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On Nishimoto's Fractional Calculus (Operator $\mathcal{N}^\nu$, Inverse of Nishimoto's Transformation and some Applications)

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Abstract

Many papers and books on the fractional calculus have been reported by the author already. In this article, the set $\{\mathcal{N}^\nu\}$ of fractional calculus operator $\mathcal{N}^\nu$ is discussed. And it is shown that the set $\{\mathcal{N}^\nu\}$ is an Abelian product group for the function $f \in \mathring{\mathcal{F}} = \{f \mid 0 \neq |f_\nu| < \infty, \nu \in R\}$, and for continuous index ν .

Moreover, " $\{\mathcal{N}^\nu\}$ is a group acting on a set $\mathring{\mathcal{F}}$ where ν is the continuous index " is shown, in Chapter 1.

In Chapter 2, the inverse of Nishimoto's integral transformation is discussed. Inverse of Goursat's transformation and that of Cauchy's one are discussed as special cases of that of Nishimoto's transformation.

In Chapter 3, some applications of the author's fractional calculus to a generalized higher order ordinary differential equation (homogeneous and nonhomogeneous) are shown.

Chapter 1. On the fractional calculus operator $\mathcal{N}^\nu$

§0. Introduction (Definition of Fractional Calculus)

(I) DEFINITION. (by K. Nishimoto) ([1], [11] Vol. 1)
Let $D=\{\underset{-}{D}, \underset{+}{D}\}$, $C=\{\underset{-}{C}, \underset{+}{C}\}$,

$\underset{-}{C}$ be a curve along the cut joining two points z and $-\infty+i\,\mathrm{Im}(z)$,

$\underset{+}{C}$ be a curve along the cut joining two points z and $\infty+i\,\mathrm{Im}(z)$,

$\underset{-}{D}$ be a domain surrounded by $\underset{-}{C}$, $\underset{+}{D}$ be a domain surrounded by $\underset{+}{C}$.

(Here D contains the points over the curve C).

Moreover, let $f=f(z)$ be a regular function in D ($z\in D$),

$$f_\nu=(f)_\nu={}_C(f)_\nu=\frac{\Gamma(\nu+1)}{2\pi i}\int_C\frac{f(\zeta)}{(\zeta-z)^{\nu+1}}d\zeta \qquad (\nu\notin Z^-), \tag{1}$$

$$(f)_{-m}=\lim_{\nu\to -m}(f)_\nu \qquad (m\in Z^+), \tag{2}$$

where $-\pi\leqslant\arg(\zeta-z)\leqslant\pi$ for $\underset{-}{C}$, $\quad 0\leqslant\arg(\zeta-z)\leqslant 2\pi$ for $\underset{+}{C}$,

$$\zeta \neq z\,,\quad z \in C\,,\quad \nu \in R\,,\quad \Gamma\text{; Gamma function,}$$

then $(f)_\nu$ is the fractional differintegration of arbitrary order ν (derivatives of order ν for $\nu > 0$, and integrals of order $-\nu$ for $\nu < 0$), with respect to z, of the function f, if $|(f)_\nu| < \infty$.

Note 1. See Figs. 1 and 2 for the integral curves $\underset{-}{C}$ and $\underset{+}{C}$, and the domains $\underset{-}{D}$ and $\underset{+}{D}$ respectively.

Note 2. More generally, if $f(z)$ is regular except the singular points in a finite (or infinite) number and there are no these singularities inside C and on C, then $f_\nu(z)$ can be defined again with the above definition.

Note 3. If $f(z)$ is a many valued regular function, we will define $f_\nu(z)$ for the principal

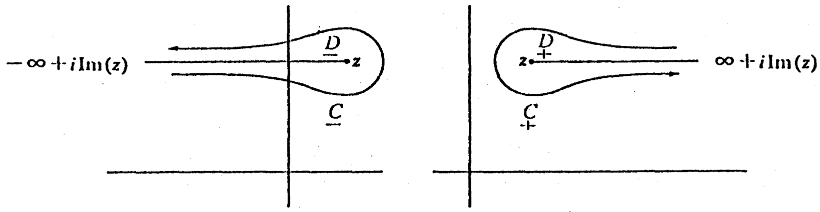

Fig. 1. Fig. 2.

values of $f(z)$.

Note 4. For the complex ν, we consider the principal value of it, and f_ν ($\mathrm{Re}(\nu) > 0$) is the fractional derivative of order $\mathrm{Re}(\nu)$, and f_ν ($\mathrm{Re}(\nu) < 0$) is the fractional integral of order $-\mathrm{Re}(\nu)$, if $|f_\nu| < \infty$.

However, as a matter of convenience, we will assume that $\nu \in R$ in this paper.

Note 5.

$$f_\nu(z) = \frac{d^\nu}{dz^\nu} f(z) \quad \text{and} \quad f_{-\nu}(z) = \int f(z)(dz)^\nu \qquad \text{for} \quad \nu > 0\,,$$

where $\nu \in R$.

Note 6. Formula (1) is a complex integral transformation of Mellin type.

Note 7. *Notations*

C:	set of a complex number	R:	set of a real number
Z:	set of an integer (contains zero)	R^+:	set of a positive real number
Z^+ $(=N)$:	set of a positive integer	R^-:	set of a negative real number
Z^-:	set of a negative integer		

(II) The set $\mathscr{F}$

We call the function $f = f(z)$ such that $|f_\nu| < \infty$ in D as fractional differintegrable functions by arbitrary order ν and denote the set of them with a notation $\mathscr{F} = \{f \,|\, |f_\nu| < \infty,\ \nu \in R\}$.

Then we have $\qquad |f_\nu| < \infty \Longleftrightarrow f \in \mathscr{F} \qquad (\text{in } D)\,.$

(III) Unification of integrations and differentiations

Notice that the definition (in the above description) for our fractional calculus means the unification of integrations and differentiations. That is, the formula (1)—having (2)—we can unify the integrations of arbitrary order and the differentiations of arbitrary order.

§1. On the fractional calculus operator $\mathcal{N}^{\nu}$

THEOREM 1. *Let fractional calculus operator (Nishimoto's operator) $\mathcal{N}^{\nu}$ be*

$$\mathcal{N}^{\nu}=\left(\frac{\Gamma(\nu+1)}{2\pi i}\int_{C}\frac{d\zeta}{(\zeta-z)^{\nu+1}}\right) \qquad (\nu \notin Z^{-}) \qquad \begin{bmatrix}\text{Refer to}\\ \S 0.\ (1)\end{bmatrix} \tag{1}$$

with

$$\mathcal{N}^{-m}=\lim_{\nu\to -m}\mathcal{N}^{\nu} \qquad (m\in Z^{+}), \tag{2}$$

and define the binary operation $\circ$ as

$$\mathcal{N}^{\beta}\circ\mathcal{N}^{\alpha}f=\mathcal{N}^{\beta}\mathcal{N}^{\alpha}f=\mathcal{N}^{\beta}(\mathcal{N}^{\alpha}f) \qquad (\alpha,\beta\in R), \tag{3}$$

then the set

$$\{\mathcal{N}^{\nu}\}=\{\mathcal{N}^{\nu}\,|\,\nu\in R\}$$

is an Abelian product group (having continuous index ν) which has the inverse transform operator $(\mathcal{N}^{\nu})^{-1}=\mathcal{N}^{-\nu}$ to the fractional calculus operator $\mathcal{N}^{\nu}$, for the function f such that $f\in\mathscr{F}=\{f\,|\,0\neq|f_{\nu}|<\infty,\ \nu\in R\}$, where $f=f(z)$ and $z\in C$. (viz. $-\infty<\nu<\infty$).

(For our convenience, we call $\mathcal{N}^{\beta}\circ\mathcal{N}^{\alpha}$ as product of $\mathcal{N}^{\beta}$ and $\mathcal{N}^{\alpha}$.)

Proof. Let ν, α, β, $\gamma\in R$ in the following.

(i) Closure; Since we have

$$\mathcal{N}^{\beta}\mathcal{N}^{\alpha}f=\mathcal{N}^{\beta}(\mathcal{N}^{\alpha}f)=\mathcal{N}^{\beta}f_{\alpha}=(f_{\alpha})_{\beta}=f_{\alpha+\beta}=\mathcal{N}^{\alpha+\beta}f=\mathcal{N}^{\gamma}f, \tag{5}$$

where $\alpha+\beta=\gamma\in R$, by the index law ([11] Vol. 1, pp. 41–45 & [12] pp. 52–54), if

$$\mathcal{N}^{\alpha}f=f_{\alpha}\neq 0, \tag{6}$$

hence we have formally

$$\mathcal{N}^{\beta}\mathcal{N}^{\alpha}=\mathcal{N}^{\alpha+\beta}=\mathcal{N}^{\gamma}\in\{\mathcal{N}^{\nu}\} \tag{7}$$

from (5), for $f\in\mathscr{F}$.

Note 1. We have

$$\mathcal{N}^{\alpha}f=\mathcal{N}^{\alpha}f(z)=\left(\frac{\Gamma(\alpha+1)}{2\pi i}\int_{C}\frac{d\zeta}{(\zeta-z)^{\alpha+1}}\right)f(\zeta)$$

$$=\frac{\Gamma(\alpha+1)}{2\pi i}\int_{C}\frac{f(\zeta)d\zeta}{(\zeta-z)^{\alpha+1}}=f_{\alpha}(z)=f_{\alpha} \tag{8}$$

for $f\in\mathscr{F}$.

Note 2.
$$(f_{\alpha})_{\beta}=\frac{\Gamma(\beta+1)}{2\pi i}\int_{C}\frac{f_{\alpha}(\eta)}{(\eta-z)^{\beta+1}}d\eta \tag{9}$$

$$=\frac{\Gamma(\beta+1)\Gamma(\alpha+1)}{(2\pi i)^{2}}\int_{C}f(\zeta)d\zeta\int_{C}\frac{d\eta}{(\zeta-\eta)^{\alpha+1}(\eta-z)^{\beta+1}} \tag{10}$$

$$=\frac{\Gamma(\alpha+\beta+1)}{2\pi i}\int_{C}\frac{f(\zeta)d\zeta}{(\zeta-z)^{\alpha+\beta+1}}=f_{\alpha+\beta}=\mathcal{N}^{\alpha+\beta}f, \tag{11}$$

where $|\Gamma(\alpha+\beta+1)/\Gamma(\alpha+1)\Gamma(\beta+1)|<\infty$, if $f\in\mathscr{F}$. ([11] Vol. 1).

(ii) Associative law; Since we have

$$\mathcal{N}^{\gamma}(\mathcal{N}^{\beta}\mathcal{N}^{\alpha})f=\mathcal{N}^{\gamma}(\mathcal{N}^{\beta+\alpha}f)=\mathcal{N}^{\gamma}f_{\alpha+\beta}=(f_{\alpha+\beta})_{\gamma}=f_{\alpha+\beta+\gamma}=\mathcal{N}^{\alpha+\beta+\gamma}f \tag{12}$$

and

$$(\mathcal{N}^{\gamma}\mathcal{N}^{\beta})\mathcal{N}^{\alpha}f=\mathcal{N}^{\gamma+\beta}f_{\alpha}=(f_{\alpha})_{\gamma+\beta}=f_{\alpha+\beta+\gamma}=\mathcal{N}^{\alpha+\beta+\gamma}f \tag{13}$$

by the index law, if $f_{\alpha}\cdot f_{\alpha+\beta}\neq 0$. Therefore we obtain formally

$$\mathcal{N}^{\gamma}(\mathcal{N}^{\beta}\mathcal{N}^{\alpha})=(\mathcal{N}^{\gamma}\mathcal{N}^{\beta})\mathcal{N}^{\alpha}=\mathcal{N}^{\alpha+\beta+\gamma}\in\{\mathcal{N}^{\nu}\} \tag{14}$$

from (12) and (13), for $f\in\mathring{\mathscr{F}}$.

(iii) Unit element; Since we have

$$\mathcal{N}^{\nu}\mathcal{N}^{0}f=\mathcal{N}^{\nu+0}f=f_{\nu+0}=f_{\nu}=\mathcal{N}^{\nu}f \tag{15}$$

and

$$\mathcal{N}^{0}\mathcal{N}^{\nu}f=\mathcal{N}^{0+\nu}f=f_{0+\nu}=f_{\nu}=\mathcal{N}^{\nu}f, \tag{16}$$

if $f\in\mathring{\mathscr{F}}$. Therefore we obtain

$$\mathcal{N}^{\nu}\mathcal{N}^{0}f=\mathcal{N}^{0}\mathcal{N}^{\nu}f=\mathcal{N}^{\nu}f \tag{17}$$

from (15) and (16), hence formally

$$\mathcal{N}^{\nu}\mathcal{N}^{0}=\mathcal{N}^{0}\mathcal{N}^{\nu}=\mathcal{N}^{\nu} \tag{18}$$

from (17). Therefore, we have formally

$$\mathcal{N}^{0}=1\in\{\mathcal{N}^{\nu}\}\,. \tag{19}$$

That is, $\mathcal{N}^{0}=1$ is the unit element of the set $\{\mathcal{N}^{\nu}\}$ for $f\in\mathring{\mathscr{F}}$.

(iv) Inverse element; Since we have

$$\mathcal{N}^{-\nu}\mathcal{N}^{\nu}f=\mathcal{N}^{-\nu}(\mathcal{N}^{\nu}f)=\mathcal{N}^{0}f=f \tag{20}$$

and

$$\mathcal{N}^{\nu}\mathcal{N}^{-\nu}f=\mathcal{N}^{\nu}(\mathcal{N}^{-\nu}f)=\mathcal{N}^{0}f=f, \tag{21}$$

if $\mathcal{N}^{\nu}f=f_{\nu}\neq 0$ and $\mathcal{N}^{-\nu}f=f_{-\nu}\neq 0$ respectively.

Therefore, we have formally

$$\mathcal{N}^{-\nu}\mathcal{N}^{\nu}=\mathcal{N}^{\nu}\mathcal{N}^{-\nu}=\mathcal{N}^{0}=1\,, \tag{22}$$

from (20) and (21). That is, $\mathcal{N}^{-\nu}$ and $\mathcal{N}^{\nu}$ are inverse element each other for $f\in\mathring{\mathscr{F}}$. Moreover, let $(\mathcal{N}^{\nu})^{-1}$ be the inverse element to the element $\mathcal{N}^{\nu}$, then

$$(\mathcal{N}^{\nu})^{-1}\mathcal{N}^{\nu}=\mathcal{N}^{\nu}(\mathcal{N}^{\nu})^{-1}=1\,. \tag{23}$$

Therefore, we have

$$\mathcal{N}^{-\nu}=(\mathcal{N}^{\nu})^{-1} \tag{24}$$

from (22) and (23), for $f\in\mathring{\mathscr{F}}$. And hence, we can see that

$$((\mathcal{N}^{\nu})^{-1})^{-1}=(\mathcal{N}^{-\nu})^{-1}=\mathcal{N}^{\nu} \tag{25}$$

from (24).

Notice that $(\mathcal{N}^{\nu})^{-1}=\mathcal{N}^{-\nu}$, then in our fractional calculus the inverse transform integral to $\mathcal{N}^{\nu}$ is given by (1) itself having $-\nu$ instead of ν. This fact is a very symbolic matter in our fractional calculus operator.

(v) Commutative law; Since we have

$$\mathcal{N}^{\beta}\mathcal{N}^{\alpha}f=\mathcal{N}^{\beta}(\mathcal{N}^{\alpha}f)=f_{\beta+\alpha} \tag{26}$$

and

$$\mathcal{N}^{\alpha}\mathcal{N}^{\beta}f=\mathcal{N}^{\alpha}(\mathcal{N}^{\beta}f)=f_{\alpha+\beta}, \tag{27}$$

if $\mathcal{N}^{\alpha}f=f_{\alpha}\neq 0$ and $\mathcal{N}^{\beta}f=f_{\beta}\neq 0$ respectively, therefore, we have formally

$$\mathcal{N}^{\alpha}\mathcal{N}^{\beta}=\mathcal{N}^{\beta}\mathcal{N}^{\alpha} \tag{28}$$

from (26) and (27), for $f\in\mathscr{F}$.

(vi) Continuity of index ν; Next, because $\nu\in R$, namely $-\infty<\nu<\infty$, then $\{\mathcal{N}^{\nu}\}$ is a set of $\mathcal{N}^{\nu}$ which has continuous index ν if $f\in\mathscr{F}$. Hence the set $\{\mathcal{N}^{\nu}\}$ has infinite elements $\mathcal{N}^{\nu}$ in number.

By the above exposition (i)~(vi), we have this theorem clearly.

Note 1. It is clear that $\mathcal{N}^{\nu}f\neq f\mathcal{N}^{\nu}$ $(\nu\neq 0)$. That is, only $\mathcal{N}^{\nu}f$ has mathematical meaning.

Note 2. Inspite of $\mathcal{N}^{\alpha}f=f_{\alpha}=0$, if we calculate as

$$\mathcal{N}^{\beta}\mathcal{N}^{\alpha}f=\mathcal{N}^{\beta+\alpha}f \tag{29}$$

formally, we can omit the condition $\mathcal{N}^{\alpha}f=f_{\alpha}\neq 0$. (See Note 3.)

Note 3. If we set $f=z^{m}$ $(m\in Z^{+}\cup\{0\})$, we have $\mathcal{N}^{\alpha}f=f_{\alpha}=0$ $(\alpha\notin Z)$ (See Lemma 2.), however, when we calculate as formally

$$\mathcal{N}^{\beta}\mathcal{N}^{\alpha}z^{m}=\mathcal{N}^{\beta}(\mathcal{N}^{\alpha}z^{m})=\mathcal{N}^{\beta}\left(e^{-i\pi\alpha}\frac{\Gamma(-m+\alpha)}{\Gamma(-m)}z^{m-\alpha}\right) \quad (m\in Z^{+},\ \alpha\notin Z) \tag{30}$$

$$=e^{-i\pi\alpha}\frac{\Gamma(-m+\alpha)}{\Gamma(-m)}\mathcal{N}^{\beta}z^{m-\alpha}=e^{-i\pi(\alpha+\beta)}\frac{\Gamma(-m+\alpha)}{\Gamma(-m)}\cdot\frac{\Gamma(-m+\alpha+\beta)}{\Gamma(-m+\alpha)}z^{m-\alpha-\beta} \tag{31}$$

$$=e^{-i\pi(\alpha+\beta)}\frac{\Gamma(-m+\alpha+\beta)}{\Gamma(-m)}z^{m-\alpha-\beta}=\mathcal{N}^{\alpha+\beta}z^{m}, \tag{32}$$

the index law holds.

Therefore, if we calculate as

$$\mathcal{N}^{\beta}\mathcal{N}^{\alpha}z^{m}=\mathcal{N}^{\alpha+\beta}z^{m} \quad (\text{inspite of } \mathcal{N}^{\alpha}z^{m}=0) \tag{33}$$

formally, we can omit the condition $\mathcal{N}^{\alpha}f=f_{\alpha}\neq 0$.

Therefore, by the above exposition (i)~(vi) and Note 2, we may have this theorem clearly, for the function $f\in\mathscr{F}=\{f\,|\,|f_{\nu}|<\infty,\ \nu\in R\}$. (When we calculate as Note 3.)

Note 4. From now we call the set $\{\mathcal{N}^{\nu}\}=\{\mathcal{N}^{\nu}\,|\,\nu\in R\}$ as "Fractional Calculus Operator Group for $f\in\mathscr{F}$" and denote this by "F.O.G." for our convenience. That is, "F.O.G." is an Abelian product group for $f\in\mathscr{F}$ which has continuous index ν".

§2. The set $\{\mathcal{N}^{\nu}\}$ and action group

(I) We have the following definition for action group ([22]. pp. 40–42, and pp. 113–133).

DEFINITION. Let $G=\{g\}$ be a group, and $A=\{a\}\neq\varnothing$ be a set. When the map from $G\times A=\{(g,a)\,|\,g\in G,\ a\in A\}$ to $A=\{a\,|\,a\in A\}$ satisfies the properties

(i) $g_1 \circ (g_2 a) = (g_1 \circ g_2) a$ for all $g_1, g_2 \in G$, $a \in A$,

(ii) $1 \circ a = a$ for all $a \in A$,

we say "G is a group acting on a set A". Then we call G as "action group". Obeying this definition, we have the following theorem.

THEOREM 2. *The "F.O.G. $\{\mathcal{N}^\nu\}$" is an "Action product group which has continuous index ν" for the set $\mathring{\mathscr{F}}$.*

Proof. Let $G = \{\mathcal{N}^\nu\}$, $A = \mathring{\mathscr{F}}$ and $a = f \in \mathring{\mathscr{F}}$ in the definition, we have then

$$\mathcal{N}^\beta(\mathcal{N}^\alpha f) = (\mathcal{N}^\beta \mathcal{N}^\alpha) f \qquad \text{for all} \quad \mathcal{N}^\beta, \mathcal{N}^\alpha \in \{\mathcal{N}^\nu\}, \quad f \in \mathring{\mathscr{F}},$$

and

$$\mathcal{N}^0 f = 1 f = f \qquad \text{for all} \quad f \in \mathring{\mathscr{F}}.$$

Therefore, we can see that the set $\{\mathcal{N}^\nu\}$ of our fractional calculus operator $\mathcal{N}^\nu$ is "a group acting on a set $\mathring{\mathscr{F}}$".

In more detail, "$\{\mathcal{N}^\nu\}$ is an Abelian group acting on a set $\mathring{\mathscr{F}}$". That is, "the set $\{\mathcal{N}^\nu\}$ is an Abelian action product group for the set $\mathring{\mathscr{F}}$", and for continuous index ν.

Chapter 2. Inverse of Nishimoto's integral transformation, inverse of Goursat's transformation and that of Cauchy's one

§ 1. A complex integral transformation and its inverse

THEOREM 1. *Let Nishimoto's complex integral transformation be*

$$\mathfrak{N}\{f(\zeta)\} = \frac{\Gamma(\mu+1)}{2\pi i} \int_C \frac{f(\zeta)}{(\zeta - z)^{\mu+1}} d\zeta = F(z), \tag{1}$$

for a given constant $\mu \in R$, then the inverse to $F(z)$ is given by

$$\mathfrak{N}^{-1}\{F(z)\} = \frac{\Gamma(-\mu+1)}{2\pi i} \int_C \frac{F(z)}{(z-\zeta)^{-\mu+1}} dz, \tag{2}$$

where $f(\zeta)$ is a regular function in D and $0 \neq |F(z)| < \infty$. (For the integral contour C and the domain D of formula (1)*, see the definition in* §0. *And when $\mu = -n (n \in Z^+)$ in* (1)*, refer to* §0. (2). *For the integral contour C and the domain D of formula* (2) *see the Fig.* 1′ *and Fig.* 2′.)

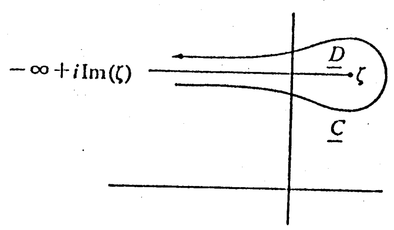

Fig. 1′.

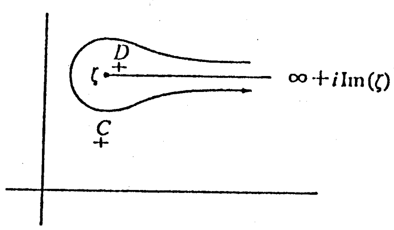

Fig. 2′.

Proof. Substitute (1) into the right hand side of (2), we have then

$$\text{R.H.S. of (2)} = \frac{\Gamma(-\mu+1)\Gamma(\mu+1)}{(2\pi i)^2} \int_C \frac{dz}{(z-\zeta)^{-\mu+1}} \int_C \frac{f(\eta)}{(\eta - z)^{\mu+1}} d\eta \tag{3}$$

$$=\frac{\Gamma(-\mu+1)\Gamma(\mu+1)}{(2\pi i)^2}\int_C f(\eta)d\eta\int_C \frac{1}{(\eta-z)^{\mu+1}(z-\zeta)^{-\mu+1}}dz\,. \tag{4}$$

Putting $z-\zeta=w$ and $\eta-\zeta=p\equiv\delta e^{i\phi}$ (δ, $\phi\in R$. $\phi=\arg p$ and $\delta\neq 0$) we have then

$$\eta-z=(\eta-\zeta)-(z-\zeta)=p-w\,.$$

Next set $w=pu$ ($u=w/p=(1/\delta)e^{-i\phi}w$, $|\phi|<\pi/2$), then we have (for $C=\underset{-}{C}$)

$$\int_{\underset{-}{C}}\frac{1}{(\eta-z)^{\mu+1}(z-\zeta)^{-\mu+1}}dz=\int_{-\infty}^{(0+)}\frac{1}{(p-w)^{\mu+1}w^{-\mu+1}}dw \tag{5}$$

$$=\frac{1}{p}\int_{-\infty e^{-i\phi}}^{(0+)}u^{-(-\mu+1)}(1-u)^{-(\mu+1)}du \tag{6}$$

$$=\frac{1}{p}\int_{-\infty}^{(0+)}u^{-(-\mu+1)}(1-u)^{-(\mu+1)}du \qquad (\text{for } |\phi|<\pi/2) \tag{7}$$

$$=\frac{1}{p}\cdot\frac{2\pi i}{\Gamma(-\mu+1)\Gamma(\mu+1)} \qquad (\text{for } -\mathrm{Re}(\mu+1)<\mathrm{Re}(-\mu)<0)\,. \tag{8}$$

Substituting (8) into (4), we obtain

$$(4)=\frac{1}{2\pi i}\int_C\frac{f(\eta)}{\eta-\zeta}d\eta=\frac{1}{2\pi i}\oint\frac{f(\eta)}{\eta-\zeta}d\eta=f(\zeta)\,. \tag{9}$$

In case of $\pi/2<|\phi|\leqslant\pi$, we have (for $C=\underset{+}{C}$)

$$\int_{\underset{+}{C}}\frac{1}{(\eta-z)^{\mu+1}(z-\zeta)^{-\mu+1}}dz=\int_{\infty}^{(0+)}\frac{1}{(p-w)^{\mu+1}w^{-\mu+1}}dw \tag{10}$$

$$=\frac{1}{p}\int_{\infty e^{-i\phi}}^{(0+)}u^{-(-\mu+1)}(1-u)^{-(\mu+1)}du \tag{11}$$

$$=\frac{1}{p}\int_{-\infty}^{(0+)}u^{-(-\mu+1)}(1-u)^{-(\mu+1)}du \qquad (\text{for } \pi/2<|\phi|\leqslant\pi) \tag{12}$$

$$=\frac{1}{p}\cdot\frac{2\pi i}{\Gamma(-\mu+1)\Gamma(\mu+1)} \qquad (\text{for } -\mathrm{Re}(\mu+1)<\mathrm{Re}(-\mu)<0)\,. \tag{13}$$

Substituting (13) into (4), we obtain (9) again.
Therefore, we have

$$\frac{\Gamma(-\mu+1)}{2\pi i}\int_C\frac{F(z)}{(z-\zeta)^{-\mu+1}}dz=f(\zeta)\,, \tag{14}$$

that is,

$$\mathfrak{N}^{-1}\{F(z)\}=f(\zeta)\,, \tag{15}$$

when

$$\mathfrak{N}\{f(\zeta)\}=F(z)\,, \tag{1}$$

for $0\neq|F(z)|<\infty$.

THEOREM 2. *We have formally*

$$\mathfrak{N}^{-1}\mathfrak{N}=\mathfrak{N}\mathfrak{N}^{-1}=1 \tag{16}$$

for $F(z) \neq 0$,

where

$$F(z)=\mathfrak{N}\{f(\zeta)\}. \tag{1}$$

Proof. We have

$$\mathfrak{N}^{-1}\{F(z)\}=\mathfrak{N}^{-1}\{\mathfrak{N}\{f(\zeta)\}\}=\mathfrak{N}^{-1}\mathfrak{N}f(\zeta), \tag{17}$$

since

$$F(z)=\mathfrak{N}\{f(\zeta)\}. \qquad (F(z)\neq 0) \tag{1}$$

Therefore, we have formally

$$\mathfrak{N}^{-1}\mathfrak{N}=1 \tag{18}$$

from (15) and (17).

Next we have

$$\mathfrak{N}\{f(\zeta)\}=\mathfrak{N}\{\mathfrak{N}^{-1}\{F(z)\}\}=\mathfrak{N}\mathfrak{N}^{-1}F(z), \tag{19}$$

since

$$f(\zeta)=\mathfrak{N}^{-1}\{F(z)\}. \qquad (F(z)\neq 0) \tag{15}$$

Therefore, we have formally

$$\mathfrak{N}\mathfrak{N}^{-1}=1 \tag{20}$$

from (1) and (19). Hence we obtain (16) from (18) and (20) formally, under the condition.

Note. Notice that $\mathfrak{N}\{f(\zeta)\}=f_\mu=F(z)$. (Refer to §0.(1) and (2))

§2. Inverse of Goursat's transformation and of Cauchy's one

(I) Inverse of Goursat's transformation

COROLLARY 1. *Let a complex integral transformation be*

$$\mathfrak{N}\{f(\zeta)\}=\frac{\Gamma(n+1)}{2\pi i}\oint\frac{f(\zeta)}{(\zeta-z)^{n+1}}d\zeta=F(z), \tag{1}$$

for a given constant $n\in Z^+$, *then the inverse to* $F(z)$ *is given by*

$$\mathfrak{N}^{-1}\{F(z)\}=\frac{\Gamma(-n+1)}{2\pi i}\oint\frac{F(z)}{(z-\zeta)^{-n+1}}dz, \qquad (\textit{see } §0.\ (2)) \tag{2}$$

under the same conditions with that of Theorem 1, *where* $\oint$ *means a complex contour integration along a closed simple Jordan curve which surrounds* z *for formula* (1) *and* ζ *for formula* (2).

Proof. When $\mu=n$ $(n\in Z^+)$ we have

$$\mathfrak{N}\{f(\zeta)\}=\frac{\Gamma(n+1)}{2\pi i}\int_C\frac{f(\zeta)}{(\zeta-z)^{n+1}}d\zeta$$

$$=\frac{n!}{2\pi i}\oint\frac{f(\zeta)}{(\zeta-z)^{n+1}}d\zeta=f^{(n)}(z)=F(z) \tag{3}$$

from § 1. (1), if $f(\zeta)$ is a regular function in D.

And we have

$$\mathfrak{N}^{-1}\{F(z)\}=\frac{\Gamma(-n+1)}{2\pi i}\int_C\frac{F(z)}{(z-\zeta)^{-n+1}}dz=\frac{\Gamma(-n+1)}{2\pi i}\oint\frac{F(z)}{(z-\zeta)^{-n+1}}dz \tag{4}$$

$$=\lim_{\mu\to n}\left(\frac{\Gamma(-\mu+1)}{2\pi i}\int_C\frac{F(z)}{(z-\zeta)^{-\mu+1}}dz\right) \tag{5}$$

from § 1. (2), having $\mu=n$ ($n\in Z^+$).

Formula (1) is Goursat's transformation, hence formula (2) is inverse Goursat's transformation.

(II) Inverse of Cauchy's transformation

COROLLARY 2. *Let a complex integral transformation be*

$$\mathfrak{N}\{f(\zeta)\}=\frac{1}{2\pi i}\oint\frac{f(\zeta)}{\zeta-z}d\zeta=f(z)=F(z), \tag{6}$$

then the inverse to $F(z)$ *is given by*

$$\mathfrak{N}^{-1}\{F(z)\}=\frac{1}{2\pi i}\oint\frac{F(z)}{z-\zeta}dz \tag{7}$$

under the same conditions with that of Theorem 1.

Proof. Set $\mu=0$ in Theorem 1.

Formula (6) is Cauchy's transformation, hence formula (7) is inverse Cauchy's transformation.

Chapter 3. Fractional calculus method to extended linear ordinary differential equations of Fuchs type

§ 1. Solutions to a Nishimoto's linear third order ordinary differential equation of Fuchs type

In a previous paper the following theorems were shown by the author [8].

THEOREM A. *If* $f\in\mathscr{F}$ *and* $f_{-\lambda}\neq 0$, *then the generalized nonhomogeneous linear third order ordinary differential equation of Fuchs type*

$L[\varphi(z), a, b, c, d, A, B, C, \lambda]$

$$=\varphi_3\cdot v+\varphi_2\cdot(\lambda v_1+g)+\varphi_1\cdot\left\{\lambda(\lambda-1)\frac{1}{2}v_2+\lambda g_1\right\}+\varphi\cdot\lambda(\lambda-1)\frac{1}{2}g_2=f \quad (v\neq 0) \tag{1}$$

has a particular solution of the form

$$\varphi=\left(\left(f_{-\lambda}\cdot\frac{1}{v}\cdot e^{P(z)}\right)_{-1}\cdot e^{-P(z)}\right)_{\lambda-2}, \tag{2}$$

where

$$v=v(z)=(zc-a)(zd-b), \tag{3}$$

$$g=g(z)=z^2A+zB+C, \tag{4}$$

$$P(z)=(g/v)_{-1}\,, \tag{5}$$

$\varphi=\varphi(z)$ $(z\in C)$, $f=f(z)$ (*an arbitrary given function*), *and* a, b, c, d, A, B, C *and* λ *are given constants.*

THEOREM B. *The generalized homogeneous linear third order ordinary differential equation of Fuchs type*

$$L[\varphi(z), a, b, c, d, A, B, C, \lambda]=0 \qquad (v \neq 0) \tag{6}$$

has solutions of the form

$$\varphi=K(e^{-P(z)})_{\lambda-2}\,, \tag{7}$$

where $P(z)$ *is the one shown by* (5) *and* K *is an arbitrary constant.*

THEOREM C. *If* $f\in\mathscr{F}$ and $f_{-\lambda}\neq 0$, *then the fractional differintegrated functions*

$$\varphi=\left(\left(f_{-\lambda}\cdot\frac{1}{v}\cdot e^{P(z)}\right)_{-1}\cdot e^{-P(z)}\right)_{\lambda-2}+K(e^{-P(z)})_{\lambda-2} \qquad (v\neq 0) \tag{8}$$

satisfy the generalized nonhomogeneous differential equation (1). *Where* $P(z)$ *is the function shown by* (5), *and* K *is an arbitrary constant.*

Note. Letting

$$\frac{g}{v}=\frac{z^2A+zB+C}{(zc-a)(zd-b)}=p+\frac{q}{zc-a}+\frac{r}{zd-b} \qquad \left(z\neq\frac{a}{c},\frac{b}{d}\right), \tag{9}$$

we obtain

$$p=A/cd \qquad (cd\neq 0)\,, \tag{10}$$

$$q=\left(A\cdot\frac{a^2}{c}+Ba+Cc\right)\Big/(ad-bc) \qquad (ad\neq bc)\,, \tag{11}$$

$$r=-\left(A\cdot\frac{b^2}{d}+Bb+Cd\right)\Big/(ad-bc) \quad (ad\neq bc)\,. \tag{12}$$

Hence it follows that

$$P(z)=\left(\frac{g}{v}\right)_{-1}=pz+\log\{(zc-a)^{q/c}\cdot(zd-b)^{r/d}\} \tag{13}$$

from (9). Therefore we have

$$\varphi=((f_{-\lambda}\cdot e^{pz}(zc-a)^{(q/c)-1}\cdot(zd-b)^{(r/d)-1})_{-1}\cdot e^{pz}\cdot(zc-a)^{-q/c}\cdot(zd-b)^{-r/d})_{\lambda-2} \tag{14}$$

for $cd\neq 0$, $ad\neq bc$, from (2), where p, q and r are the ones shown by (10), (11) and (12) respectively.

§2. Extension of the Theorems A, B and C [9]

With the help of Nishimoto's fractional calculus ([11]~[12]) we can derive the following theorems for more generalized linear higher order ordinary differential equation than the one which are shown in §1.

THEOREM 1. *If* $f\in\mathscr{F}$ *and* $f_{-\lambda}\neq 0$, *then the extended nonhomogeneous differintegral* equation of Fuchs type (see Note 3 for*)*

$$L[\varphi(z), m, n; a, b, c, d; A, B, C; \lambda]$$

$$=\varphi_m\cdot v+\sum_{k=1}^{n}\varphi_{m-k}\{G(\lambda, k)v_k+G(\lambda, k-1)g_{k-1}\}+\varphi_{m-n-1}\cdot G(\lambda, n)g_n=f \qquad (v\neq 0) \tag{1}$$

has a particular solution of the form

$$\varphi=\left(\left(f_{-\lambda}\cdot\frac{1}{v}\cdot e^{P(z)}\right)_{-1}\cdot e^{-P(z)}\right)_{\lambda-m+1}, \tag{2}$$

where

$$v=v(z, n)=z^n cd-z^{n-1}(bc+ad)+z^{n-2}ab, \tag{3}$$

$$g=g(z, n)=z^n A+z^{n-1}B+z^{n-2}C, \tag{4}$$

$$P(z)=(g/v)_{-1} \qquad (v\neq 0), \tag{5}$$

$$G(\lambda, k)=\frac{\Gamma(\lambda+1)}{\Gamma(\lambda+1-k)\Gamma(k+1)}, \tag{6}$$

$m\in Z$, $(n-2)\in Z^+\cup\{0\}$, $z\in C$, $\varphi_0=\varphi=\varphi(z)$, $f=f(z)$ *is known (an arbitrary given function), and* a, b, c, d, A, B, C *and* λ *are given constants.*

Proof. Letting

$$\varphi=w_\lambda, \tag{7}$$

(1) becomes

$$w_{m+\lambda}\cdot v+\sum_{k=1}^{n}w_{m-k+\lambda}\{G(\lambda, k)v_k+G(\lambda, k-1)g_{k-1}\}+w_{m-n-1+\lambda}G(\lambda, n)g_n=f \qquad (v\neq 0). \tag{8}$$

Here we have

$$(w_m\cdot v)_\lambda=\sum_{k=0}^{n}G(\lambda, k)w_{m+\lambda-k}v_k \tag{9}$$

and

$$(w_{m-1}\cdot g)_\lambda=\sum_{k=0}^{n}G(\lambda, k)w_{m-1+\lambda-k}g_k=\sum_{k=1}^{n+1}G(\lambda, k-1)w_{m+\lambda-k}g_{k-1}, \tag{10}$$

since

$$(w_m\cdot z^n)_\lambda=\sum_{k=0}^{n}G(\lambda, k)w_{m+\lambda-k}(z^n)_k \qquad (n\in Z^+\cup\{0\}) \tag{11}$$

(by the generalized Leibnitz rule).

Substituting (9) and (10) into the left hand side of (8), yields

$$(w_m\cdot v)_\lambda+(w_{m-1}\cdot g)_\lambda=f, \tag{12}$$

hence

$$w_m+w_{m-1}\cdot\frac{g}{v}=f_{-\lambda}\cdot\frac{1}{v}. \tag{13}$$

Set

$$w_{m-1}=u=u(z)\,, \tag{14}$$

we have then

$$u_1+u\cdot\frac{g}{v}=f_{-\lambda}\cdot\frac{1}{v}\,. \tag{15}$$

A particular solution to this linear first order ordinary differential equation is given by

$$u=\left(f_{-\lambda}\cdot\frac{1}{v}\cdot e^{(g/v)_{-1}}\right)_{-1}\cdot e^{-(g/v)_{-1}}\,. \tag{16}$$

Thus we obtain (2) from (16), (14) and (7).
Inversely, substituting (2) into the left hand side of (1), we obtain

$$\text{L.H.S. of (1)}=u_{1+\lambda}v+\sum_{k=1}^{n}u_{1+\lambda-k}\{G(\lambda,k)v_k+G(\lambda,k-1)g_{k-1}\}+u_{\lambda-n}\cdot G(\lambda,n)g_n \tag{17}$$

$$=\sum_{k=0}^{n}G(\lambda,k)(u_{1+\lambda-k}\cdot v_k+u_{\lambda-k}\cdot g_k)$$

$$=(u_1\cdot v+u\cdot g)_\lambda \tag{18}$$

$$=\left(f_{-\lambda}\cdot\frac{1}{v}\cdot e^{P(z)}\cdot e^{-P(z)}\cdot v+\left(f_{-\lambda}\cdot\frac{1}{v}e^{P(z)}\right)_{-1}\cdot(-P_1(z))e^{-P(z)}\cdot v\right.$$

$$\left.+\left(f_{-\lambda}\cdot\frac{1}{v}e^{P(z)}\right)_{-1}\cdot e^{-P(z)}\cdot g\right)_\lambda \tag{19}$$

$$=(f_{-\lambda})_\lambda$$

$$=f, \tag{20}$$

since

$$P(z)=(g/v)_{-1}\qquad(v\neq 0)\,. \tag{5}$$

This Theorem 1 is an extended one from Theorem A.

Note 1. Here we used the notations v_k and g_k instead of $(v(z,n))_{k(z)}$ and $(g(z,n))_{k(z)}$ respectively for the sake of our convenience.

For the notation $(f(z_1,z_2))_{v_k(z_k)}$ (which means a fractional and partial differintegration of arbitrary order v_k, with respect to z_k, of the function $f=f(z_1,z_2)$) refere to ([11], Vol. 3) and ([12]pp. 160–163) for example.

2. Remember that v and g contain n respectively in (2).
3. For $m\geqq n+1$, (1) is a differential equation,
for $n+1>m>0$, (1) is a differintegral equation, and
for $m\leqq 0$, (1) is an integral equation.

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