数理解析研究所講究録
第 891 巻 1995 年 157-167

Solutions to a Singular Diffusion Equation

陳 蘊剛 (Yun-Gang Chen)
儀我 美一 (Yoshikazu Giga)
佐藤 剛 (Koh Sato)

§1. Introduction

We consider the following equation

$$u_t = |\,u_x|^{-\alpha}u_{xx}, \qquad (t,x)\in(0,T)\times(0,1) \quad \text{(E1)}$$
$$u(t,0)=u(t,1)=0, \qquad t\in[0,T] \quad \text{(E2)}$$
$$u(0,x)=u_0(x), u_0(x)\geq 0, \quad 0<x<1 \quad \text{(E3)}$$

where $\alpha \geq 0$ and u_0 is a smooth function on $(0,1)$. We may recognize this equation as a diffusion equation whose diffusion coefficient is $|u_x|^{-\alpha}$. So if there is a point where u_x is close to 0, then we can guess that very strong diffusion would be happen at that point. This is a simple example of p-Laplace equations. We refer to [EDB] for general regularity properties of solutions.

If we differentiate both sides of (E1)(with respect to x), and set $v=u_x$, then the equation would be described as follows

$$v_t = c(|v|^{p-2}v)_{xx} \quad \text{(P)}$$

where $p=2-\alpha$ and c is a certain constant. The property of the equation depends on p. If $p>2$, this equation is called porous medium equation which presents a model of the diffusion in porous media. In the case of $1<p<2$, the equation is called plasma equation since it was derived from the model describing the behavior of the plasma in strong magnetic field. The later case (which corresponds to the case of $0<\alpha<1$ in (E1)), Berryman and Holland showed that all positive solution to the equation (P) vanishes in finite time (i.e. $\exists t_*$ such that $v(t,\cdot)\to 0$ as $t\to t_*$) under the Dirichlet boundary condition. Furthermore, the profile of each solution tends to that of a certain separable solution as $t\to t_*$ ([B],[BH1]).

But those results on (P) can not be applied directly to our problem. Because boundary conditions are different and another restriction ($\int_0^1 v(x)dx=0$) is required (so $v(x)$ must be negative in some interval).

We first look for non-negative separable solutions of (E1)-(E2) (§2), then we construct a stable difference scheme which approximates (E1) (§3). The result of the numerical experiment gives us a hint that the solutions to (E1)-(E3) also vanishes in finite time (We shall prove this fact in our forthcoming paper [OS]). We apply a rescaling technique to the scheme to obtain more precise value of the vanishing time and the asymptotic profile of the solutions (§4).

§2. Separable Solution

First, we look for a non-negative separable solution $u(t,x) = U(x) \cdot T(t)$ of

$$u_t | u_x|^\alpha = u_{xx} \quad (2.1)$$

which is derived from (E1), where $U(x)$ and $T(t)$ are assumed to be non-negative C^2 *functions*. Thus we get the following equations,

$$T^{\alpha-1}(t)T'(t) = U^{-1}(x)|U'(x)|^{-\alpha}U''(x) = -c \quad (S)$$

where $c > 0$, since $U''(x) \leq 0$ can be obtained from $U(0) = U(1) = 0$ and $U \geq 0$.

Then we are led to the following equations for $U(x)$ and $T(t)$

$$\begin{aligned} &T(t)^{\alpha-1} \cdot T'(t) = -c && (S1) \\ &U''(x) = -cU(x)|U'(x)|^\alpha. && (S2) \end{aligned}$$

Note if we assume $\tilde{U}(x) = \beta U(x)$ and $\tilde{T}(t) = \beta^{-1}T(t)$ where β is a positive constant, then $U(x)T(t) = \tilde{U}(x)\tilde{T}(t)$ and

$$\tilde{T}^{\alpha-1}(t)\tilde{T}'(t) = \tilde{U}^{-1}(x)|\tilde{U}'(x)|^{-\alpha}\tilde{U}''(x) = -\beta^{-\alpha}c.$$

Now, we may assume $c = 1/\alpha$ without loss of generality. Then from (S1), we can easily see that the separable solution can be written as follows

$$u(t,x) = (t_* - t)^{1/\alpha}U(x)$$

where $t_* > 0$ is the vanishing time and $U(x)$ is a solution of the following equation

$$\begin{aligned} &U''(x) = -\frac{1}{\alpha}U(x)|U'(x)|^\alpha, \quad U(x) > 0, \quad 0 < x < 1 && \text{(D1)} \\ &U(0) = U(1) = 0. && \text{(D2)} \end{aligned}$$

Thus, we can find $U(x)$ from (D1), (D2) by the following way.

Proposition. Suppose $V(x) \geq 0$ satisfies the following equations

$$\begin{aligned} &V''(x) = -\frac{1}{\alpha}V(x)(V'(x))^{\alpha}, \ V'(x) \geq 0, \ 0 \leq x \leq 1/2 && \text{(V1)} \\ &V(0) = 0 && \text{(V2)} \\ &V'(1/2) = 0. && \text{(V3)} \end{aligned}$$

Then

$$U(x) = \begin{cases} V(x), & 0 \leq x \leq 1/2 \\ V(1-x), & 1/2 < x \leq 1 \end{cases}$$

is a symmetric solution of (D1), (D2).

Proof. Multiply both sides of (V1) by $(V'(x))^{1-\alpha}$ and integrate them from 0 to x. Then with a standard calculation, we can obtain the following

$$V'(x) = (V'(0)^{2-\alpha} - c_\alpha V(x)^2)^{\frac{1}{2-\alpha}} \quad (2.2)$$

where $c_\alpha = \dfrac{2-\alpha}{2\alpha}$, and get

$$V(x) = V'(0)^{1-\frac{\alpha}{2}} c_\alpha^{-\frac{1}{2}} W^{-1}(V'(0)^{\frac{\alpha}{2}} c_\alpha^{\frac{1}{2}} x).$$

Here $W^{-1}(x)$ is the inverse function of a non-decreasing function W such that

$$W(y) := \int_0^y (1-s^2)^{-\frac{1}{2-\alpha}} ds, \ 0 \leq y \leq 1.$$

We note that the integral is convergent at $y = 1$ and we put $W(1) = M_\alpha(< \infty)$. But this solution only satisfies (V1) and (V2) in a certain interval not necessarily $[0, 1/2]$. To satisfy (V3), V has to attain its maximum value at $x_* \leq 1/2$. We can express the x_* at which $V(x)$ attains its maximum as follows

$$x_* = V'(0)^{-\alpha/2} c_\alpha^{-1/2} M_\alpha.$$

And if $V'(0)$ is sufficiently large so that $x_* \leq 1/2$, we can write the solution to (V1)-(V3) as follows

$$V(x) = \begin{cases} V'(0)^{1-\frac{\alpha}{2}} c_\alpha^{\frac{1}{2}} W^{-1}(V'(0)^{\frac{\alpha}{2}} c_\alpha^{\frac{1}{2}} x), & 0 \leq x \leq x_* \\ V'(0)^{1-\alpha/2} c_\alpha^{1/2}, & x_* < x \leq 1/2. \end{cases}$$

Moreover, since $V'(x_*) = V''(x_*) = 0$ (because $\frac{d}{dx}W^{-1}(M) = 0$), we obtain $U(x) \in C^2(0,1)$. Obviously $U(x)$ satisfies (D1) in $(0, 1/2)$ and $U(x)$ satisfies (D2). Since the following equations

$$\frac{d}{dx}V(1-x) = -V'(1-x), \ \frac{d^2}{dx^2}V(1-x) = V''(1-x)$$

hold, $U(x)$ satisfies (D1) also in $(1/2, 1)$.

Remark. $U(x)$ can be the profile of the separable solution of (E1)-(E2) (not (2.1)) only if $V(x)$ attains its unique maximum value at $x = 1/2$. We can show this, for instance, by using the theory of viscosity solution.

§3 Difference Scheme

To calculate the numerical solution of (E), we introduce a modified equation (E′) to overcome the difficulty in computing which occurs when the value of the u_x in (E1) reaches 0.

$$u_t = |\, u_x^2 + \delta|^{-\alpha/2} u_{xx}, \quad (t,x) \in (0,T) \times (0,1), \quad (\mathrm{E}')$$

This equation is an approximation of (E1).

Now we introduce our difference scheme for (E′).

$$\frac{u_j^{n+1} - u_j^n}{\tau} = \{(\frac{u_{j+1}^n - u_{j-1}^n}{2h})^2 + \delta\}^{-\alpha/2} \cdot \frac{u_{j+1}^{n+1} - 2u_j^{n+1} + u_{j-1}^{n+1}}{h^2}, \quad (3.1)$$

$$u_0^n = u_N^n = 0, \quad (3.2)$$

$$u_j^0 = u_0(jh), \quad (3.3)$$

$$1 \le j \le N-1, n \ge 0,$$

where N is the number of meshes, $h = 1/N$ is the mesh size, $\tau > 0$ is the discrete time increment, and u_j^n is the value of the numerical solution at net point $(n\tau, jh) \in [0,T] \times [0,1]$.

We can show that the difference scheme (3.1)-(3.3) has L^∞-stability.

Proposition. (Stability of the scheme) Let $\{u_j^n\}$ be the solution of (3.1)-(3.3). Then $\|u^n\|_\infty \le \|u^0\|_\infty$ for $\forall n > 0$ where $\|u^n\|_\infty = \max_j |u_j^n|$.

Proof. We prove it by showing the following inequalities hold.

$$\max_j \ u_j^n \le \max_j \ u_j^0 \quad (3.4)$$

$$\min_j \ u_j^n \ge \min_j \ u_j^0. \quad (3.5)$$

First, we rewrite (3.3) as follows

$$-\lambda_j^n u_{j+1}^{n+1} + (1+2\lambda_j^n)u_j^{n+1} - \lambda_j^n u_{j-1}^{n+1} = u_j^n$$

where

$$\lambda_j^n = \{(\frac{u_{j+1}^n - u_{j-1}^n}{2h})^2 + \delta\}^{-\alpha/2} \cdot \frac{\tau}{h^2}.$$

Suppose for a fixed n, $u_m^n = \max_j u_j^n$. Then we can easily see that

$$\begin{aligned} \sup_j u_j^n &= u_m^n \\ &\leq -\lambda_m^{n-1} u_{m+1}^n + (1+2\lambda_m^{n-1})u_m^n - \lambda_{m-1}^{n-1} u_{m-1}^n \\ &= u_m^{n-1} \\ &\leq \sup_j u_j^{n-1} \end{aligned}$$

and thus we can obtain (3.4). (3.5) can be shown in the same way.

Such a stability results is proved in [CGHH] for a singular equation related to a level set method for geometric evolutions in [CGG]. However our equation (E1) is not included in [CGHH].

§4 Rescaling

We computed the numerical solution for (3.1)-(3.3) with several cases of (E1). From the computation, we can see that the solution to (E1)-(E3) vanishes in finite time. So we tried to calculate $u(t,x)$ more accurately especially when it vanishes, by using a "rescaling" approach for u and variable time increment τ in the following way.

Suppose

$$v(t,x) = Mu(M^{-\alpha}t, x). \quad (3.6)$$

It is easy to see that if $u(t,x)$ satisfies (2.1) then so does $v(t,x)$. So we observe the value of $\|u^n\|_\infty$ at every time step. If the value become smaller than the given threshold (for instance 1/2), then we rescale u and τ by (3.6) with $M = \|u^n\|_\infty^{-1}$ (i.e. multiply u_j^n by M and τ by $M^{-\alpha}$), and continue to calculate the values of the solution for the next time step, watching in the same way. Thus, we can keep the value of δ small enough compared with u, and τ would get smaller and smaller as t_n gets close to the vanishing time.

In this way, we can obtain more accurate value of u and its vanishing time numerically.

For semilinear heat equations, such a rescaling technique is applied in [Ch].

§5. Result of Numerical Experiment

We computed the asymptotic behavior of the solutions to (E1)-(E3) by the scheme (3.1)-(3.3) with $h = 1/64$ $(N = 64)$, $\tau = 2.5 \cdot h^2$ and $\delta = 10^{-100}$. Note that we took 0.5 as a threshold of the rescaling, so if $\|u(t,\cdot)\|_\infty < 1/2$ then rescaling would be done, so the $u(t,\cdot)$ would be magnified and the value of τ becomes smaller. We present rescaled profiles of u for 3 different initial data with $\alpha = 0.5$ (Figure 1 - 3). Each figure contains its initial data u_0 (labeled `t0`), and rescaled profile of $u(t,\cdot)$ (labeled `t1`-`t5`). The profile `t`j is obtained by rescaling j times of original picture. The time T_j of the profile `t`j is displayed below the each figures.

Initial data for each figures are as follows.

Figure 1. $u_0(x) = \sin \pi x$

Figure 2. $u_0(x) = \dfrac{64}{27}x^3(1-x)$

Figure 3. $u_0(x) = \dfrac{16}{15}\mu(\mu(x))$ where $\mu(x) = \dfrac{15}{4}x(1-x)$.

The rescaled profile of numerical solutions to (E1)-(E3) can be seen to approach asymptotically to a certain profile.

All computed asymptotic profiles of u with different initial data (plotted in Figure 1 - 3) coincide, at least in plotting resolution as shown in Figure 4.

Furthermore, the asymptotic profile (labeled `pde`) also coincide with the numerical solution for (D1)-(D2) (labeled `ode`) obtained by solving (V1) - (V3) by shooting method (Figure 5).

Reference

[B] J.G.Berryman, *Evolution of a stable profile for a class of nonlinear diffusion equations with fixed boundaries,* Journal of Mathematical Physics, 18(1977),2108-2115.

[BH1] J.G.Berryman and C.J.Holland, *Evolution of a stable profile for a class of nonlinear diffusion equations II,* Journal of Mathematical Physics, 19(1978),2476-2480.

[Ch] Y.-G. Chen, *Asymptotic behaviors of blowing-up solutions for finite difference analogue of* $u_t = u_{xx} + u^{1+\alpha}$, J. Fac. Sci. Univ. Tokyo, Sect. IA, Math. 33(1986), 541-574.

[CGG] Y.-G. Chen, Y. Giga and S.Goto, *Uniqueness and existence of viscosity solutions of generalized mean curvature flow equations,* Journal of Differential Geometry, 33(1991),749-786.

[CGHH] Y.-G. Chen, Y. Giga, T.Hitaka and M. Honma *A stable Difference Scheme for Computing Motion of Level Surfaces by the Mean Curvature,* preprint.

[EDB] E.DiBenedetto, *Degenerate Parabolic Equations,* Springer-Verlag, 1993.

[OS] M. Ohnuma and K. Sato, *Remarks on Degenerate Parabolic Equations,* in preparation.

Hokkaido Tokai University

Sapporo 005, Japan

and

Department of Mathematics

Hokkaido University

Sapporo 060, Japan

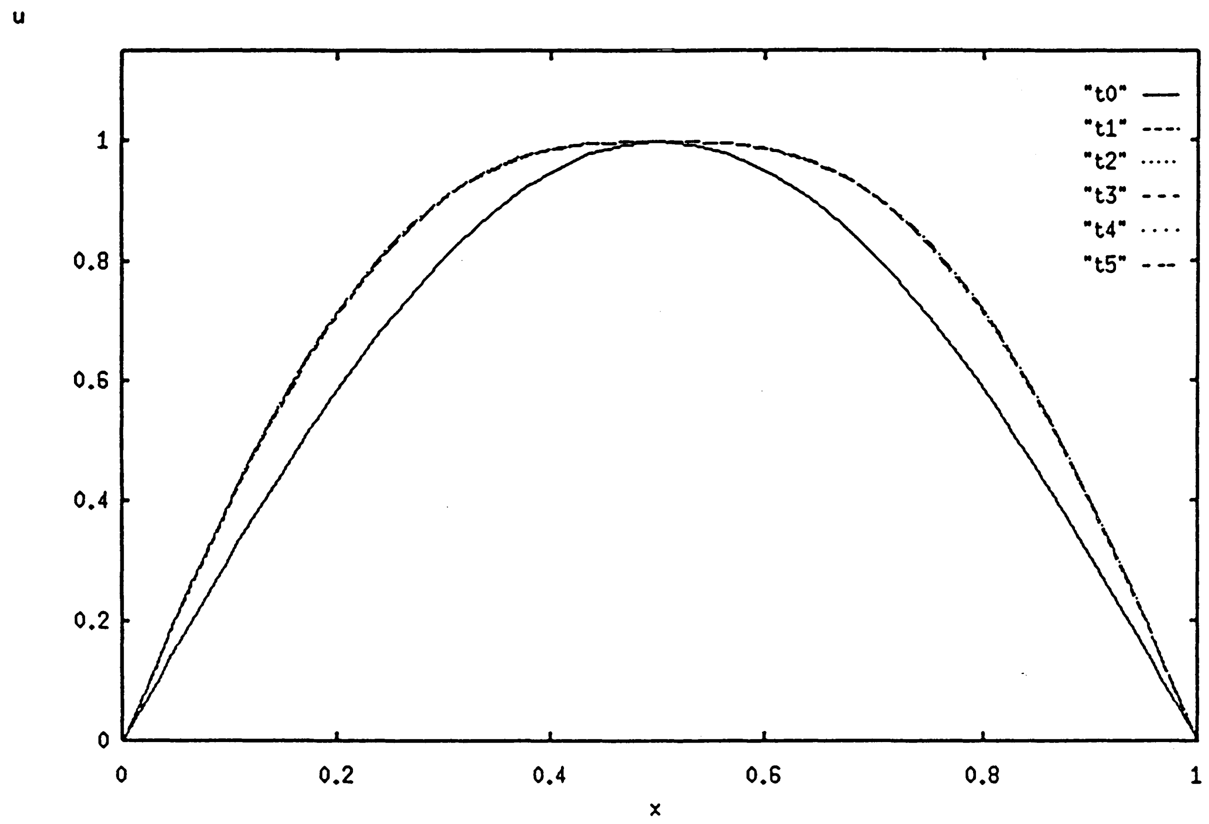

Figure 1.

$$
\begin{aligned}
T_0 &= 0\\
T_1 &= 0.04516602\\
T_2 &= 0.08173087\\
T_3 &= 0.1078698\\
T_4 &= 0.1262872\\
T_5 &= 0.1392649
\end{aligned}
$$

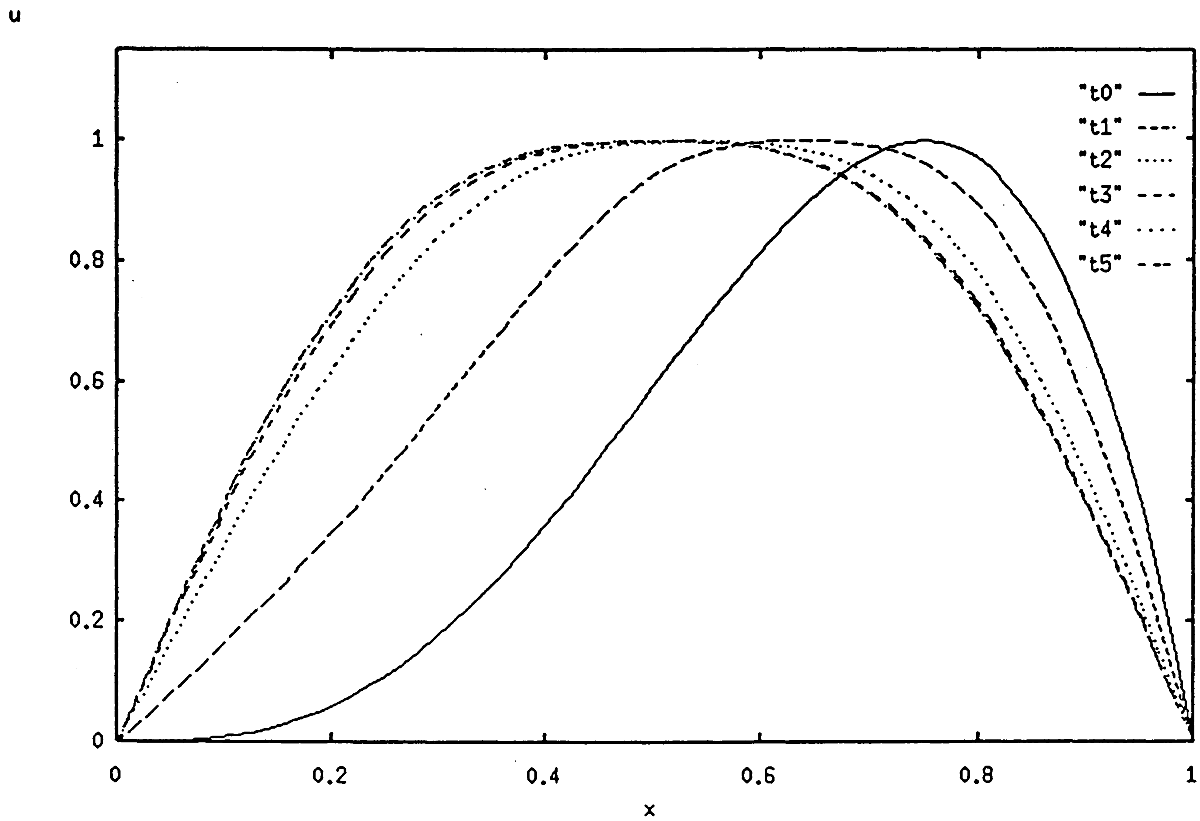

Figure 2.

$$
\begin{aligned}
T_0 &= 0\\
T_1 &= 0.02868652\\
T_2 &= 0.06010765\\
T_3 &= 0.08589717\\
T_4 &= 0.1042887\\
T_5 &= 0.1170933
\end{aligned}
$$

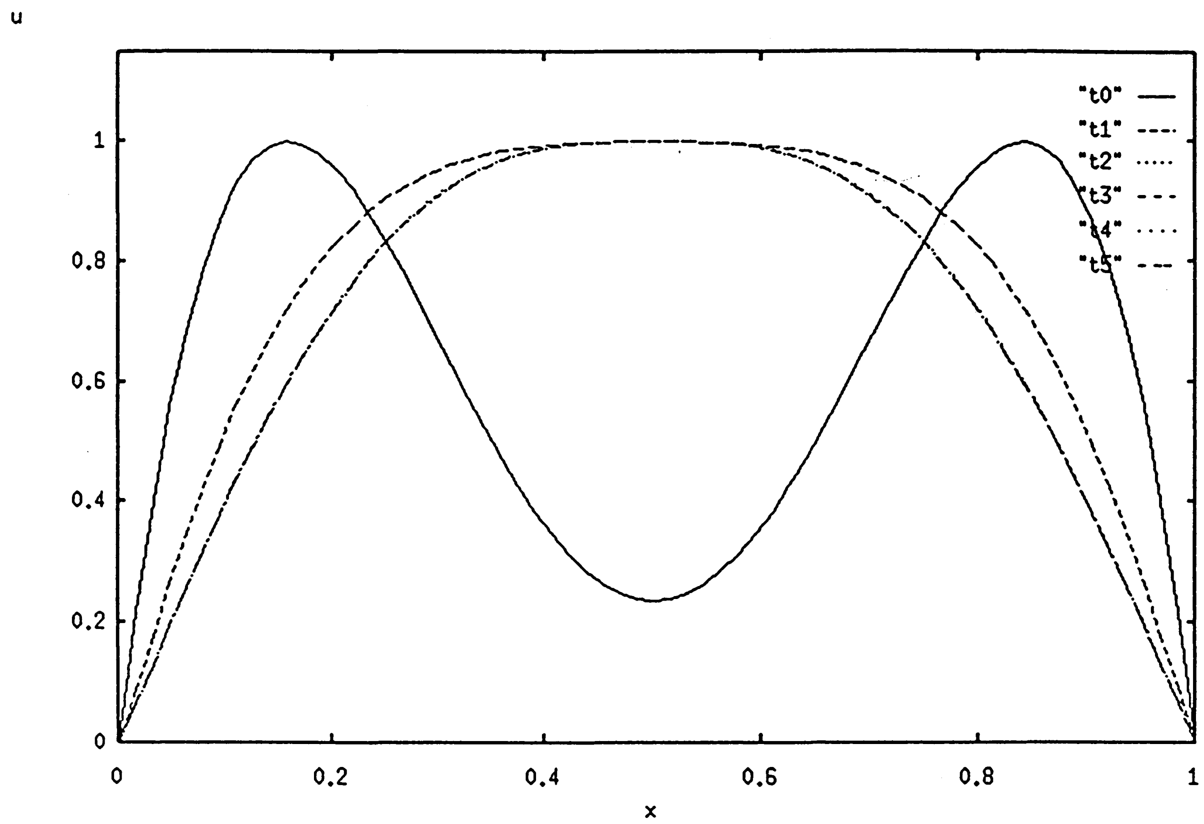

Figure 3.

$$
\begin{aligned}
T_0 &= 0 \\
T_1 &= 0.02624512 \\
T_2 &= 0.06630449 \\
T_3 &= 0.09247656 \\
T_4 &= 0.1109373 \\
T_5 &= 0.1239462
\end{aligned}
$$

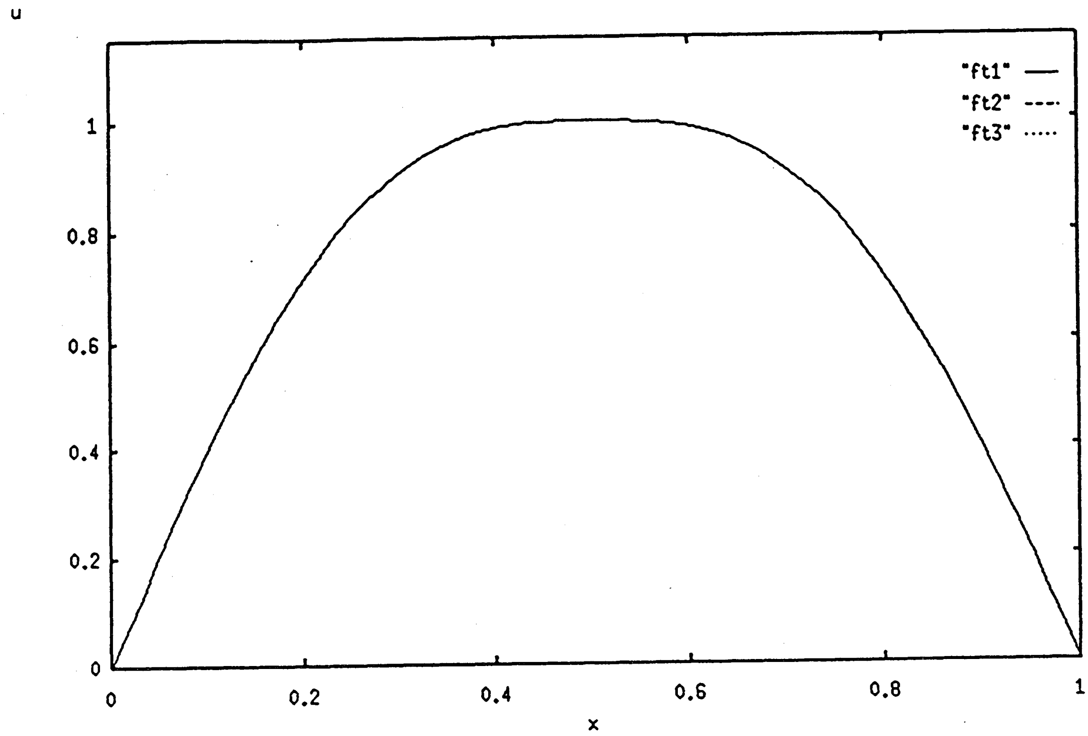

Figure 4. The asymptotic profiles of Figure 1 - 3 (labeled ft1 - ft3). They coincide with in plotting resolution.

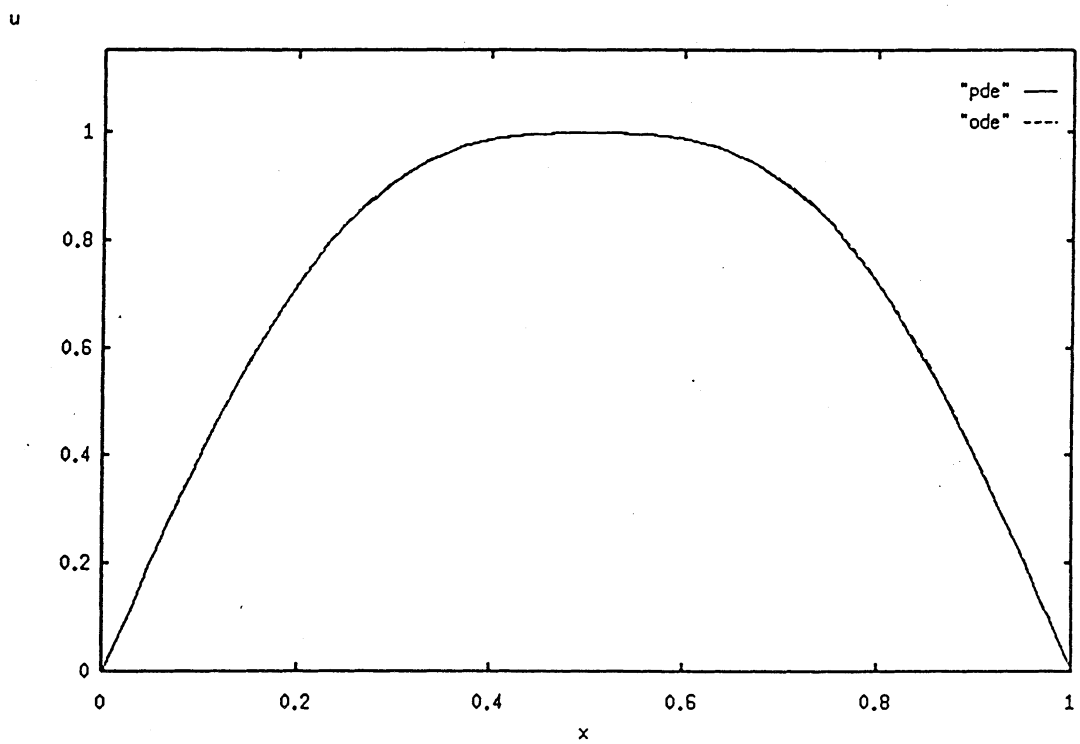

Figure 5. The asymplotic profiles of Figure 1 - 3 (labeled pde) and rescaled profile of separable solution obtained by solving (V1)-(V3) numerically. They alose coincide with in plotting resolution.