separate examples of integral equations with convolution (1.6) were considered previously in [7], [16], which involve the operator (1.6) as follows:

$$(\mathfrak{K}f)(x) = \int_0^\infty K(x,u)f(u)du, \tag{1.7}$$

where we fixed some function $g(y)$ and calculated the kernel $K(x,u)$ by integral

$$K(x,u) = \frac{1}{2x}\int_0^\infty \exp\left(-\frac{1}{2}\left[\frac{xu}{y} + \frac{xy}{u} + \frac{yu}{x}\right]\right) g(y)dy \quad (x > 0). \tag{1.8}$$

We will touch these questions below and will demonstrate interesting examples of integral equations and their solutions.

2. Inversion of the Kontorovich-Lebedev Transform in $L_{\nu,p}$

Throughout of this section we take $f(x)$ from the weighted space $L_{\nu,p}(\mathbf{R}_+)$ with $\nu \in \mathbf{R}$ and $p \geqq 1$. First we observe from the integral representation (1.1) and the definition of the Macdonald function (1.2) that K-L transform is an even function of the real variable τ and without loss of generality we can consider it only for the nonnegative variable τ. From the asymptotic behavior of the Macdonald function [1] and the Hölder inequality (1.4) we immediately obtain that the integral (1.1) is absolutely convergent for any function $f(x) \in L_{\nu,p}(\mathbf{R}_+)$ with $\nu < 1$. Namely we have:

Lemma 2.1. *Let $f(x) \in L_{\nu,p}(\mathbf{R}_+)$ $(\nu < 1)$. Then for K-L transform (1.1) there holds the uniform estimate by $\tau \geqq 0$*

$$|(\mathfrak{KL}f)(\tau)| \leqq C||f||_{L_{\nu,p}(\mathbf{R}_+)}, \tag{2.1}$$

where

$$C = \left(\int_0^\infty K_0^q(y) y^{(1-\nu)q-1} dy\right)^{1/q} \quad \left(q = \frac{p}{p-1}\right) \tag{2.2}$$

and $K_0(x)$ is the Macdonald function of order zero.

Proof. To establish this estimate we can appeal to the simple inequality $|K_{i\tau}(x)| \leqq K_0(x)$ $(x > 0)$ that naturally arises from the formula (1.2) if we put there $\delta = 0$. Hence invoking to the Hölder inequality (1.4) we have

$$\begin{aligned}
(2.3)\ |(\mathfrak{KL}f)(\tau)| &\leqq \int_0^\infty K_0(y)|f(y)|dy \\
&\leqq \left(\int_0^\infty K_0^q(y) y^{(1-\nu)q-1} dy\right)^{1/q} \left(\int_0^\infty |f(y)|^p y^{\nu p-1} dy\right)^{1/p} = C||f||_{L_{\nu,p}(\mathbf{R}_+)}.
\end{aligned}$$

Indeed, according to the known asymptotic of the function

$$\begin{cases} K_0(x) = O(\log x) \quad (x \to +0), \qquad K_\mu(x) = O(x^{-|\Re\mu|}) \quad (x \to +0,\ \mu \neq 0); \\ K_\mu(x) = O\left(\dfrac{e^{-x}}{\sqrt{x}}\right) \quad (x \to \infty), \end{cases} \tag{2.4}$$

the integral in (2.2) is obviously convergent when $\nu < 1$. ■

Lemma 2.1 shows that K-L transform of $L_{\nu,p}$-functions is at least continuous function of $\tau \in \mathbf{R}_+$ in view of the uniform convergence of the integral (1.1). Moreover we can deduce its differential properties. Performing the differentiation by τ of arbitrary order $k = 0, 1, \cdots$ under the integral sign in the formula (1.2) with $\delta = 0$ we arrive at the formula

$$\frac{\partial^k}{\partial \tau^k} K_{i\tau}(x) = \frac{1}{2} \int_{-\infty}^\infty e^{-x\cosh u} e^{i\tau u} (iu)^k du \tag{2.5}$$

by the Lebesgue theorem and evidently

$$\left|\frac{\partial^k}{\partial \tau^k} K_{i\tau}(x)\right| \leqq \int_0^\infty e^{-x\cosh u} u^k du. \tag{2.6}$$

Lemma 2.2. *Under assumptions of Lemma 2.1 K-L transform is infinitely differentiable function on $\mathbf{R}_+$ and we have the uniform estimate*

$$\left|\frac{d^k}{d\tau^k}(\mathfrak{KL}f)(\tau)\right| \leqq B_k\|f\|_{L_{\nu,p}(\mathbf{R}_+)} \quad (k=0,1,\cdots), \tag{2.7}$$

where

$$B_k = q^{\nu-1}\Gamma^{1/q}(q(1-\nu))\int_0^\infty \frac{u^k}{\cosh^{1-\nu}u}\,du < +\infty. \tag{2.8}$$

Proof. As in the previous Lemma 2.1 by making use of the Hölder inequality (1.4), we obtain

$$\left|\frac{d^k}{d\tau^k}(\mathfrak{KL}f)(\tau)\right| \leqq \left(\int_0^\infty \left|\frac{\partial^k}{\partial\tau^k}K_{i\tau}(y)\right|^q y^{(1-\nu)q-1}dy\right)^{1/q}\|f\|_{L_{\nu,p}(\mathbf{R}_+)}. \tag{2.9}$$

Invoking to the generalized Minkowski inequality (1.5) and using the estimate (2.6), we continue

$$\left(\int_0^\infty \left|\frac{\partial^k}{\partial\tau^k}K_{i\tau}(y)\right|^q y^{(1-\nu)q-1}dy\right)^{1/q} \leqq \int_0^\infty u^k\left(\int_0^\infty e^{-qy\cosh u}y^{(1-\nu)q-1}dy\right)^{1/q}du \tag{2.10}$$
$$= q^{\nu-1}\Gamma^{1/q}(q(1-\nu))\int_0^\infty \frac{u^k}{\cosh^{1-\nu}u}\,du = B_k < +\infty.$$

■

From these properties for K-L transform it follows that we can discuss its belonging to the space $L_r(\mathbf{R}_+)$ for some $1 \leqq r \leqq \infty$ investigating only its behavior at infinity. The answer can be obtained by applying more careful estimate of the Macdonald function (1.2), that is

$$|K_{i\tau}(x)| \leqq \frac{1}{2}e^{-\delta\tau}\int_{-\infty}^\infty e^{-x\cos\delta\cosh u}du = e^{-\delta\tau}K_0(x\cos\delta) \quad \left(\delta\in\left[0,\frac{\pi}{2}\right)\right). \tag{2.11}$$

Lemma 2.3. *K-L transform* (1.1) *is a bounded mapping from* $L_{\nu,p}(\mathbf{R}_+)$ $(\nu < 1, p \geqq 1)$ *into* $L_r(\mathbf{R}_+) \equiv L_{1/r,r}(\mathbf{R}_+)$, *where* $r \geqq 1$ *and parameters* p *and* r *have no dependence.*

Proof. Using the estimate (2.11) and treating like Lemma 2.1, we have the inequality

$$|(\mathfrak{KL}f)(\tau)| \leqq e^{-\delta\tau}\int_0^\infty K_0(y\cos\delta)|f(y)|dy \tag{2.12}$$
$$\leqq e^{-\delta\tau}\left(\int_0^\infty K_0^q(y\cos\delta)y^{(1-\nu)q-1}dy\right)^{1/q}\left(\int_0^\infty |f(y)|^p y^{\nu p-1}dy\right)^{1/p}$$
$$= C_\delta e^{-\delta\tau}\|f\|_{L_{\nu,p}(\mathbf{R}_+)},$$

where the constant $C_\delta > 0$ depends on $\delta \in [0,\pi/2)$. It is obvious to see that the norm for K-L transform (1.1) in the space $L_r(\mathbf{R}_+)$ $(r \geqq 1)$ is finite for fixed $\delta \in (0,\pi/2)$. Moreover we established the fact that K-L transform belongs to the weigted space $L_r(\mathbf{R}_+;\rho)$, if the weigt function $\rho(\tau)$ satisfies the condition

$$\int_0^\infty \rho(\tau)e^{-\delta r\tau}d\tau < \infty. \tag{2.13}$$

So we received the desired result. ∎

These lemmas show that K-L transform (1.1) of $L_{\nu,p}$-functions $f(x)$ possesses both the smoothness and L_r-properties and besides, the range of K-L transform

$$KL(L_{\nu,p}) = \{g : g(\tau) = (\mathfrak{KL}f)(\tau),\ f \in L_{\nu,p}(\mathbf{R}_+)\} \quad (\nu < 1, p \geq 1) \tag{2.14}$$

does not coincide with the space $L_r(\mathbf{R}_+)$. In fact, we know that K-L transform also belongs to the weighted space $L_r(\mathbf{R}_+;\rho)$ with condition (2.13). But choosing a different weight we can easily verify that there exists some function belonging to $L_r(\mathbf{R}_+)$ which does not belong to the space $L_r(\mathbf{R}_+;\rho)$, and vice versa. Thus it is necessary to describe the range of K-L transform (1.1) more definitely.

For this purpose we will use the inverse operator that was introduced in a slightly different form in [16]. Let us consider the operator

$$(I_\varepsilon g)(x) = \frac{2}{\pi^2}\, x^{\varepsilon-1} \int_0^\infty \tau \sinh([\pi-\varepsilon]\tau)K_{i\tau}(x)g(\tau)d\tau \quad (\varepsilon \in (0,\pi)). \tag{2.15}$$

Theorem 2.1. *Let $g(\tau) = (\mathfrak{KL}f)(\tau)$ for the density $f(y) \in L_{\nu,p}(\mathbf{R}_+)$ $(\nu < 1,$ $1 \leqq p \leqq \infty)$, then the operator* (2.15) *has the form*

$$(I_\varepsilon g)(x) = \frac{\sin\varepsilon}{\pi}\, x^\varepsilon \int_0^\infty \frac{K_1([x^2+y^2-2xy\cos\varepsilon]^{1/2})}{(x^2+y^2-2xy\cos\varepsilon)^{1/2}}\, yf(y)dy \ (x>0), \tag{2.16}$$

where $K_1(z)$ is the Macdonald function (1.2) *of order* 1.

Proof. Substituting the value of $g(\tau)$ as K-L transform (1.1) into the formula (2.15) and appealing to the inequality (2.11) we have the estimate

$$\begin{aligned} |(I_\varepsilon g)(x)| \leqq\ & \frac{2}{\pi^2}\, x^{\varepsilon-1}K_0(x\cos\delta_1)\int_0^\infty \tau\sinh([\pi-\varepsilon]\tau)e^{-(\delta_1+\delta_2)\tau}d\tau \\ & \times \int_0^\infty K_0(y\cos\delta_2)|f(y)|dy, \end{aligned} \tag{2.17}$$

as far as we choose $\delta_1+\delta_2+\varepsilon > \pi$. Obviously two integrals in (2.17) are convergent (the second one is provided by Lemma 2.3). Hence we can apply the Fubini theorem. As a result we use the formula [10, Vol.2, (2.16.51.8)] and we have

$$\begin{aligned} & \int_0^\infty \tau\sinh((\pi-\varepsilon)\tau)K_{i\tau}(x)K_{i\tau}(y)dt \\ & = \frac{\pi xy\sin\varepsilon}{2}\frac{K_1([x^2+y^2-2xy\cos\varepsilon]^{1/2})}{(x^2+y^2-2xy\cos\varepsilon)^{1/2}}, \end{aligned} \tag{2.18}$$

which gives the representation (2.16). ∎

The inversion formula of K-L transform (1.1) on the space $L_{\nu,p}(\mathbf{R}_+)$ is established by:

Theorem 2.2. *Let $g(\tau) = (\mathfrak{KL}f)(\tau)$ and $f(y) \in L_{\nu,p}(\mathbf{R}_+)$ $(0 < \nu < 1, 1 \leq p \leqq \infty)$. Then*

$$f(x) = (Ig)(x), \tag{2.19}$$

where $(Ig)(x)$ is understood as

$$(Ig)(x) = \underset{\varepsilon\to+0}{\text{l.i.m.}}\, (I_\varepsilon g)(x) \quad (x > 0), \tag{2.20}$$

where the limit in (2.20) is meant in terms of the norm in $L_{\nu,p}(\mathbf{R}_+)$. Moreover, the limit in (2.20) exists almost everywhere on $\mathbf{R}_+$.

Proof. Replacing the variable $y = x(\cos\varepsilon + t\sin\varepsilon)$ in the integral (2.16), we arrive at the equality

$$(I_\varepsilon g)(x) = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{R(x,t,\varepsilon)(\cos\varepsilon + t\sin\varepsilon)}{t^2+1} f(x[\cos\varepsilon + t\sin\varepsilon])dt, \tag{2.21}$$

where

$$R(x,t,\varepsilon) = \begin{cases} x^{\varepsilon+1}\sqrt{t^2+1}\sin\varepsilon\ K_1(x\sqrt{t^2+1}\sin\varepsilon), & t \geqq -\cot\varepsilon, \\ 0, & t < -\cot\varepsilon. \end{cases} \tag{2.22}$$

From the asymptotic behavior of the Macdonald function $K_1(z)$ [16] we obtain $R(x,t,\varepsilon) < C$ uniformly as a function of three variables $t \in \mathbf{R}$, $x \in \mathbf{R}_+$ and $\varepsilon \in (0,\pi)$, and we observe the limit

$$\lim_{\varepsilon\to+0} R(x,t,\varepsilon) = 1.$$

Further, if we use the approximation property of the Poisson kernel

$$P(t) = \frac{1}{\pi}\frac{1}{t^2+1}$$

in order to estimate the $L_{\nu,p}$-norm of the difference $I_\varepsilon g - f$ by applying the generalized Minkowski inequality (1.5), then

$$\begin{aligned} &\|I_\varepsilon g - f\|_{L_{\nu,p}(\mathbf{R}_+)} \\ &\leqq \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{1}{t^2+1}\|f(x[\cos\varepsilon + t\sin\varepsilon])(\cos\varepsilon + t\sin\varepsilon)R(x,t,\varepsilon) - f(x)\|_{L_{\nu,p}(\mathbf{R}_+)}dt \to 0 \\ &\qquad (\varepsilon \to +0). \end{aligned} \tag{2.23}$$

Indeed, from (2.21) we have the estimate

$$\begin{aligned} \|(I_\varepsilon g)\|_{L_{\nu,p}(\mathbf{R}_+)} &\leqq \frac{C}{\pi}\int_{-\cot\varepsilon}^{\infty} \frac{1}{t^2+1}\|f(x[\cos\varepsilon + t\sin\varepsilon])(\cos\varepsilon + t\sin\varepsilon)\|_{L_{\nu,p}(\mathbf{R}_+)}dt \\ &\leqq C\|f\|_{L_{\nu,p}(\mathbf{R}_+)}\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{(1+|t|)^{1-\nu}}{t^2+1}dt \\ &= C_1\|f\|_{L_{\nu,p}(\mathbf{R}_+)} \qquad (0 < \nu < 1), \end{aligned} \tag{2.24}$$

where C_1 is a positive absolute constant because the integral by t is convergent under condition on the parameter ν. Thus from the Lebesgue theorem and the continuity of the $L_{\nu,p}$-norm [16] we proved the equality (2.20). The existence of the limit almost everywhere on $\mathbf{R}_+$ follows from the radial property of the Poisson kernel $P(t) = P(|t|) \in L_1(\mathbf{R}_+)$. ■

Theorem 2.2 yields also the inequality

$$||I_\varepsilon g||_{L_{\nu,p}(\mathbf{R}_+)} \leqq C||Ig||_{L_{\nu,p}(\mathbf{R}_+)}, \tag{2.25}$$

where $g \in KL(L_{\nu,p})$ $(0 < \nu < 1, 1 \leqq p \leqq \infty)$. It follows from Theorem 2.2 that $(\mathfrak{KL}f)(\tau) \equiv 0$ for $f(y) \in L_{\nu,p}(\mathbf{R}_+)$ $(0 < \nu < 1, 1 \leqq p \leqq \infty)$, iff $f(y) \equiv 0$. So, in the space $KL(L_{\nu,p})$ we can introduce a norm by the equality

$$||g||_{KL(L_{\nu,p})} = ||f||_{L_{\nu,p}} \quad \text{for} \quad g = (\mathfrak{KL}f)(\tau). \tag{2.26}$$

As it is evident, the space $KL(L_{\nu,p})$ is a Banach one with the norm (2.26) and as an isometric to $L_{\nu,p}$.

The next theorem gives a characterization of the space $KL(L_{\nu,p})$ in terms of the operator (2.15).

Theorem 2.3. *The necessary and sufficient condition for $g(\tau)$ belongs to $KL(L_{\nu,p})$ $(0 < \nu < 1, 1 \leq p \leqq \infty)$ is $g(\tau) \in L_r(\mathbf{R}_+)$ $(1 \leqq r \leqq \infty)$ and*

$$\underset{\varepsilon \to +0}{\text{l.i.m.}} (I_\varepsilon g) \in L_{\nu,p}(\mathbf{R}_+). \tag{2.27}$$

Proof. The necessity is a simple fact as a corollary of Lemma 2.3, of Theorem 2.2 and of the inequality (2.25). The sufficiency part is more complicated.

Let $g(\tau) \in L_r(\mathbf{R}_+)$ and assume that the condition (2.27) is valid. We have to show that there exists a function $f \in L_{\nu,p}$ such that

$$g = (\mathfrak{KL}f)(\tau). \tag{2.28}$$

From the condition (2.27) we conclude that $(I_\varepsilon g) \in L_{\nu,p}\mathbf{R}_+$ for sufficiently small $\varepsilon > 0$ and we can calculate the composition

$$(\mathfrak{KL}(I_\varepsilon g))(\tau) = \int_0^\infty K_{i\tau}(y)\,(I_\varepsilon g)\,(y)dy. \tag{2.29}$$

Let a function $g(\tau)$ be taken from $C_0^\infty(\mathbf{R}_+)$ being dense in L_r, then we have by substituting (2.15) into (2.29) the possibility to change the order of integration by the Fubini theorem. Using the value of the integral [10, Vol.2, (2.16.33.2)]

$$\int_0^\infty y^{\varepsilon-1}K_{i\tau}(y)K_{i\beta}(y)dy = \frac{2^{\varepsilon-3}}{\Gamma(\varepsilon)}\left|\Gamma\left(\frac{\varepsilon+i[\tau+\beta]}{2}\right)\Gamma\left(\frac{\varepsilon+i[\tau-\beta]}{2}\right)\right|^2, \tag{2.30}$$

we obtain

$$\begin{aligned} g_\varepsilon(\tau) \equiv (\mathfrak{KL}\,(I_\varepsilon g))(\tau) = \frac{2^{\varepsilon-2}}{\pi^2\Gamma(\varepsilon)}\int_0^\infty \beta\sinh((\pi-\varepsilon)\beta) \\ \times\left|\Gamma\left(\frac{\varepsilon+i[\tau+\beta]}{2}\right)\Gamma\left(\frac{\varepsilon+i[\tau-\beta]}{2}\right)\right|^2 g(\beta)d\beta. \end{aligned} \tag{2.31}$$

In order to prove the validity of the equality (2.31) for all $g \in L_r(\mathbf{R}_+)$, we have to prove the boundedness of the operator in the right-hand side of (2.31). From the asymptotic formula for the gamma-function [1] the kernel of the integrand in (2.31) is equal to

$$O(e^{(\pi/2-\varepsilon)\beta-\pi|\tau-\beta|/2-\pi\tau/2}) \quad (\beta \to \infty, \tau \to \infty, \varepsilon \in (0,\pi)). \tag{2.32}$$

Hence we have the following estimate

$$\begin{aligned} |g_\varepsilon(\tau)| &\leqq Ce^{-\pi\tau/2}\int_0^\infty e^{(\pi/2-\varepsilon)\beta-\pi|\beta-\tau|/2}|g(\beta)|d\beta \\ &\leqq Ce^{(\delta-\pi/2)\tau}\int_0^\infty e^{(\pi/2-\varepsilon-\delta)\beta}|g(\beta)|d\beta, \end{aligned} \tag{2.33}$$

where the parameter δ is taken from the interval $(\pi/2-\varepsilon, \pi/2)$. So from the estimate (2.33) with the aid of the Hölder inequality, we get the boundedness of the operator in the right-hand side of (2.31) in the space $L_r(\mathbf{R}_+)$ $(1 \le r \leqq \infty)$.

Now let us calculate the limit of the right-hand side of (2.31), as $\varepsilon \to +0$ in norm of the space $L_r(\mathbf{R}_+)$. We begin by representing the function $g_\varepsilon(\tau)$ by the substitution $\beta = \tau + \varepsilon t$

$$g_\varepsilon(\tau) = \frac{1}{\pi}\int_{-\infty}^{\infty}\frac{g(\tau+\varepsilon t)}{t^2+1}\,h(\tau,t,\varepsilon)dt, \tag{2.34}$$

where

$$\begin{aligned} h(\tau,t,\varepsilon) = H(\tau+\varepsilon t)\,&\frac{2^{\varepsilon-2}\varepsilon(\tau+\varepsilon t)(t^2+1)\sinh([\pi-\varepsilon][\tau+\varepsilon t])}{\pi\Gamma(\varepsilon)} \\ &\times\left|\Gamma\left(i\tau+\frac{\varepsilon}{2}[1+it]\right)\Gamma\left(\frac{\varepsilon}{2}[1-it]\right)\right|^2 \end{aligned} \tag{2.35}$$

and $H(x)$ is the Heaviside function. From the previous discussion, we conclude that the function $h(\tau,t,\varepsilon)$ is uniformly bounded for all parameters $\tau > 0, t \in \mathbf{R}, \varepsilon \in (0,\pi)$. Moreover, from the supplement formula for the gamma-function $\Gamma(z+1) = z\Gamma(z)$ there holds the relation

$$\lim_{\varepsilon\to+0} h(\tau,t,\varepsilon) = 1. \tag{2.36}$$

Hence we obtain the following estimate for the norm of the function $g_\varepsilon(\tau)$ in the space $L_r(\mathbf{R}_+)$

$$\begin{aligned} &||g_\varepsilon(\tau)-g(\tau)||_{L_r(\mathbf{R}_+)} \\ &\leqq \frac{1}{\pi}\int_{-\infty}^{\infty}\frac{1}{t^2+1}\,||g(\tau+\varepsilon t)h(\tau,t,\varepsilon)-g(\tau)||_{L_r(\mathbf{R}_+)}dt \to 0 \quad (\varepsilon \to +0). \end{aligned} \tag{2.37}$$

But, on the other side, appealing to estimate (2.12), we see that K-L transform (1.1) is a bounded mapping in $L_{\nu,p}$ with $0 < \nu < 1, 1 \leqq p \leqq \infty$, because due to the inequality (2.13) the weight $\rho(\tau) = \tau^{\nu p-1}$ satisfies this condition. Thus there exists a limit in the sense of the $L_{\nu,p}$-norm

$$\operatorname*{l.i.m.}_{\varepsilon\to+0}\left(\mathfrak{KL}\left(I_\varepsilon g\right)\right)(\tau) = \left(\mathfrak{KL}\left[\operatorname*{l.i.m.}_{\varepsilon\to+0} I_\varepsilon g\right]\right)(\tau) = (\mathfrak{KL}f)(\tau), \tag{2.38}$$

where $f = Ig \in L_{\nu,p}$. Since the operator $(\mathfrak{KL}(I_\epsilon g))(\tau)$ converges in the norm L_r, too, then the limit functions must coincide almost everywhere on $\mathbf{R}_+$. Thus, from the equality (2.38) we obtain (2.28). ■

3. The Convolution for the Kontorovich-Lebedev Transform

We already defined the convolution operator (1.6) and called it the convolution for K-L transform (1.1). We will see below the direct connection of these objects. Now we start to study mapping properties of the convolution (1.6). First we observe from the definition that the convolution (1.6) is symmetrical (commutative)

$$f * g = g * f. \tag{3.1}$$

Second, if $f(x) > 0, g(x) > 0$ (or $f(x) < 0, g(x) < 0$) for $x \in \mathbf{R}_+$ then $(f * g)(x) > 0$, and for $f(x) > 0, g(x) < 0$, (or $f(x) < 0, g(x) > 0$) the inequality $(f * g)(x) < 0$ is justified. Now we obtain some estimates for the convolution (1.6) in the Lebesgue space L_p applying in all cases the Fubini theorem.

Theorem 3.1. *Let $f(x), g(x) \in L_{1/2,1}(\mathbf{R}_+)$. Then the convolution* (1.6) *exists and satisfies the estimate*

$$|(f * g)(x)| \leqq \frac{e^{-x}}{2\sqrt{2x}} ||f||_{1/2,1} ||g||_{1/2,1}. \tag{3.2}$$

Proof. Using the elementary inequalities

$$e^{-x} \leqq \frac{1}{1+x} \quad (x > 0), \tag{3.3}$$

$$a^2 + b^2 \geqq 2ab, \tag{3.4}$$

we have

$$\begin{aligned} |(f * g)(x)| &\leqq \frac{1}{2x} \int_0^\infty \int_0^\infty \frac{\exp(-x[u^2 + t^2]/2ut)}{1 + ut/2x} |f(t)g(u)| dtdu \\ &\leqq \frac{e^{-x}}{2\sqrt{2x}} \int_0^\infty \int_0^\infty \frac{|f(t)|}{\sqrt{t}} \frac{|g(u)|}{\sqrt{u}} dtdu \\ &= \frac{e^{-x}}{2\sqrt{2x}} ||f||_{1/2,1} ||g||_{1/2,1}. \end{aligned} \tag{3.5}$$

■

Theorem 3.2. *Let $f(x)$, $g(x) \in L_{0,1}(\mathbf{R}_+)$. Then the convolution* (1.6) *exists for each $x > 0$, and satisfies the estimate*

$$|(f * g)(x)| \leqq e^{-x} ||f||_{0,1} ||g||_{0,1}. \tag{3.6}$$

Proof. Similar application of (3.2) with the proof of Theorem 3.1 yields

$$\begin{aligned}|(f*g)(x)| &\leqq \int_0^\infty\int_0^\infty \exp\left(-x\frac{u^2+t^2}{2ut}\right)\frac{1}{2x+ut}|f(t)g(u)|dtdu \\ &\leqq e^{-x}\int_0^\infty \frac{|f(t)|}{t}dt\int_0^\infty\frac{|g(u)|}{u}du.\end{aligned}$$

■

Theorem 3.3. *Let $p \geqq 1, q \geqq 1$ and $1/p+1/q=1$. Let $f(x), g(x) \in L(\mathbf{R}_+;\rho_1(x))$, where $\rho_1(x) = x^{-1/2}\exp(-x/[2\min(p,q)])$. Then the convolution $(f*g)(x)$ exists for each $x>0$, and satisfies the estimate*

$$(3.7)\qquad \begin{aligned}|(f*g)(x)| \leqq& \sqrt{\frac{\max(p,q)}{8x}}\exp\left(-\frac{x}{2}\left[1+\frac{1}{\max(p,q)}\right]\right)\\ &\times\|f\|_{L(\mathbf{R}_+;\rho_1(x))}\,\|g\|_{L(\mathbf{R}_+;\rho_1(x)}\end{aligned}$$

Proof. According to the definition of the convolution, we have

$$(3.8)\qquad \begin{aligned}(f*g)(x) =& \frac{1}{2x}\int_0^\infty\int_0^\infty \exp\left(-\frac{1}{2p}\left[\frac{xt}{u}+\frac{xu}{t}+\frac{ut}{x}\right]\right)\\ &\times\exp\left(-\frac{1}{2q}\left[\frac{xt}{u}+\frac{xu}{t}+\frac{ut}{x}\right]\right)f(t)g(u)dtdu\\ =& \frac{1}{2x}\int_0^\infty\int_0^\infty \exp\left(-x\frac{u^2+t^2}{2ut\max(p,q)}-\frac{ut}{2x\max(p,q)}\right)\\ &\times\exp\left(-\frac{1}{2\min(p,q)}\left[\frac{xt}{u}+\frac{xu}{t}+\frac{ut}{x}\right]\right)f(t)g(u)dtdu.\end{aligned}$$

Using inequalities (3.3) - (3.4) and the inequality

$$(3.9)\qquad \frac{xu}{t}+\frac{xt}{u}+\frac{ut}{x} \geqq x+u+t,$$

we obtain the estimate

$$\begin{aligned}|(f*g)(x)| \leqq& \sqrt{\frac{\max(p,q)}{8x}}\exp\left(-\frac{x}{\max(p,q)}-\frac{x}{2\min(p,q)}\right)\\ &\times\int_0^\infty\frac{|f(x)|}{\sqrt{t}}\exp\left(-\frac{t}{2\min(p,q)}\right)dt\int_0^\infty\frac{|g(u)|}{\sqrt{u}}\exp\left(-\frac{u}{2\min(p,q)}\right)du,\end{aligned}$$

and hence the inequality (3.7). ■

Theorem 3.4. *Let $f(x), g(x) \in L(\mathbf{R}_+;e^{-x/2})$. Then the convolution (1.6) exists for each $x>0$, and satisfies the estimate*

$$(3.10)\qquad |(f*g)(x)| \leqq \frac{e^{-x}}{2x}\|f\|_{L(\mathbf{R}_+;e^{-x/2})}\|g\|_{L(\mathbf{R}_+;e^{-x/2})}.$$

Proof. The desired result follows from the inequality (3.9). As is evidently to seek from above estimates the convolution (1.6) is continuous for each $x \in \mathbf{R}_+$ and exponentially decreasing function near infinity. ■

Theorem 3.5. *Let the function $x^\alpha g(x)$ be bounded for $\Re\alpha < 1$ on $\mathbf{R}_+$, and $f(x) \in L_{\alpha,1}(\mathbf{R}_+)$. Then the convolution $(f * g)(x)$ exists for each $x > 0$, and satisfies the estimate*

$$|(f * g)(x)| \leqq \frac{M}{2^\alpha}\Gamma(1-\alpha)x^{-\alpha}e^{-x}||f||_{L_{\alpha,1}(\mathbf{R}_+)}, \tag{3.11}$$

where $M > 0$ is a constant.

Proof. Indeed,

$$\begin{aligned}
|(f * g)(x)| &\leqq \frac{1}{2x}\int_0^\infty\int_0^\infty \exp\left(-\frac{1}{2}\left[\frac{xu}{t}+\frac{ut}{x}+\frac{tx}{u}\right]\right)|f(t)u^\alpha g(u)|u^{-\alpha}dtdu \\
&\leqq \frac{M}{2x}\int_0^\infty |f(t)|dt\int_0^\infty \exp\left(-x\frac{u^2+t^2}{2ut}\right)\exp\left(-\frac{ut}{2x}\right)u^{-\alpha}du \\
&\leqq \frac{M\Gamma(1-\alpha)}{2x}e^{-x}\int_0^\infty |f(t)|\left(\frac{t}{2x}\right)^{\alpha-1}dt,
\end{aligned}$$

which leads to the estimate (3.11). ■

Theorem 3.6. *Let the function $x^\alpha g(x)$ be bounded for $\Re\alpha < 1$ on $\mathbf{R}_+$ and $f(x) \in L_2(\mathbf{R}_+)$. Then the convolution $(f * g)(x)$ exists and satisfies the estimate*

$$|(f * g)(x)| \leqq M_1 x^{\alpha-2}\exp\left(-\frac{x}{p}\right)||f||_{L_2(\mathbf{R}_+)}, \tag{3.12}$$

where $p > 1$ is an arbitrary number and $M_1 > 0$ is a constant.

Proof. Applying the Hölder inequality (1.4) to the convolution (1.6), we obtain the representation

$$\begin{aligned}
|(f * g)(x)| \leqq{}& \frac{1}{2x}\left(\int_0^\infty\left|\int_0^\infty \exp\left(-\frac{1}{2}\left[\frac{xu}{t}+\frac{ut}{x}+\frac{xt}{u}\right]\right)g(u)du\right|^2 dt\right)^{1/2} \\
&\times\left(\int_0^\infty |f(t)|^2dt\right)^{1/2}.
\end{aligned} \tag{3.13}$$

We first estimate the inner integral in (3.13) by denoting it by I_1. Taking the parameters $p > 1$ and $q > 1$ with $1/p + 1/q = 1$ and we represent this integral in the form

$$I_1 = \int_0^\infty \exp\left(-\frac{1}{2p}\left[\frac{xu}{t}+\frac{ut}{x}+\frac{xt}{u}\right]\right)\exp\left(-\frac{1}{2q}\left[\frac{xu}{t}+\frac{ut}{x}+\frac{xt}{u}\right]\right)g(u)du.$$

Further, using the inequality (3.4), we obtain the estimate

$$|I_1| \leqq \exp\left(-\frac{x}{p}\right)\exp\left(-\frac{t}{q}\right)\int_0^\infty \exp\left(-\frac{u}{2}\left[\frac{t}{px}+\frac{x}{qt}\right]\right)|g(u)|du. \tag{3.14}$$

According to the assumption, we get

$$(3.15)\quad \int_0^\infty \exp\left(-\frac{u}{2}\left[\frac{t}{px}+\frac{x}{qt}\right]\right)|g(u)|du \leqq C\int_0^\infty \exp\left(-\frac{u}{2}\left[\frac{t}{px}+\frac{x}{qt}\right]\right)u^{-\alpha}du$$

$$= C\Gamma(1-\alpha)2^{1-\alpha}\left(\frac{t}{px}+\frac{x}{qt}\right)^{\alpha-1}$$

$$= C_1\Gamma(1-\alpha)x^{1-\alpha}\left(\frac{t}{qt^2+px^2}\right)^{1-\alpha}$$

$$\leqq C_2\Gamma(1-\alpha)x^{\alpha-1}t^{1-\alpha},$$

where C, C_1 and C_2 are constants. Returning to the inequality (3.13) and using relations (3.14), (3.15), we get the form (3.12). ■

Most important result is contained in the following theorem, which gives an estimate of $L_{\nu,p}$-norm of the convolution (1.6).

Theorem 3.7. *Let $f(x), g(x) \in L_p(\mathbf{R}_+)$ $(1 \leqq p \leqq \infty)$. Then the convolution* (1.6) *exists, and belongs to $L_{\nu,q}(\mathbf{R}_+)$, $(q = p/(p-1), \nu > 1/p)$and there holds the estimate*

$$(3.16)\qquad ||(f*g)(x)||_{\nu,q} \leqq C||g||_p||f||_p,$$

where $C > 0$ is a constant.

Proof. Making use of the generalized Minkowski inequality (1.5), we have

$$(3.17)\ ||(f*g)(x)||_{\nu,q}$$

$$= \frac{1}{2}\left(\int_0^\infty x^{q(\nu-1)-1}\left|\int_0^\infty\int_0^\infty \exp\left(-\frac{1}{2}\left[\frac{xu}{t}+\frac{xt}{u}+\frac{ut}{x}\right]\right)f(u)g(t)dudt\right|^q dx\right)^{1/q}$$

$$\leqq \int_0^\infty\int_0^\infty |f(u)g(t)|\frac{1}{2}\left(\int_0^\infty x^{q(\nu-1)-1}\exp\left(-\frac{q}{2}\left[\frac{xu}{t}+\frac{xt}{u}+\frac{ut}{x}\right]\right)dx\right)^{1/q}dudt.$$

The integral by x can be calculated by using formula [10, Vol.1 (2.3.16.1)] and we obtain

$$(3.18)\qquad \frac{1}{2}\left(\int_0^\infty x^{q(\nu-1)-1}\exp\left(-\frac{q}{2}\left[\frac{xu}{t}+\frac{xt}{u}+\frac{ut}{x}\right]\right)dx\right)^{1/q}$$

$$= \left(\frac{ut}{\sqrt{u^2+t^2}}\right)^{\nu-1}K^{1/q}_{q(\nu-1)}\left(q\sqrt{u^2+t^2}\right).$$

Hence we have

$$(3.19)\quad ||(f*g)(x)||_{\nu,q} \leqq \int_0^\infty\int_0^\infty\left(\frac{ut}{\sqrt{u^2+t^2}}\right)^{\nu-1}K^{1/q}_{q(\nu-1)}\left(q\sqrt{u^2+t^2}\right)|f(u)g(t)|dudt.$$

By invoking to the analogue of Hölder inequality (1.4) for double integrals with the weight $\rho(u,t) \equiv 1$, the inequality (3.19) can be written as

$$(3.20)\ ||(f*g)(x)||_{\nu,q} \leqq \left\{\int_0^\infty\int_0^\infty\left(\frac{ut}{\sqrt{u^2+t^2}}\right)^{q(\nu-1)}K_{q(\nu-1)}\left(q\sqrt{u^2+t^2}\right)dudt\right\}^{1/q}$$

$$\times\left(\int_0^\infty |f(u)|^p du\right)^{1/p}\left(\int_0^\infty |g(t)|^p dt\right)^{1/p}$$

Let us consider the double integral in (3.20) with the Macdonald function. To show its convergence we apply polar coordinates $u = r\cos\varphi, t = r\sin\varphi$ $(r > 0, \varphi \in (0, \pi/2))$ and we find

$$\int_0^\infty \int_0^\infty \left(\frac{ut}{\sqrt{u^2+t^2}}\right)^{q(\nu-1)} K_{q(\nu-1)}\left(q\sqrt{u^2+t^2}\right) dudt \tag{3.21}$$
$$= 2^{q(1-\nu)} \int_0^{\pi/2} \sin^{q(\nu-1)} 2\varphi d\varphi \int_0^\infty r^{q(\nu-1)+1} K_{q(\nu-1)}(qr)dr.$$

If we account the asymptotic behavior of the Macdonald function $K_\mu(x) = O(x^{-|\Re\mu|})$ $(x \to +0)$ and $K_\mu(x) = O(e^{-x}/\sqrt{x})$ $(x \to \infty)$, then the integral by r is convergent for any ν, whereas the integral by φ is covergent only for $\nu > 1/p$. Hence denoting the integral (3.21) by C we arrive at the inequality (3.16). ■

It is natural to seek that the convolution (1.6) belongs to the conjugate space $L_q(\mathbf{R}_+)$ if we put in the inequality (3.16) $\nu = 1/q, q < p$ which means $1 \leqq q < 2, p \geqq 2$ or $1 \leqq q \leqq 2, p > 2$.

Corollary 3.1. *The convolution operator* (1.7) *with kernel* (1.8) *is bounded from the space $L_p(\mathbf{R}_+)$ $(p \geqq 1)$ into the space $L_{\nu,q}(\mathbf{R}_+)$ $(q = p/(p-1), \nu > 1/p)$, if the characteristic function $g(x)$ of the kernel belongs to $L_p(\mathbf{R}_+)$.*

Using the Hölder inequality let us estimate now the kernel $K(x,u)$ in (1.8) of the operator (1.7) provided that $g(x) \in L_p(\mathbf{R}_+)$. We have

$$|K(x,u)| \leqq \frac{1}{2x}\int_0^\infty \exp\left(-\frac{1}{2}\left[\frac{xu}{y}+\frac{xy}{u}+\frac{uy}{x}\right]\right)|g(y)|dy \tag{3.22}$$
$$\leqq \frac{1}{2x}\left(\int_0^\infty \exp\left(-\frac{q}{2}\left[\frac{xu}{y}+\frac{xy}{u}+\frac{uy}{x}\right]\right)dy\right)^{1/q} ||g||_p,$$

and recalling the formula (3.18), we obtain the estimate

$$|K(x,u)| \leq ||g||_p x^{-1/p}\left(\frac{u}{\sqrt{u^2+x^2}}\right)^{1/q} K_1^{1/q}\left(q\sqrt{u^2+x^2}\right), \tag{3.23}$$

where $K_1(z)$ is the Macdonald function of the order 1. As it can be found from (3.23) that the kernel $K(x,u)$ has an integrable singularity at the point $x = u$, $K(x,x) = O\left(x^{-1}\right)$ $(x \to +0)$.

As it was mentioned above, the convolution (1.6) is closely related to K-L transform (1.1). Now we establish this connection by means of the factorization property of the convolution (1.6) and an analogue of the Parseval equality.

Theorem 3.8. *Under conditions of Theorem 3.7 and $1/p < \nu < 1, p > 1$ K-L transform of the convolution $(f * g)(x)$ for $f(x)$ and $g(x)$ exists and is equal to the product of K-L transforms for these functions*

$$(\mathfrak{KL}[f*g])(\tau) = (\mathfrak{KL}f)(\tau)(\mathfrak{KL}g)(\tau). \tag{3.24}$$

Furthermore the Parseval type equality

$$(3.25)\qquad (f*g)(x)=\frac{2}{\pi^2}\int_0^\infty \tau\sinh(\pi\tau)\frac{K_{i\tau}(x)}{x}(\mathfrak{KL}f)(\tau)(\mathfrak{KL}g)(\tau)d\tau,$$

holds valid for any $x>0$ and the integral (3.25) is absolutely convergent.

Proof. The existence of K-L transform of the convolution (1.6) follows from the previous theorem, because according to (2.14) $K_{i\tau}[(f*g)]\in KL(L_{\nu,q})$. Hence applying K-L operator to the convolution (1.6), and we obtain the iterated integral
(3.26)

$$(\mathfrak{KL}[f*g])(\tau)=\frac{1}{2}\int_0^\infty \frac{K_{i\tau}(y)}{y}\int_0^\infty\int_0^\infty \exp\left(-\frac{1}{2}\left[\frac{yt}{u}+\frac{yu}{t}+\frac{ut}{y}\right]\right)f(u)g(t)dudtdy.$$

Changing the order of integrations, the inner integral in y is calculated by means of the known formula deduced from [10, Vol.2 (2.16.9.1)]

$$(3.27)\qquad K_\mu(u)K_\mu(t)=\frac{1}{2}\int_0^\infty \exp\left(-\frac{1}{2}\left[\frac{ut}{y}+\frac{yu}{t}+\frac{yt}{u}\right]\right)K_\mu(y)\frac{dy}{y}.$$

Such a change may guaranteed by the similar arguments with that for (2.3). Thus, motivating it for example by estimate (2.11) and Hölder inequality like (2.3), we use the Macdonald formula (3.27) and obtain (3.24). Furthermore, taking into account the integral representation [10, Vol.2 (2.16.56.1)]

$$(3.28)\qquad \exp\left(-\frac{1}{2}\left[\frac{xy}{u}+\frac{xu}{y}+\frac{yu}{x}\right]\right)=\frac{4}{\pi^2}\int_0^\infty \tau\sinh(\pi\tau)K_{i\tau}(x)K_{i\tau}(y)K_{i\tau}(u)d\tau$$

and substituting it in the formula (1.6), we change the order of integrations by the Fubini theorem by using the estimate

$$(3.29)\ \int_0^\infty \tau\sinh(\pi\tau)|K_{i\tau}(x)K_{i\tau}(y)K_{i\tau}(u)|d\tau$$
$$\leqq K_0(x\cos\delta_1)K_0(y\cos\delta_2)K_0(u\cos\delta_3)\int_0^\infty \tau\sinh(\pi\tau)e^{-\tau(\delta_1+\delta_2+\delta_3)}d\tau<+\infty,$$

which follows from (2.11) and where $\delta_i\in[0,\pi/2)$ $(i=1,2,3)$ can be choosen to converge the integral (3.29). Then applying the Hölder inequality like (2.12), we have

$$(3.30)\ |(f*g)(x)|\ \leqq\ \frac{2}{\pi^2}\frac{K_0(x\cos\delta_1)}{x}\int_0^\infty \tau\sinh(\pi\tau)e^{-\tau(\delta_1+\delta_2+\delta_3)}d\tau$$
$$\times||f||_p||g||_p\left(\int_0^\infty K_0^q(y\cos\delta_2)dy\right)^{1/q}\left(\int_0^\infty K_0^q(u\cos\delta_3)du\right)^{1/q}.$$

Hence we verify that it can be changed the order of integration and thus we establish the Parseval equality (3.25). ■

Now we estimate the convolution (1.6) in the special weighted space $L^\alpha\equiv L(\mathbf{R}_+;K_\alpha(x))$ $(\alpha\geqq 0)$ by putting $\rho(t)=K_\alpha(t)$ $(p=1)$ in (1.3) and slightly modifying results from [12],

[16], where $K_\alpha(t)$ is the Macdonald function of index α. We show that this space of absolutely integrable functions on $\mathbf{R}_+$ with forms a normed ring or a Banach algebra with the norm

$$\|f\|_{L^\alpha} = \int_0^\infty K_\alpha(t)|f(t)|dt < +\infty. \tag{3.31}$$

We draw parallel results here with [16]. First from the asymptotic of the Macdonald function we observe the embedding

$$L^{\alpha_1} \subseteq L^{\alpha_2}, \text{ iff } \alpha_1 \geqq \alpha_2. \tag{3.32}$$

Theorem 3.9. *Let $f(x), g(x) \in L^\alpha$. Then the convolution* (1.6) *exists and belongs to the class L^α and there holds*

$$\|f * g\|_{L^\alpha} \leqq \|f\|_{L^\alpha}\|g\|_{L^\alpha}. \tag{3.33}$$

Proof. By definitions of the norm L^α and of the convolution, and performing to change the order of integrations by the Fubini theorem and using the Macdonald formula (3.27), we obtain

$$\begin{aligned} \|f * g\|_{L^\alpha} &= \int_0^\infty K_\alpha(x)|(f * g)(x)|dx \\ &\leqq \frac{1}{2}\int_0^\infty \frac{K_\alpha(x)}{x}\int_0^\infty \int_0^\infty \exp\left(-\frac{1}{2}\left[\frac{xy}{u}+\frac{uy}{x}+\frac{xu}{y}\right]\right)|f(u)g(y)|dudydx \\ &= \frac{1}{2}\int_0^\infty \int_0^\infty |f(u)g(y)|\int_0^\infty K_\alpha(x)\exp\left(-\frac{1}{2}\left[\frac{xy}{u}+\frac{uy}{x}+\frac{xu}{y}\right]\right)\frac{dx}{x}dudy \\ &= \int_0^\infty K_\alpha(u)|f(u)|du\int_0^\infty K_\alpha(y)|g(y)|dy = \|f\|_{L^\alpha}\|g\|_{L^\alpha}. \end{aligned} \tag{3.34}$$

■

This property of integrability of the convolution (1.6) with positive weight $K_\alpha(x)$ shows that it takes finite values for almost all $x > 0$.

Theorem 3.10. *Let $f(x), g(x) \in L^\alpha$. Then K-L transform* (1.1) *of the convolution $(f * g)(x)$ exists and is equal to the product of K-L transforms of $f(x)$ and $g(x)$, that is, there holds the formula* (3.24).

Proof. The existence of K-L transform of the convolution follows from previous theorem and we have the estimate

$$\begin{aligned} |K_{i\tau}[f * g]| &\leqq \int_0^\infty K_0(y)|(f * g)(y)|dy \\ &= \int_0^\infty K_0(u)|f(u)|du\int_0^\infty K_0(y)|g(y)|dy < \infty \end{aligned} \tag{3.35}$$

due to the embedding (3.32). This allows us to change the order of integrations in the respective iterated integral and to use the Macdonald formula (3.27). ■

Let us now consider a subspace of L^α, which is $L^\alpha_\beta \equiv L_1(\mathbf{R}_+; K_\alpha(\beta x))$ $(\alpha \geqq 0, 0 < \beta \leqq 1)$. The embedding

$$L^\alpha_\beta \subseteq L^\alpha$$

follows from the asymptotic of the Macdonald function and $L^\alpha_1 \equiv L^\alpha$.

Theorem 3.11. *Let $f(x), g(x) \in L^\alpha_{\cos\delta}$ $(\pi/3 < \delta < \pi/2)$. Then the Parseval equality (3.25) holds true.*

Proof. Indeed by using (3.28) and (2.11) we have the estimate

$$(3.36) \qquad |(f * g)(x)| \leqq \frac{2}{\pi^2}\frac{K_0(x\cos\delta)}{x}\int_0^\infty \tau\sinh(\pi\tau)e^{-3\tau\delta}d\tau \times \int_0^\infty K_0(u\cos\delta)|f(u)|du \int_0^\infty K_0(y\cos\delta)|g(y)|dy,$$

in which integrals by u and y are finite, because the same embedding with (3.32) for the set of spaces L^α_β is true when β is a fixed number. According to the condition on δ, the integral by τ is convergent. Hence changing the order of integrations and invoking to (1.1) we arrive at the Parseval equality (3.25). ■

Let us note now that in the Macdonald formula (3.27) the range of the parameter μ is the complex number field. Thus K-L transform (1.1) for $f \in L^\alpha$ may be defined with the index $\mu = \Re\mu + i\tau$ from the strip $|\Re\mu| \leqq \alpha$:

$$(3.37) \qquad (\mathfrak{KL}f)(\mu) = \int_0^\infty K_{\Re\mu+i\tau}(y)f(y)dy$$

and the equation (3.24) can be extended for such transform due to the assumption, the behavior of the Macdonald function and the absolutely convergency of corresponding integrals.

Considering the question of the one-to-one correspondence of the function between the space L^α and its K-L transform $K_\mu[f]$ of complex index μ, we obtain the following theorem.

Theorem 3.12. *If K-L transform $(\mathfrak{KL}f)(\mu)$ of the function $f(x) \in L^\alpha$ $(|\Re\mu| \leqq \alpha)$ is identically zero, then $f(x)$ is equal to zero almost everywhere on $\mathbf{R}_+$.*

Proof. From the formula (1.2) we have the representation of the Macdonald function is true

$$(3.38) \qquad K_\mu(x) = \frac{1}{2}\int_{-\infty}^\infty e^{-x\cosh u - \mu u}du.$$

The assumption $f(x) \in L(\mathbf{R}_+; K_\alpha(x))$ leads to the estimate

$$(3.39) \qquad \left|\int_0^\infty K_\mu(t)f(t)dt\right| = \frac{1}{2}\left|\int_0^\infty f(t)dt\int_{-\infty}^\infty e^{-t\cosh u - \mu u}du\right| \leqq \int_0^\infty |f(t)|K_{\Re\mu}(t)dt < +\infty$$

for $|\Re\mu| \leqq \alpha$, by noting the asymptotic behavior of the Macdonald function near the point zero. Thus using the Fubini theorem, we have the composition representation

$$(3.40) \qquad (\mathfrak{KL}f)(\mu) = \sqrt{\frac{\pi}{2}}\mathfrak{F}\left\{e^{-(\Re\mu)u}\mathfrak{L}\{f(x); \cosh u\}; \tau\right\},$$

where

$$(3.41)\qquad \mathfrak{F}\{f(u);\tau\} = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(u)e^{-i\tau u}du.$$

is the classical Fourier transform and

$$(3.42)\qquad \mathfrak{L}\{f(x);\cosh u\} = \int_0^{\infty} f(x)e^{-x\cosh u}dx.$$

is the Laplace transform calculated at $\cosh u$. From (3.38) and the estimate

$$(3.43)\qquad \int_{-\infty}^{\infty} e^{-(\Re\mu)u}\left|\int_0^{\infty} f(t)e^{-t\cosh u}dt\right| du \leqq 2\int_0^{\infty} |f(t)|K_{\Re\mu}(t)dt < \infty,$$

for $|\Re\mu| \leqq \alpha$, we find that $e^{-(\Re\mu)u}\mathfrak{L}\{f(x);\cosh u\} \in L_1(\mathbf{R})$. Thus K-L transform $(\mathfrak{KL}f)(\mu)$ is an analytic function on the strip $|\Re\mu| < \alpha$. Moreover, according to the well-known property of the Fourier transform of absolutely integrable functions [11], it follows from equality

$$(3.44)\qquad (\mathfrak{KL}f)(\mu) \equiv 0 \quad (\mu = \Re\mu + i\tau)$$

that

$$(3.45)\qquad \int_0^{\infty} e^{-x\cosh u}f(x)dx = 0$$

for almost all $u \in \mathbf{R}$. It is easy to note from properties of the space $L(\mathbf{R}_+; K_\alpha(x))$, that the integral in (3.45) converges absolutely and uniformly for $u \geqq u_0 > 0$, then it defines a continuous function. Hence, it follows that equality (3.45) is an identity for $u \geq u_0 > 0$. Further, that there is an $\varepsilon > 0$ such that $v = \cosh u - 1 - \varepsilon > 0$ implies that the equality (3.45) takes the form

$$(3.46)\qquad \int_0^{\infty} e^{-vx}f(x)e^{-(1+\varepsilon)x}dx \equiv 0.$$

This relation and the assumption of the theorem yields the Laplace transform of the absolutely integrable function $f(x)e^{-(1+\varepsilon)x}$ is identically zero on some closed interval $0 < a \leqq v \leqq b$ and, moreover, on the right half-plane $\Re(z) \geqq a$. Then from the uniqueness theorem for analytic functions we have

$$(3.47)\qquad \int_0^{\infty} e^{-zx}f(x)e^{-(1+\varepsilon)x}dx \equiv 0 \quad (\Re(z) \geqq a).$$

Using the inverse theorem for the Laplace transform [11] we obtain

$$(3.48)\qquad \int_0^{x} f(x)e^{-(1+\varepsilon)x}dx \equiv 0.$$

Thus from the properties of primaries of summable functions, $f(x) = 0$ for almost every $x \in \mathbf{R}_+$. ∎

For our applications of the convolution operator (1.6) to integral equations we need to consider the space L^α in view of the theory of commutative normed rings [8]. Here we repeat some results from [16] for the modified convolution (1.6). Our purpose to prove an analogue of the Wiener theorem on the existence of an inverse element of the given normed ring. Obviously, the space L^α is isometric to $L_1(\mathbf{R}_+)$ and is a Banach one with the norm (3.31). Due to Theorem 3.9, we can define an operation of multiplication for elements $f(x)$ and $g(x)$ in the form of the convolution (1.6) in the space L^α. According to the definition of the convolution, this operation of multiplication is commutative in the class L^α (see (3.1)). Using the Fubini theorem, we can establish its associativity and distributivity

$$(f*(g*h))(x) = ((f*g)*h)(x), \tag{3.49}$$

$$(f*(g+h))(x) = (f*g)(x) + (f*h)(x) \tag{3.50}$$

for $f, g, h \in L^\alpha$. Thus, the space L^α forms a commutative Banach ring with the operation of multiplication in the form of the convolution (1.6). We note some properties of the ring L^α.

Theorem 3.13. *The ring L^α does not contain the unit with respective to the operation of convolution* (1.6).

Proof. We first show that the convolution $(f*g)(x)$ of a bounded function $g(x)$ with $f(x) \in L^\alpha$ is a continuous function for $x \geqq x_0 > 0$. Indeed, let $|g(x)| \leqq C$, where $C > 0$ is a constant and $f(x) \in L^\alpha$. Then we have

$$|(f*g)(x)| \leqq \frac{C}{2x}\int_0^\infty\int_0^\infty \exp\left(-\frac{1}{2}\left[\frac{xy}{u}+\frac{xu}{y}+\frac{uy}{x}\right]\right)|f(y)|dydu. \tag{3.51}$$

Calculating the integral by u in (3.51) by using (3.18), we obtain

$$|(f*g)(x)| \leqq C\int_0^\infty \frac{yK_1\left(\sqrt{x^2+y^2}\right)}{\sqrt{x^2+y^2}}|f(y)|dy, \tag{3.52}$$

where the integral converges absolutely and uniformly for each function $f(x) \in L^\alpha$ $(\alpha \geqq 0)$ and for $x \geqq x_0 > 0$. In fact, since $f(x) \in L^\alpha$, then due to the property of embedding (3.32) $f(x)$ is an element of the space L^0, and by its definition we get

$$\int_0^\infty \frac{yK_1\left(\sqrt{x^2+y^2}\right)}{\sqrt{x^2+y^2}}|f(y)|dy \leqq \int_0^\infty \frac{K_1\left(\sqrt{x_0^2+y^2}\right)}{K_0(y)}K_0(y)|f(y)|dy \leqq C_1\|f\|_{L^0}, \tag{3.53}$$

with a constant C_1 depending on x_0. Hence, $(f*g)(x)$ is a continuous function for $x \geqq x_0 > 0$. If the ring L^α contains the unit for the convolution (1.6), then each bounded function from L^α must coincide almost everywhere with some function continuous in $x \geqq x_0 > 0$ as a function which is obtained as its convolution with the unit. But it is evident that the Lebesgue space L^α contains bounded discontinuous functions which differ from the functions being continuous for $x \geq x_0 > 0$ on a set of positive measure. The function which is equal to 1 on the interval (a, b) and equal to zero outside of (a, b) with $x_0 < a$ is a such simple example. This contradiction shows that the class L^α does not contain the unit. ■

Let us denote by $V^\alpha \equiv V(\mathbf{R}_+; K_\alpha(x))$ the commutative ring, obtained by means of formal addition of a unit to L^α. Thus, V^α consists of elements $\xi = \lambda e + f(t)$, where e is the unit, λ is an arbitrary complex number, and $f(t)$ is any element from L^α. We introduce the norm in V^α

$$\|\xi\|_{V^\alpha} = |\lambda| + \|f\|_{L^\alpha}. \tag{3.54}$$

Now we need some preliminary information from the theory of ideals of the rings [8].

Definition 1. A subset I_l of a ring R is called a *left ideal,* if

1. $I_l \neq R$;
2. $x + y \in I_l$ for $x, y \in I_l$;
3. $z \cdot x \in I_l$ for $x \in I_l, z \in R$.

The *right ideal* is defined analogously.

Definition 2. A subset I of a ring R is called a *bilateral ideal* or *ideal in R,* if I is the left and right ideal, simultaneously.

Definition 3. A bilateral ideal is called a *maximal ideal,* if it is not contained in any other bilateral ideal of the ring R.

Our problem is now to find all maximal ideals of the ring V^α. Directly from Definitions 1-3 it follows that the ring L^α is a certain maximal ideal in the ring V^α. For each $\xi = \lambda e + f(x) \in V^\alpha$ let us set

$$(\mathcal{F}\xi)(\mu) = \lambda + (\mathfrak{K}\mathfrak{L}f)(\mu), \tag{3.55}$$

where $\lambda, \mu \in \mathbf{C}$, $(\mathfrak{K}\mathfrak{L}f)(\mu)$ is K-L transform (3.37). By using Theorem 3.10 we can show that the mapping $\xi \to (\mathcal{F}\xi)(\mu)$ is a homomorphism of the ring V^α onto $\mathbf{C}$. As it is known in [8] that the maximal ideal M_μ generated by this mapping contains the set of elements $\xi = \lambda e + f(x)$ of the ring V^α such that $(\mathcal{F}\xi)(\mu) = 0$.

The following theorem states on all maximal ideals of the ring V^α.

Theorem 3.14. *There does not exist any maximal ideal in the ring V^α except for L^α and M_μ with $|\Re\mu| \leqq \alpha$. Moreover, two ideals M_{μ_1} and M_{μ_2} coincide iff $\mu_1 = \pm\mu_2$.*

Proof. Let M be a maximal ideal in V^α which differs from L^α. Then we correspond by $F(f)$ some linear functional on V^α. But we can regard it as a linear functional in the space of summable functions with the weight $K_\alpha(x)$ $(\alpha \geqq 0)$. Therefore from the Hopf lemma we have

$$F(f) = \int_0^\infty f(x)\omega(x)dx, \tag{3.56}$$

where $\omega(x)/K_\alpha(x)$ is an essentially bounded function or it belongs to L_∞ with $\omega(x) \not\equiv 0$. Since the mapping $f \to F(f)$ is a homomorphism and $F(f_1 \cdot f_2) = F(f_1)F(f_2)$, where the operation of multiplication in the ring is the convolution (1.6). That is, we can rewrite this property as follows

$$
\begin{aligned}
(3.57)\quad & \int_0^\infty f_1(x)\omega(x)dx \int_0^\infty f_2(y)\omega(y)dy \\
&= \frac{1}{2}\int_0^\infty \frac{\omega(u)}{u}\int_0^\infty\int_0^\infty \exp\left(-\frac{1}{2}\left[\frac{xy}{u}+\frac{xu}{y}+\frac{uy}{x}\right]\right) f_1(x)f_2(y)dxdydu \\
&= \int_0^\infty\int_0^\infty f_1(x)f_2(y)\int_0^\infty \exp\left(-\frac{1}{2}\left[\frac{xy}{u}+\frac{xu}{y}+\frac{uy}{x}\right]\right)\frac{\omega(u)}{2u}dudxdy.
\end{aligned}
$$

Thus we conclude that the function $\omega(x)$ must satisfy the functional equation

$$
(3.58)\qquad \omega(x)\omega(y) = \frac{1}{2}\int_0^\infty \exp\left(-\frac{1}{2}\left[\frac{xy}{u}+\frac{xu}{y}+\frac{uy}{x}\right]\right)\frac{\omega(u)}{u}du
$$

for almost all positive x and y. As it follows from the Macdonald formula (3.27), the function $\omega(x) = K_\mu(x)$ satisfies the equation (3.58), where μ is some complex number. Here, we have the reverse of this fact, which is stated as a lemma.

Lemma 1. *Let the function $\omega(x)/K_\alpha(x)$ be essentially bounded for $x > 0$ and let $\omega(x) \not\equiv 0$. Then the solution of the functional equation (3.58) is the Macdonald function $K_\mu(x)$ with $|\Re\mu| \leqq \alpha$.*

Proof. First we show that the integral

$$
(3.59)\qquad I(x,y) \equiv \int_0^\infty \exp\left(-\frac{1}{2}\left(\frac{xy}{u}+\frac{xu}{y}+\frac{uy}{x}\right)\right)\frac{\omega(u)}{u^\gamma}du \quad (\gamma > 1)
$$

converges uniformly in the region $\{x \geqq x_0 > 0, y \geqq y_0 > 0\}$ for any $x_0 > 0, y_0 > 0$. Indeed,

$$
\begin{aligned}
(3.60)\qquad |I(x,y)| &\leqq \int_0^\infty \exp\left(-\frac{1}{2}\left[\frac{xy}{u}+\frac{xu}{y}+\frac{uy}{x}\right]\right)\frac{K_\alpha(u)}{u^\gamma}du \left\|\frac{\omega(u)}{K_\alpha(u)}\right\|_{L_\infty} \\
&\leqq \left[\int_0^1 \exp\left(-\frac{x_0y_0}{2u}\right)\frac{K_\alpha(u)}{u^\gamma}du\right. \\
&\quad \left. +\exp\left(-\frac{x_0y_0}{2}\right)\int_1^\infty \exp(-u)\frac{K_\alpha(u)}{u^\gamma}du\right]\left\|\frac{\omega(u)}{K_\alpha(u)}\right\|_{L_\infty} \\
&\leqq C,
\end{aligned}
$$

where $C > 0$ is a constant. Then $I(x,y)$ can be differentiated with respect to parameters $x \geqq x_0 > 0$ and $y \geqq y_0 > 0$. We have from (3.58)

$$
\begin{aligned}
(3.61)\qquad \omega'(x)\omega(y) &= \frac{1}{4}\int_0^\infty \left(-\frac{y}{u}-\frac{u}{y}+\frac{uy}{x^2}\right)\exp\left(-\frac{1}{2}\left[\frac{xy}{u}+\frac{xu}{y}+\frac{uy}{x}\right]\right)\frac{\omega(u)}{u}du \\
&= \left(\frac{y}{4x^2}-\frac{1}{4y}\right)I_1(x,y) - \frac{y}{4}I_2(x,y),
\end{aligned}
$$

where we set

$$(3.62)\qquad \begin{cases} I_1(x,y) = \displaystyle\int_0^\infty \exp\left(-\frac{1}{2}\left[\frac{xy}{u}+\frac{xu}{y}+\frac{uy}{x}\right]\right)\omega(u)du; \\ I_2(x,y) = \displaystyle\int_0^\infty \exp\left(-\frac{1}{2}\left[\frac{xy}{u}+\frac{xu}{y}+\frac{uy}{x}\right]\right)\frac{\omega(u)}{u^2}du. \end{cases}$$

In a similar manner we have

$$(3.63)\qquad \omega(x)\omega'(y) = \left(\frac{x}{4y^2}-\frac{1}{4x}\right)I_1(x,y) - \frac{x}{4}I_2(x,y).$$

From these equalities it follows that

$$(3.64)\qquad y\omega'(y)\frac{\omega(x)}{x} = \left(\frac{1}{4y}-\frac{y}{4x^2}\right)I_1(x,y) - \frac{y}{4}I_2(x,y),$$

$$(3.65)\qquad x\omega'(x)\frac{\omega(y)}{y} = \left(\frac{1}{4x}-\frac{x}{4y^2}\right)I_1(x,y) - \frac{x}{4}I_2(x,y).$$

Differentiating left and right sides of (3.64) with respect to y and (3.65) with respect to x and subtracting them, we obtain the differential equation for $\omega(x)$

$$(3.66)\qquad (y\omega'(y))'\frac{\omega(x)}{x} - (x\omega'(x))'\frac{\omega(y)}{y} = \left(\frac{y}{x}-\frac{x}{y}\right)\omega(x)\omega(y),$$

which leads to

$$(3.67)\qquad \frac{y\,(y\omega'(y))'}{\omega(y)} - y^2 = \frac{x\,(x\omega'(x))'}{\omega(x)} - x^2 = \mu^2.$$

since the variables x and y are arbitrary. But both sides in the chain of equalities (3.67) are Bessel equation [1]

$$\omega''(y) + \frac{1}{y}\omega'(y) - \left(1+\frac{\mu^2}{y^2}\right)\omega(y) = 0$$

with the solution being the Macdonald function $K_\mu(y)$ for which $|\Re\mu| \leqq \alpha$ due to the assumptions. ∎

Proof of Theorem 3.14 (continued) Now we prove the last result in Theorem 3.14. Since $\omega(x) = K_\mu(x)$, the maximal ideal M generated by homomorphism (3.55) coincides with M_μ due to the formula (3.56), where the parameter μ is taken in the strip $|\Re\mu| \leqq \alpha$. By the evenness of the Macdonald function $K_\mu(x)$ with respect to the index two maximal ideals corresponding to the numbers μ_1 and μ_2 coincide if and only if $\mu_1 = \pm\mu_2$. Thus Theorem 3.14 is completely proved. ∎

The mentioned properties of the ring L^α allow us to obtain the following analogue of the Wiener theorem.

Theorem 3.15. *If the function $\mathcal{F}(\mu)$ defined by the relation* (3.55) *does not vanish nowhere in the closed strip $|\Re\mu| \leqq \alpha$ including infinity, then there is a unique element $q(x)$ in the ring L^α such that*

$$\frac{1}{\mathcal{F}(\mu)} = \lambda + K_\mu[q]. \tag{3.68}$$

Proof. Since $\mathcal{F}(\mu)$ does not vanish nowhere in the strip $|\Re\mu| \leqq \alpha$, then $f(x)$ does not belong to any maximal ideal and such a set is completely exhausted by Theorem 3.14. As it is known from [8], such an element $f(x)$ has a unique inverse $q(x)$ in the ring V^α, because the mapping $\mathcal{F}(\mu)$ is a homomorphism. Thus we obtain the equality (3.68).■

Finally we establish the analogue of the Titchmarsh theorem [11] on the absence of the divisors of zero for the convolution (1.6).

Theorem 3.16. *Let the functions $f(x), g(x)$ be from the ring L^α and $(f * g)(x) \equiv 0, x > 0$. Then at least one of the functions $f(x)$ and $g(x)$ is equal to zero almost everywhere on $\mathbf{R}_+$.*

Proof. As already noted, the equality (3.24) can be extended for K-L transform (3.37) of the index μ in the strip $|\Re\mu| \leqq \alpha$

$$(\mathfrak{KL}(f * g))(\mu) = (\mathfrak{KL}f)(\mu)(\mathfrak{KL}g)(\mu). \tag{3.69}$$

So, we find that the right side of equality (3.69) is equal to zero. Since the functions $(\mathfrak{KL}f)(\mu), (\mathfrak{KL}g)(\mu)$ are analytic as functions of the complex variable μ in the strip $|\Re\mu| < \alpha$, at least one of them is identically zero and Theorem 3.12 leads to that $f(x)$ or $g(x)$ is equal to zero almost everywhere on $\mathbf{R}_+$. ■

4. Convolution Hilbert spaces

Now we return to study mapping properties of the convolution operator (1.6) and weighted space L_β^α being define as so-called convolution Hilbert space by means of completion of pre-Hilbert space with the inner product as the convolution (1.6). The most important results for this purpose are respective Theorems 3.9 and 3.10 for subspace L_β^α. Let us consider at first the mapping properties of the convolution (1.6) at the subspace L_β^α with $0 < \beta < 1$.

Theorem 4.1. *Let $f(x), g(x) \in L_\beta^\alpha$, $(0 < \beta \leqq 1, \alpha \geqq 0)$. Then the convolution* (1.6) *exists and belongs to the space L_β^α and satisfies the estimation*

$$||f * g||_{L_\beta^\alpha} \leqq C_\beta ||f||_{L_\beta^\alpha} ||g||_{L_\beta^\alpha}, \tag{4.1}$$

where C_β is a positive constant depending only on β.

Proof. By the definition of the norm in the space L_β^α we have

$$\begin{aligned} (4.2)\ ||f * g||_{L_\beta^\alpha} &= \int_0^\infty K_\alpha(\beta x)|(f * g)(x)|dx \\ &\leqq \frac{1}{2}\int_0^\infty \frac{K_\alpha(\beta x)}{x}\int_0^\infty\int_0^\infty \exp\left(-\frac{1}{2}\left[\frac{xv}{u} + \frac{uv}{x} + \frac{xu}{v}\right]\right)|f(u)g(v)|dudvdx. \end{aligned}$$

By interchanging the oreder of integrations, the integral by x corresponds to formula [10, Vol.2, (2.16.9.1)] which gives

$$(4.3)\qquad \frac{1}{2}\int_0^\infty \frac{K_\alpha(\beta x)}{x}\exp\left(-\frac{1}{2}\left[\frac{xv}{u}+\frac{uv}{x}+\frac{xu}{v}\right]\right)dx$$
$$= K_\alpha\left(\frac{1}{2}\left[\sqrt{u^2+v^2+2uv\beta}+\sqrt{u^2+v^2-2uv\beta}\right]\right)$$
$$\times K_\alpha\left(\frac{1}{2}\left[\sqrt{u^2+v^2+2uv\beta}-\sqrt{u^2+v^2-2uv\beta}\right]\right).$$

Thus we have the norm estimate of the convolution (1.6)

$$(4.4)\ ||f*g||_{L^\alpha_\beta} \leqq \int_0^\infty\int_0^\infty K_\alpha\left(\frac{1}{2}\left[\sqrt{u^2+v^2+2uv\beta}+\sqrt{u^2+v^2-2uv\beta}\right]\right)$$
$$\times K_\alpha\left(\frac{1}{2}\left[\sqrt{u^2+v^2+2uv\beta}-\sqrt{u^2+v^2-2uv\beta}\right]\right)|f(u)g(v)|dudv.$$

Let us now establish the uniform boundedness of the function of two variables

$$(4.5)\qquad F(u,v)=\frac{K_\alpha\left(\left[\sqrt{u^2+v^2+2uv\beta}+\sqrt{u^2+v^2-2uv\beta}\right]/2\right)}{K_\alpha(\beta u)}$$
$$\times\frac{K_\alpha\left(\left[\sqrt{u^2+v^2+2uv\beta}-\sqrt{u^2+v^2-2uv\beta}\right]/2\right)}{K_\alpha(\beta v)}$$
$$(0<\beta\leqq 1, u, v>0)$$

From the asymptotic behavior of the Macdonald function (2.4) it is easy to see that $F(u,v)<C_\beta$ for $0<u,v<\infty$. Further

$$(4.6)\qquad F(u,v) = O\left(\exp\left(\beta(u+v)-\sqrt{u^2+v^2+2uv\beta}\right)\right)$$
$$\leqq C_\beta\exp\left((\beta-\sqrt{\beta})(u+v)\right) = O(1)\quad (u+v\to\infty).$$

Under these circumstances we can change the order of integrations in (4.2) and get

$$(4.7)\qquad ||f*g||_{L^\alpha_\beta}\leqq \int_0^\infty K_\alpha(\beta u)|f(u)|du\int_0^\infty K_\alpha(\beta v)|g(v)|dv = ||f||_{L^\alpha_\beta}||g||_{L^\alpha_\beta}$$

■

For such subspaces L^α_β the embedding of type

$$(4.8)\qquad L^\alpha_{\beta_1}\subseteq L^\alpha_{\beta_2}\quad (\beta_1\leqq\beta_2)$$

is true, and for the convolution (1.6) the previous theorem gives the validity of the equality (3.24).

In order to introduce the convolution Hilbert space it is more suitable to consider the space L^{α}_{β}. Let $\omega(x)$ $(x \in \mathbf{R}_+)$ be an arbitrary positive function satisfying the conditions

$$\omega(x) \in L_1\left((0,1); \frac{\log x}{x}\right), \quad \omega(x) \in L_1\left((1,\infty); e^{-\beta x}\right) (0 < \beta \leqq 1). \tag{4.9}$$

Then K-L transform (1.1) of the function $\omega(x)/x$ exists as it is seen from the asymptotic behavior of the Macdonald function and the inequality (2.11):

$$\begin{aligned} \left|\mathfrak{KL}\left[\frac{\omega(x)}{x}\right]\right| &\leqq \int_0^\infty |K_{i\tau}(y)|\frac{\omega(y)}{y}dy \\ &\leqq e^{-\delta\tau}\left(\int_0^1 K_0(y\cos\delta)\frac{\omega(y)}{y}dy + \int_1^\infty K_0(y\cos\delta)\frac{\omega(y)}{y}dy\right) \\ &\leqq e^{-\delta\tau}\left(C_1\int_0^1 \log y\frac{\omega(y)}{y}dy + C_2\int_1^\infty e^{-y\cos\delta}\omega(y)dy\right) < \infty, \end{aligned} \tag{4.10}$$

where $\delta \in [0,\pi/2)$ and we can put $\cos\delta = \beta$. In our further considerations we need to impose some additional conditions on the function $\omega(x)$ for the positiveness of K-L transform $(\mathfrak{KL}[\omega(x)/x])(\tau)$ for $\tau \geqq 0$. Of course, from the representation of the Macdonald function $K_{i\tau}(x)$ through the integral (1.2) (letting there for instance $\delta = 0$) it follows that this function is real. Further we have the composition representation like (3.40) under conditions (4.9):

$$\left(\mathfrak{KL}\left[\frac{\omega(x)}{x}\right]\right)(\tau) = \sqrt{\frac{\pi}{2}}\mathfrak{F}_c\left\{\mathfrak{L}\left\{\frac{\omega(x)}{x};\cosh u\right\};\tau\right\}, \tag{4.11}$$

where we mean $\mathfrak{F}_c\{f(u);\tau\}$ the cosine-Fourier transform

$$\mathfrak{F}_c\{f(u);\tau\} = \sqrt{\frac{2}{\pi}}\int_0^\infty f(u)\cos(\tau u)du. \tag{4.12}$$

Let us recall now to a useful result from [11, Theorem 124] to get a sufficient condition for the positiveness of the composition (4.11). By the assumptions (4.9) the Laplace transform (3.42) of the function $\omega(x)/x$ is bounded function on $\mathbf{R}_+$ depending on the variable $\cosh u$ $(u \geqq 0)$ and steadily decreases to zero as u diverges to infinity. Indeed, for $u_1 > u_2$ we obtain

$$\begin{aligned} \mathfrak{L}\left\{\frac{\omega(x)}{x};\cosh u_1\right\} &= \int_0^\infty \frac{\omega(y)}{y}e^{-y\cosh u_1}dy \\ &\leqq \int_0^\infty \frac{\omega(y)}{y}e^{-y\cosh u_2}dy \\ &\leqq C_1\int_0^1 \log y\frac{\omega(y)}{y}dy + C_2\int_1^\infty e^{-y\beta}\omega(y)dy < \infty, \end{aligned} \tag{4.13}$$

and this Laplace transform tends to 0 as $u \to +\infty$ by virtue of the Lebesgue theorem. Moreover we can differentiate this integral with respect to u under the integral and we have

$$\frac{d}{du}\mathfrak{L}\left\{\frac{\omega(x)}{x};\cosh u\right\} = -\sinh u\int_0^\infty \omega(y)e^{-y\cosh u}dy. \tag{4.14}$$

Theorem 4.2. *Let assume the condition (4.9) for the function $\omega(x)$, and let the integral in right side of the equality (4.14) be positive and non-increasing and tend to a limit at infinity. Then K-L transform (4.11) is positive function for all $\tau \geqq 0$.*

Proof. The proof follows from [11, Theorem 124], because under above assumptions the Laplace transform $\mathfrak{L}\{\omega(x)/x; \cosh u\}$ is bounded function, which decreases steadily to zero at infinity. Thus we conclude that the Laplace transform $\mathfrak{L}\{\omega(x)/x; \cosh u\}$ is a convex downwards function of the variable u and therefore the composition (4.11) with the cosine-Fourier transform is positive for $\tau \geqq 0$. ■

Let us give several concrete examples of function $\omega(x)$ and corresponding K-L transforms (1.1). Evidently the function $\omega(x) \equiv x$ satisfies conditions (4.9). Invoking to the formula [10, Vol.2, (2.16.2.1)], we obtain the expression for K-L transform

$$(4.15) \qquad \left(\mathfrak{K}\mathfrak{L}1\right)(\tau) = \int_0^\infty K_{i\tau}(y)dy = \frac{\pi}{2\cosh(\pi\tau/2)}.$$

Letting $\omega(x) \equiv x^\gamma$ $(\gamma > 0)$, we have

$$(4.16) \qquad \left(\mathfrak{K}\mathfrak{L}x^{\gamma-1}\right)(\tau) = \int_0^\infty y^{\gamma-1}K_{i\tau}(y)dy = 2^{\gamma-2}\left|\Gamma\left(\frac{\gamma+i\tau}{2}\right)\right|^2$$

by virtue of the integral [10, Vol.2, (2.16.2.2)]. Next we consider the function $\omega(x) \equiv e^{-x}x^\gamma$ $(\gamma > 0)$. Appealing to the integral [10, Vol.2, (2.16.6.4)], and we obtain

$$(4.17) \qquad \left(\mathfrak{K}\mathfrak{L}(e^{-x}x^{\gamma-1})\right)(\tau) = \int_0^\infty y^{\gamma-1}e^{-y}K_{i\tau}(y)dy = 2^{-\gamma}\sqrt{\pi}\frac{|\Gamma(\gamma+i\tau)|^2}{\Gamma(\gamma+1/2)}.$$

As the last example, we demonstrate for the function $\omega(x) \equiv xe^{-\gamma x}$ $(0 \leqq \gamma < 1)$. Making use of the formula [10, Vol.2, (2.16.6.1)] we have

$$(4.18) \qquad \left(\mathfrak{K}\mathfrak{L}e^{-x}\right)(\tau) = \int_0^\infty e^{-\gamma y}K_{i\tau}(y)dy = \frac{\pi\sinh\left(\tau\cos^{-1}\gamma\right)}{\sinh(\pi\tau)\sqrt{1-\gamma^2}}.$$

So we consider below the function

$$(4.19) \qquad q(\tau) = \left(\mathfrak{K}\mathfrak{L}\left[\frac{\omega(x)}{x}\right]\right)(\tau) \quad (\tau \geqq 0),$$

as a weight function for respective Lebesgue spaces.

Let us take in general two complex-valued functions $f(x)$ and $g(x)$ from the space $L^\alpha_{\cos\delta} \subset L^\alpha$ $(\pi/3 < \delta < \pi/2, \alpha \geqq 0)$. Then according to Theorem 4.1 the convolution $(f * \overline{g})(x)$ exists and belongs to the space $L^\alpha_{\cos\delta}$. Moreover the factorization equality (3.24) is true and due to Theorem 3.11 the Parseval relation (3.25) holds. Multiplying left and right parts of (3.25) by $\omega(x)$ and integrating on $\mathbf{R}_+$, we obtain the equality

$$(4.20) \qquad \int_0^\infty (f*\overline{g})(x)\omega(x)dx = \frac{2}{\pi^2}\int_0^\infty \frac{\omega(x)dx}{x}\int_0^\infty \tau\sinh(\pi\tau)K_{i\tau}(x)\left(\mathfrak{K}\mathfrak{L}f\right)(\tau)\left(\mathfrak{K}\mathfrak{L}\overline{g}\right)(\tau).$$

Hence, using the estimate (3.36) and the condition (4.9) for $\omega(x)$, we have

$$(4.21) \qquad \int_0^\infty |(f*\overline{g})(x)|\omega(x)dx \leqq \frac{2}{\pi^2}\int_0^\infty K_0(x\cos\delta)\frac{\omega(x)}{x}dx\int_0^\infty \tau\sinh(\pi\tau)e^{-3\tau\delta}d\tau$$
$$\times\int_0^\infty K_0(u\cos\delta)|f(u)|du\int_0^\infty K_0(y\cos\delta)|g(y)|dy,$$

where all integrals are convergent under the assumptions. Thus finally we apply the Fubini theorem that enables us to change the order integrations at the right part of (4.20), and we come to the equality

$$\int_0^\infty (f*\overline{g})(x)\omega(x)dx = \frac{2}{\pi^2}\int_0^\infty \tau\sinh(\pi\tau)q(\tau)\left(\mathfrak{KL}f\right)(\tau)\left(\mathfrak{KL}\overline{g}\right)(\tau)d\tau \tag{4.22}$$

where the weight function $q(\tau)$ is defined by the formula (4.19). Let us denote the left part of (4.22) as

$$\int_0^\infty (f*\overline{g})(x)\omega(x)dx = \langle f,g\rangle. \tag{4.23}$$

It is not difficult to see from the equality (4.22) and Theorem 3.12 that $\langle f,g\rangle$ possesses all properties of the inner product. With this inner product the set of functions $L^\alpha_{\cos\delta}$ becomes the pre-Hilbert space. Its completion we call as the convolution Hilbert space and is denoted by S_q. From this inner product $\langle f,g\rangle$ the norm $||f||_S = \sqrt{\langle f,f\rangle}$ is defined. If $f,g\in L^\alpha_{\cos\delta}$, then we have

$$\begin{aligned}\langle f,g\rangle &= \int_0^\infty (f*\overline{g})(x)\omega(x)dx \\ &= \frac{1}{2}\int_0^\infty \frac{\omega(x)}{x}dx\int_0^\infty\int_0^\infty \exp\left(-\frac{1}{2}\left[\frac{xu}{v}+\frac{xv}{u}+\frac{vu}{x}\right]\right)f(u)\overline{g}(v)dudv \\ &= \int_0^\infty\int_0^\infty S_\omega(u,v)f(u)\overline{g}(v)dudv,\end{aligned} \tag{4.24}$$

where

$$S_\omega(u,v) = \frac{1}{2}\int_0^\infty \frac{\omega(x)}{x}\exp\left(-\frac{1}{2}\left[\frac{xu}{v}+\frac{xv}{u}+\frac{vu}{x}\right]\right)dx. \tag{4.25}$$

Of course we can perform to change the order of integrations by the above estimate and the Fubini theorem. Thus if $f(x)$ satisfies the condition

$$\int_0^\infty\int_0^\infty S_\omega(u,v)|f(u)f(v)|dudv < \infty, \tag{4.26}$$

then $||f||_S<\infty$ and $f\in S_q\supset L^\alpha_{\cos\delta}$. Furthermore if $f(x)$ and $g(x)$ satisfy the condition (4.26), then from the Cauchy inequality

$$|\langle f,g\rangle| \leqq ||f||_S||g||_S, \tag{4.27}$$

it follows that the integral

$$\int_0^\infty\int_0^\infty S_\omega(u,v)|f(u)g(v)|dudv$$

is convergent and the equality (4.24) is valid.

Let us now introduce the weighted Hilbert space $H_q\equiv L_2(\mathbf{R}_+;2\tau\sinh(\pi\tau)q(\tau)/\pi^2)$ with the norm

$$||h||_{H_q} = \frac{\sqrt{2}}{\pi}\left(\int_0^\infty \tau\sinh(\pi\tau)q(\tau)|h(\tau)|^2d\tau\right)^{1/2}, \tag{4.28}$$

where $q(\tau)$ is the weight function (4.19). As it follows from (4.22) the operator of K-L transform (1.1) maps the space $L^{\alpha}_{\cos\delta}$ into H_q and moreover

$$(4.29) \qquad ||\mathfrak{KL}f||^2_{H_q} = \frac{2}{\pi^2}\int_0^\infty \tau\sinh(\pi\tau)q(\tau)|(\mathfrak{KL}f)(\tau)|^2 d\tau$$
$$= \int_0^\infty (f*\overline{f})(x)\omega(x)dx = ||f||^2_S.$$

According to the known Banach theorem we can extend K-L operator on all $f \in S_q$. So K-L transform is defined for all $f \in S_q$, its range $KL(S_q)$ belongs to H_q and there holds the property for any $f \in S_q$

$$(4.30) \qquad ||f||_S = ||\mathfrak{KL}f||_{H_q}, \quad \left(\mathfrak{KL}f\right)(\tau) = 0 \text{ iff } f = 0.$$

Consequently we conclude that there exists the inverse bounded operator $\left(\mathfrak{KL}^{-1}h\right)(x)$. Due to the definition of the norm (4.28) and the equality (4.24), the inner product of two elements $\varphi, \psi \in H_q$ can be written by the formula

$$(4.31) \qquad ((\varphi,\psi)) = \frac{2}{\pi^2}\int_0^\infty \tau\sinh(\pi\tau)q(\tau)\varphi(\tau)\overline{\psi}(\tau)d\tau.$$

Returning to considered examples of function $\omega(x)$ we get the respective examples of Hilbert spaces S_q and H_q with the relation (4.22) and the condition (4.26) for each case. For instance, putting $\omega(x) \equiv x$ using the formula (4.15) and the equality (4.22), we have

$$(4.32) \qquad \int_0^\infty (f*\overline{g})(x)xdx = \frac{2}{\pi}\int_0^\infty \tau\sinh\left(\frac{\pi\tau}{2}\right)\left(\mathfrak{KL}f\right)(\tau)\left(\mathfrak{KL}\overline{g}\right)(\tau)d\tau.$$

To obtain the corresponding condition (4.26) we calculate the integral (4.25) by means of the formula (3.23) and it gives the boundedness

$$(4.33) \qquad \int_0^\infty\int_0^\infty \frac{uy}{\sqrt{u^2+y^2}}K_1(\sqrt{u^2+y^2})|f(u)f(y)|dudy < \infty.$$

Similarly we can treat other examples (4.16) - (4.18).

As we noted above if $f(x), g(x) \in L^{\alpha}_{\cos\delta}$, then $(f*g)(x) \in L^{\alpha}_{\cos\delta}$ and the equality (3.24) is valid. Hence there holds the representation

$$(4.34) \qquad (f*g)(x) = \left(\mathfrak{KL}^{-1}\left[\left(\mathfrak{KL}f\right)(\tau)\left(\mathfrak{KL}g\right)(\tau)\right]\right)(x).$$

If for two elements f, g of the convolution Hilbert space S_q the function $\left(\mathfrak{KL}f\right)(\tau)\left(\mathfrak{KL}g\right)(\tau)$ $= \varphi(\tau)\psi(\tau)$ belongs to $KL(S_q)$, then the element $\mathfrak{KL}^{-1}[\varphi(\tau)\psi(\tau)]$ may be called the generalized convolution of the elements f, g and denoted by $f*g$. Let us prove that for any $f \in S_q$ and $g \in L^{\alpha}_{\cos\delta}$ the convolution (1.6) exists and there holds the inequality

$$(4.35) \qquad ||f*g||_S \leqq \sup_{\tau>0}|\psi(\tau)|\ ||f||_S,$$

where

$$(4.36) \qquad \psi(\tau) \equiv \left(\mathfrak{KL}g\right)(\tau) = \int_0^\infty K_{i\tau}(y)g(y)dy.$$

Indeed since the function $g(x)$ belongs to $L^{\alpha}_{\cos\delta}$, then due to the estimate

$$|\psi(\tau)| \leqq e^{-\delta\tau} \int_0^{\infty} K_0(y\cos\delta)|g(y)|dy \tag{4.37}$$

we have that $\sup_{\tau>0}|\psi(\tau)| = M < \infty$ and consequently, $\varphi(\tau)\psi(\tau) \in H_q$ for $\varphi(\tau) \equiv \left(\mathfrak{KL}f\right)(\tau)$. Let us now prove $\varphi(\tau)\psi(\tau) \in KL(S_q)$. There exists some sequence $f_n(x) \in L^{\alpha}_{\cos\delta}$ such that $||f - f_n||_S \to 0$ as $n \to \infty$. Hence we obtain according to Theorem 4.1 that $h_n(x) = f_n(x) * g(x) \in L^{\alpha}_{\cos\delta}$ and denoting by $\varphi_n(\tau) = \left(\mathfrak{KL}f_n\right)(\tau)$, we have (see (4.29))

$$\begin{aligned} ||h_n - h_m|| &= ||(f_n - f_m) * g||_S = ||(\mathfrak{KL}[f_n - f_m])(\mathfrak{KL}g)||_{H_q} \\ &= ||(\varphi_n - \varphi_m)\psi||_{H_q} \leqq M||\varphi_n - \varphi_m||_{H_q} = M||f_n - f_m||_S. \end{aligned} \tag{4.38}$$

Thus the sequence $\{h_n\}$ is convergent at the Hilbert space S_q. Let the corresponding limit be h. Then we obtain

$$\left(\mathfrak{KL}h\right)(\tau) = \left(\mathfrak{KL}f\right)(\tau)\left(\mathfrak{KL}g\right)(\tau) = \varphi(\tau)\psi(\tau), \tag{4.39}$$

and the product $\varphi(\tau)\psi(\tau)$ belongs to $KL(S_q)$.

We now turn to establish the inversion formula for K-L transform at the convolution Hilbert space.

Theorem 4.3. *Let the weight function $\omega(x)$ satisfy the condition (4.9) with $\beta = \cos\delta$ $(\pi/3 < \delta < \pi/2)$. Then for a function $f(x)$ from the convolution Hilbert space S_q there holds the inversion formula of K-L transform*

$$\begin{aligned} \left(f * \frac{\omega(x)}{x}\right)(x) &= \frac{2}{\pi^2 x}\frac{d}{dx}\int_0^{\infty} \tau\sinh(\pi\tau)q(\tau) \\ &\times \Re\left[\frac{x^{1-i\tau}2^{i\tau}\Gamma(i\tau)}{1-i\tau}\,{}_1F_2\left(\frac{1-i\tau}{2}; 1-i\tau, \frac{3-i\tau}{2}; \frac{x^2}{4}\right)\right]\left(\mathfrak{KL}f\right)(\tau)d\tau, \end{aligned} \tag{4.40}$$

where ${}_1F_2(a;b,c;z)$ means the hypergeometric function [1] and notation $\Re[F(i\tau)] = (F(i\tau) + F(-i\tau))/2$ gives the real part of arbitrary function $F(i\tau)$. Besides, if $f(x) \in L^{\alpha}_{\cos\delta} \subset S_q$, then the formula (4.40) takes the classical form

$$\left(f * \frac{\omega(x)}{x}\right)(x) = \frac{2}{\pi^2}\int_0^{\infty} \tau\sinh(\pi\tau)q(\tau)\frac{K_{i\tau}(x)}{x}\left(\mathfrak{KL}f\right)(\tau)d\tau. \tag{4.41}$$

Proof. In order to prove the formula (4.40) we start from formula (4.22). Letting there $g(y) = 1$ $(0 < y \leqq x)$; $g(y) = 0$ $(y > x)$, we transform the right-hand side of the equality (4.22) by calculating the respective integral $\left(\mathfrak{KL}g\right)(\tau)$ in view of the formula [10, Vol.2, (1.12.1.2)] as

$$\int_0^x K_{i\tau}(y)dy = \Re\left[\frac{x^{1-i\tau}2^{i\tau}\Gamma(i\tau)}{1-i\tau}\,{}_1F_2\left(\frac{1-i\tau}{2}; 1-i\tau, \frac{3-i\tau}{2}; \frac{x^2}{4}\right)\right]. \tag{4.42}$$

The left-hand side of the equality (4.22) is an absolute convergent integral under conditions of the theorem from the inequality (4.27). So, by denoting the right-hand side as $R(\tau, x)$, temporary, the formula (4.22) becomes

$$\int_0^x \int_0^\infty S_\omega(u,y) f(u) du dy = \frac{2}{\pi^2} \int_0^\infty \tau \sinh(\pi\tau) q(\tau) R(\tau, x) \left(\mathfrak{KL} f\right)(\tau) d\tau, \tag{4.43}$$

by the notation (4.25). Moreover this enables us to perform differentiation of the left- and right-hand sides of (4.43) to obtain

$$\int_0^\infty S_\omega(u,x) f(u) du = \frac{2}{\pi^2} \frac{d}{dx} \int_0^\infty \tau \sinh(\pi\tau) q(\tau) R(\tau, x) \left(\mathfrak{KL} f\right)(\tau) d\tau. \tag{4.44}$$

But the left-hand side of (4.44) is equal to $x(f * [\omega(x)/x])(x)$, which leads to (4.40). The formula (4.41) can be easily deduced by performing the differentiation under the integral with respect to τ in the right-hand side of (4.44), provided that $f(x) \in L^\alpha_{\cos\delta}$ invoking to the representation (4.42) and the inequality (3.36) meaning there instead of g the function $\omega(x)/x \in L^\alpha_{\cos\delta}$ under condition of the present theorem. ■

Putting $\omega(x) \equiv x$ at the formula (4.40), we attract our attention to corresponding inversion formula for the space (4.33), namely

$$(f * 1)(x) = \frac{2}{\pi x} \frac{d}{dx} \int_0^\infty \tau \sinh\left(\frac{\pi\tau}{2}\right) R(\tau, x) \left(\mathfrak{KL} f\right)(\tau) d\tau \tag{4.45}$$

Let us note that if the range of K-L transform $KL(S_q)$ coincides with the space H_q, then for the existence of the convolution $(f * g)(x)$ of $f, g \in S_q$ it is necessary and sufficient that the product $\left(\mathfrak{KL} f\right)(\tau) \left(\mathfrak{KL} g\right)(\tau)$ belongs to the space H_q. However it is true that $KL(S_q) = H_q$.

Theorem 4.4. *The range of K-L transform $KL(S_q)$ coincides with the weighted Hilbert space H_q.*

Proof. We note that there exists in H_q no element except zero that is orthogonal to $KL(S_q)$. In fact, let us assume $(\varphi_0, \left(\mathfrak{KL} g\right)) = 0$ for arbitrary $g \in S_q$, where the inner product means the formula (4.31). In particular, we take the function g as in the Theorem 4.3 that $g(y) = 1$ $(0 < y \leqq x)$; $g(y) = 0$ $(y > x)$. The equality

$$\left(\varphi_0, \int_0^x K_{i\tau}(y) dy\right) = \frac{2}{\pi^2} \int_0^\infty \tau \sinh(\pi\tau) q(\tau) \varphi_0(\tau) \int_0^x K_{i\tau}(y) dy d\tau = 0 \tag{4.46}$$

after differentiation with respect to x yields

$$\int_0^\infty \tau \sinh(\pi\tau) q(\tau) \varphi_0(\tau) K_{i\tau}(x) d\tau = 0 \tag{4.47}$$

for all $x > 0$. The last operation is possible due to its absolute and uniform convergence of the integral (4.47) in view of the estimate

$$\int_0^\infty \tau \sinh(\pi\tau) q(\tau) |\varphi_0(\tau) K_{i\tau}(x)| d\tau \tag{4.48}$$
$$\leqq \frac{\pi K_0(x\cos\delta)}{\sqrt{2}} ||\varphi_0||_{H_q} \left(\int_0^\infty \tau \sinh(\pi\tau) q(\tau) e^{-2\delta\tau} d\tau\right)^{1/2}.$$

So this estimate shows that the left hand-side of (4.47) is a function in $L_1(\mathbf{R}_+)$ and according to Fourier integrals theory [11] we can take the cosine-Fourier transform (4.12) of both sides of the equality (4.47). Changing the order of integrations by the Fubini theorem and calculating the inner integral by formula [10, Vol.2, (2.16.14.1)] we obtain a new equality

$$\int_0^\infty \tau \sinh\left(\frac{\pi\tau}{2}\right) q(\tau)\varphi_0(\tau)\cos\left(\tau\log(x+\sqrt{x^2+1})\right) d\tau \equiv 0. \tag{4.49}$$

In view of the above estimates, observing the integrand in (4.49) from the space $L_1(\mathbf{R}_+)$ by τ, applying the Hölder inequality and using known theorem [11] about uniqueness of cosine Fourier transform of summable functions of $L_1(\mathbf{R}_+)$, we obtain $\varphi_0(\tau) = 0$ almost everywhere. ■

5. Kontorovich-Lebedev convolution integral equations

In this last section we consider some class of integral equations with the kernel (1.8) which contains the inner integral of the Kontorovich-Lebedev convolution (1.6). Such equations were first mentioned in [7] and were exhibited in details recently in [16]. Comparing with usual convolution equations of Fourier, Mellin or Laplace type [11] it is not so easy to recognize the convolution property of the operator (1.7). Nevertheless this class of integral equations is also worth mentioning in connection with some applications to problems of mathematical physics [7]. First it was described in [12] using the algebra of the introduced convolution (1.6). Here we give some examples of Kontorovich-Lebedev type convolution integral equations and their solutions in slightly different form than in [16] in view of the considered convolution operator (1.6) and its new properties. The operational method of solutions of such equations was demonstrated in [15] and [16].

The most familiar form of the integral equation is

$$f(x) = h(x) + \lambda\int_0^\infty K(x,u)f(u)dy \ (x>0), \tag{5.1}$$

where λ is some complex parameter, $h(x)$ and $K(x,u)$ are given functions, and $f(x)$ is to be determined. We will call such equation as usually the integral equation of the second kind. It can be solved by means of the K-L integrals in certain special cases meaning such ones in which the integral operator (5.1) is the convolution operator (1.7) with the kernel (1.8).

Let us consider some examples of concrete kernels (1.8) choosing different functions $g(x)$ and calculating the respective integrals. Let $g(x) = \exp(-x\cos\mu)x^{\gamma-1}$ $(0 \leqq \mu < \pi, \Re\gamma > 0)$. Then due to the formula [10, Vol.1, (2.3.16.1)], the function $K(x,u)$ is

$$K(x,u) = \frac{x^{\gamma-1}u^\gamma}{(x^2+u^2+2xu\cos\mu)^{\gamma/2}} K_\gamma\left(\sqrt{x^2+u^2+2xu\cos\mu}\right) \tag{5.2}$$

and the equation (5.1) takes the form

$$f(x) = h(x) + \lambda\int_0^\infty \frac{x^{\gamma-1}u^\gamma K_\gamma\left(\sqrt{x^2+u^2+2xu\cos\mu}\right)}{(x^2+u^2+2xu\cos\mu)^{\gamma/2}} f(u)du. \tag{5.3}$$

If we set $\gamma = 1/2$, then by invoking to the fact that the Macdonald function $K_{1/2}(z)$ is equal to $e^{-z}\sqrt{\pi/2z}$ [1], we have the equation (5.3) of the form

$$f(x) = h(x) + \lambda\sqrt{\frac{\pi}{2x}}\int_0^\infty \frac{\exp\left(-\sqrt{x^2+u^2+2xu\cos\mu}\right)}{(x^2+u^2+2xu\cos\mu)^{1/2}}u^{1/2}f(u)du. \tag{5.4}$$

The simplest case was first considered in [7] when $\mu = 0$ as

$$f(x) = h(x) + \lambda\sqrt{\frac{\pi}{2x}}\int_0^\infty \frac{\exp(-x-u)}{x+u}u^{1/2}f(u)du. \tag{5.5}$$

Let $g(x) = (x+a)^{-1}$ with a parameter $a > 0$. Then the integral (1.8) can be evaluated by the formula [10, Vol.1, (2.3.16.4)]

$$K(x,u) = \frac{\sqrt{\pi/a}}{2x}\exp\left(a\frac{x^2+u^2}{2xu} + \frac{xu}{2a}\right)\operatorname{erfc}\left(\sqrt{\frac{xu}{2a}} + a\frac{\sqrt{x^2+u^2}}{2xu}\right), \tag{5.6}$$

where

$$\operatorname{erfc}(x) = \frac{2}{\pi}\int_x^\infty e^{-t^2}dt \tag{5.7}$$

is the error function [1].

Let us return to the general convolution operator (1.7). We already noted in Corollary 3.1 its mapping properties and behavior of the kernel $K(x,u)$ at the neighborhood of the point $(0,0)$. The considered examples of the kernel $K(x,u)$ confirm that it contains removable singularity at the origin $(0,0)$. We begin from the homogeneous equation

$$f(x) = \lambda(\tau)\int_0^\infty K(x,u)f(u)du \quad (x > 0), \tag{5.8}$$

where $\lambda(\tau)$ is a continuous function on $\mathbf{R}_+$ of variable τ.

Theorem 5.1. *Let $g(x) \in L^0 \equiv L(\mathbf{R}_+K_0(x))$. If $1/\lambda(\tau) = \left(\mathfrak{K}\mathfrak{L}g\right)(\tau)$, then the function $K_{i\tau}(x)/x$ satisfies the equation* (5.8).

Proof. Substituting $K_{i\tau}(x)/x$ in the right-hand side of the equality (5.8) and taking into account the inequality $|K_{i\tau}(x)| \leqq K_0(x)$, we obtain the estimate

$$\left|\lambda(\tau)\int_0^\infty K(x,u)K_{i\tau}(u)\frac{du}{u}\right| \leqq |\lambda(\tau)|\int_0^\infty |K(x,u)|\frac{K_0(u)}{u}du. \tag{5.9}$$

Due to the Macdonald formula (3.27), we have

$$\int_0^\infty |K(x,u)|K_0(u)\frac{du}{u} \leqq \frac{K_0(x)}{x}\int_0^\infty K_0(y)|g(y)|dy = \frac{K_0(x)}{x}||g||_{L^0}. \tag{5.10}$$

Hence we perform to change the order of integrations by the Fubini theorem and obtain the identity (5.8) with $f(x) = K_{i\tau}(x)/x$. ■

The partial solution of the equation (5.1) is given by the following:

Theorem 5.2. *Let* $h(x), g(x) \in L_p(\mathbf{R}_+)$ $(p \geqq 1)$ *and* $\sup_{\tau \geqq 0} |(\mathfrak{K}\mathfrak{L}g)(\tau)| < 1/|\lambda|$. *Then the function*

$$f(x) = \frac{2}{\pi^2} \underset{\varepsilon \to 0+}{\mathrm{l.i.m.}} \int_0^\infty \frac{\tau \sinh((\pi-\varepsilon)\tau) (\mathfrak{K}\mathfrak{L}h)(\tau)}{1 - \lambda (\mathfrak{K}\mathfrak{L}g)(\tau)} \frac{K_{i\tau}(x)}{x} d\tau \tag{5.11}$$

is a partial L_p*-solution of the equation* (5.1) *and the limit is meant in the norm of* $L_p(\mathbf{R}_+)$.

Proof. First let us show that the space $L_p(\mathbf{R}_+)$ $(p \geqq 1)$ is a subspace of the space $L^0_{\cos\delta}$ $(\delta \in (0, \pi/2))$. Indeed, if $g \in L_p(\mathbf{R}_+)$, then by the usual Hölder inequality we have

$$\int_0^\infty K_0(y \cos\delta)|g(y)|dy \leqq \left(\int_0^\infty K_0^q(y\cos\delta)dy\right)^{1/q} ||g||_{L_p} < \infty \tag{5.12}$$

which gives the desired result. On the other hand, according to Theorem 2.2 for the function $h \in L_p(\mathbf{R}_+)$ we have the limit relation

$$h(x) = \frac{2}{\pi^2} \underset{\varepsilon \to 0+}{\mathrm{l.i.m.}} \int_0^\infty \tau \sinh((\pi-\varepsilon)\tau) \frac{K_{i\tau}(x)}{x} (\mathfrak{K}\mathfrak{L}h)(\tau) d\tau \tag{5.13}$$

and the estimate due to the fact $h(x) \in L^0_{\cos\delta}$ as

$$|(\mathfrak{K}\mathfrak{L}h)(\tau)| \leqq e^{-\delta\tau} ||h||_{L^0_{\cos\delta}} \quad \left(\delta \in \left(0, \frac{\pi}{2}\right)\right). \tag{5.14}$$

Hence denoting the right-hand side of (5.11) as $(I_\varepsilon f)(x)$, we obtain under the assumption for $(\mathfrak{K}\mathfrak{L}g)(\tau)$ that for each $\varepsilon > 0$

$$|(I_\varepsilon f)(x)| \leqq C \frac{K_0(x\cos\delta)}{x} ||h||_{L^0_{\cos\delta}} \int_0^\infty \tau \sinh((\pi-\varepsilon)\tau) e^{-2\delta\tau} d\tau, \tag{5.15}$$

where C is an absolute positive constant and δ is chosen from the interval $((\pi-\varepsilon)/2, \ \pi/2)$. Therefore there exists the convolution $(g * (I_\varepsilon f))(x)$ provided by the estimate

$$\begin{aligned} |(g * (I_\varepsilon f))(x)| \ &\leqq \ \frac{C}{x} ||h||_{L^0_{\cos\delta}} \int_0^\infty |g(y)| dy \\ &\times \int_0^\infty \exp\left(-\frac{1}{2}\left[\frac{xu}{y} + \frac{xy}{u} + \frac{yu}{x}\right]\right) \frac{K_0(u\cos\delta)}{u} du \\ &\leqq \ C \frac{K_0(x\cos\delta)}{x} ||h||_{L^0_{\cos\delta}} \int_0^\infty K_0(y\cos\delta)|g(y)|dy \\ &\leqq \ C_\delta \frac{K_0(x\cos\delta)}{x} ||h||_{L^0_{\cos\delta}} ||g||_p, \end{aligned} \tag{5.16}$$

where C_δ is a positive constant. To get this estimate we used the inequality (4.5) as well as the usual Hölder inequality. Hence from the assumption we have

$$\frac{(\mathfrak{K}\mathfrak{L}h)(\tau)}{1 - \lambda (\mathfrak{K}\mathfrak{L}g)(\tau)} \in KL(L_p), \tag{5.17}$$

in view of the result of Theorem 2.3 such that the condition (2.27) holds. Invoking to the Macdonald formula (3.27), Theorem 2.2 and the Lebesgue theorem, we have the limit equality

$$(5.18)\quad \lim_{\varepsilon\to 0+} \lambda(g*(I_\varepsilon f))(x) = (f*g)(x)$$

$$= \lim_{\varepsilon\to 0+} \frac{2\lambda}{\pi^2 x}\int_0^\infty \frac{\tau\sinh((\pi-\varepsilon)\tau)\left(\mathfrak{K}\mathfrak{L}h\right)(\tau)}{1-\lambda\left(\mathfrak{K}\mathfrak{L}g\right)(\tau)}\left(\mathfrak{K}\mathfrak{L}g\right)(\tau)K_{i\tau}(x)d\tau$$

$$= -\lim_{\varepsilon\to 0+} \frac{2}{\pi^2 x}\int_0^\infty \tau\sinh((\pi-\varepsilon)\tau)K_{i\tau}(x)\left(\mathfrak{K}\mathfrak{L}h\right)(\tau)d\tau$$

$$+\lim_{\varepsilon\to 0+} \frac{2}{\pi^2 x}\int_0^\infty \frac{\tau\sinh((\pi-\varepsilon)\tau)\left(\mathfrak{K}\mathfrak{L}h\right)(\tau)}{1-\lambda\left(\mathfrak{K}\mathfrak{L}g\right)(\tau)}K_{i\tau}(x)d\tau = -h(x)+f(x)$$

almost everywhere, where the last equality is easy seen from Theorem 2.2. ∎

Now we consider an equation similar to (5.1) with the kernel (1.8) and $\lambda = -1$

$$(5.19)\qquad h(x) = f(x) + \int_0^\infty K(x,u)f(u)du$$

with respect to the function $f(x)$ in the class $L(\mathbf{R}_+; K_\alpha(x))$, where the given functions $h(x)$ and $g(x)$ in (1.8) belong to normed ring $L(\mathbf{R}_+; K_\alpha(x))$. Applying K-L transform (1.1) with the index $\eta = \Re\eta + i\tau$ in the strip $|\Re\eta| \leqq \alpha$ to the both sides of the equality (5.19), we obtain from the factorization equality (3.69) for the convolution $(f*g)(x)$ the following algebraic equation

$$(5.20)\qquad \left(\mathfrak{K}\mathfrak{L}h\right)(\eta) = \left(\mathfrak{K}\mathfrak{L}f\right)(\eta)(1+\left(\mathfrak{K}\mathfrak{L}g\right)(\eta)) \quad (|\Re\eta| \leqq \alpha).$$

If the condition

$$(5.21)\qquad 1+\left(\mathfrak{K}\mathfrak{L}g\right)(\eta) \neq 0, \quad (|\Re\eta| \leqq \alpha),$$

holds, then by the analogue of the Wiener Theorem 3.15, there is a unique function $q(x) \in L^\alpha$ such that

$$(5.22)\qquad \frac{1}{\left(\mathfrak{K}\mathfrak{L}g\right)(\eta)} = 1+\left(\mathfrak{K}\mathfrak{L}q\right)(\eta).$$

Then from (5.20), we obtain the equality

$$(5.23)\qquad \left(\mathfrak{K}\mathfrak{L}f\right)(\eta) = \frac{\left(\mathfrak{K}\mathfrak{L}h\right)(\eta)}{1+\left(\mathfrak{K}\mathfrak{L}g\right)(\eta)} = (1+\left(\mathfrak{K}\mathfrak{L}q\right)(\eta))\left(\mathfrak{K}\mathfrak{L}h\right)(\eta) \quad (|\Re\eta| \leqq \alpha),$$

which is equivalent to

$$(5.24)\qquad f(x) = h(x) + \int_0^\infty K_q(x,u)h(u)du \quad (x>0),$$

where $K_q(x,u)$ is a new kernel similar to (1.8) with the function $q(x)$. It is easily seen that, conversely, the function $f(x)$ in the formula (5.24) gives the solution of the equation (5.19) for any function $h(x)$ from the ring L^α only under condition (5.21). Thus we have proved:

Theorem 5.3. *Let functions $g(x), h(x) \in L(\mathbf{R}_+; K_\alpha(x))$. Then the equation (5.19) is solvable in the class L^α if and only if condition (5.21) holds. Moreover, its unique solution is represented by the formula (5.24).*

Corollary 5.1. *The equation (5.3) with $\lambda = -1$ is solvable in the ring $L(\mathbf{R}_+; K_0(x))$ if and only if*

$$1 + \sqrt{\frac{\pi}{2}}\Gamma(\gamma - i\tau)\Gamma(\gamma + i\tau)\sin^{1/2-\gamma}\mu P^{1/2-\gamma}_{-1/2+i\tau}(\cos\mu) \neq 0 \quad (\tau \in \mathbf{R}), \tag{5.25}$$

where $P^{1/2-\gamma}_{-1/2+i\tau}(\cos\mu)$ is the Legendre function [1].

Proof. Actually the inequality (5.25) means the condition (5.21) for K-L transform (1.1) of the function $g(x) = e^{-x\cos\mu}x^{\gamma-1}$. To evaluate this we use the integral [10, Vol.2, (2.16.6.3)] and we have

$$\begin{aligned}(5.26)\quad &\int_0^\infty x^{\gamma-1}e^{-x\cos\mu}K_{i\tau}(x)dx \\ &= \sqrt{\frac{\pi}{2}}\Gamma(\gamma - i\tau)\Gamma(\gamma + i\tau)\sin^{1/2-\gamma}\mu P^{1/2-\gamma}_{-1/2+i\tau}(\cos\mu) \neq 0 \quad (\tau \in \mathbf{R}),\end{aligned}$$

which leads to (5.25). ∎

Corollary 5.2. *The Lebedev's equation (5.5) is solvable for $\lambda = -2/\pi^2$ in the class $L(\mathbf{R}_+; K_\alpha(x))$ $(0 \leqq \alpha < 1/2)$, and moreover, its unique solution has the form*

$$f(x) = h(x) + \frac{2}{\pi^2}\int_0^\infty \frac{uK_1(u)K_0(x) - xK_1(x)K_0(u)}{x^2 - u^2}uh(u)du. \tag{5.27}$$

Conversely, the equation (5.27) is solvable in the ring $L(\mathbf{R}_+; K_\alpha(x))$ $(0 \leqq \alpha < 1/2)$ and its unique solution has the form (5.5) in the case $\lambda = -2/\pi^2$.

Proof. The proof of can be obtained by using the same integral (5.26), when $\mu = 0, \gamma = 1/2$. In this case the result is reduced to $\left(\mathfrak{KL}g\right)(\eta) = \pi^{3/2}/(\sqrt{2}\cos(\pi\eta))$, where $g(x) = e^{-x}x^{-1/2}$. Hence we get the equation (5.23) in the form

$$\left(\mathfrak{KL}f\right)(\eta) = \frac{\cos(\pi\eta)}{1+\cos(\pi\eta)}\left(\mathfrak{KL}h\right)(\eta) = (1+\left(\mathfrak{KL}q\right)(\eta))\left(\mathfrak{KL}h\right)(\eta) \quad (|\Re\eta| \leqq \alpha), \tag{5.28}$$

where the value of the function $q(x)$ can be obtained by using the formula [10, Vol.2, (2.16.33.2)] and $q(x) = -(2/\pi^2)K_0(x)$. The respective kernel $K_q(x,u)$ is evaluated from the formula (1.8) and the integral [10, Vol.2, (2.16.9.1)], which leads to the kernel in the solution (5.27)

$$\begin{aligned}(5.29)\quad K_q(x,u) &= -\frac{1}{\pi^2 x}\int_0^\infty \exp\left(-\frac{1}{2}\left[\frac{xy}{u} + \frac{xu}{y} + \frac{uy}{x}\right]\right)K_0(y)dy \\ &= \frac{2}{\pi^2}\frac{u(uK_1(u)K_0(x) - xK_1(x)K_0(u))}{x^2 - u^2}.\end{aligned}$$

The condition $0 \leqq \alpha < 1/2$ arises from the convergence of the integral (3.31) for $||g||_{L^\alpha} < +\infty$, where $g(x) = e^{-x}x^{-1/2}$. ■

Concerning the convolution equation of the first kind like (1.7)

$$\int_0^\infty K(x,u)f(u)du = h(x) \tag{5.30}$$

its solution can be written based on the factorization equality (3.24) for K-L transform and its range for corresponding space of functions. Thus, for example, if we look for a solution in the convolutional Hilbert space S_q, we have to take the given function $h(x)$ from S_q as well as the kernel function $g(x)$ (see (1.8)). According to Theorem 4.4 the range $KL(S_q)$ coincides with the Hilbert space H_q. So from (5.30) we have the algebraic equation in terms of K-L transform

$$\left(\mathfrak{KL}f\right)(i\tau)\left(\mathfrak{KL}g\right)(i\tau) = \left(\mathfrak{KL}h\right)(i\tau). \tag{5.31}$$

Hence

$$\left(\mathfrak{KL}f\right)(i\tau) = \frac{\left(\mathfrak{KL}h\right)(i\tau)}{\left(\mathfrak{KL}g\right)(i\tau)} \tag{5.32}$$

and the solution of the equation (5.30) on the space S_q can be written by the formula (4.40) if and only if the right-hand side of (5.31) belongs to the space H_q.

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