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Generalized Airy-Weber Functions

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1 Introduction

Let $\varphi(q,z)$ be a r-reticular phase function on $Q\times C^{m}$ satisfying"non-degeneracy condition", where Q is a manifold of the dimension n. We call a generalized Airy-Weber function with monodromy exponent s a function defined on Qx$\tilde{C}^{*}$ where $\tilde{C}^{*}$ is the universal covering of $C^{*}=C\setminus(0)$ by

$$\mathcal{A}_{s,\Gamma}^{\varphi}(q,x)=\int_{\Gamma}e^{-x\varphi(q,z)}z_{1}^{s_{1}}\ldots.z_{r}^{s_{r}}\ dz_{1}\ldots dz_{m} \tag{0}$$

where

$$s=(s_{1},\ldots,s_{r})\in C^{r},x=-ik,$$

and Γ is non-compact m- cycle of $\tilde{C}^{*r}\times C^{m-r}$ depending continuously on $Argx$ and q and verifying " the Steepest descent condition" (so that the integrand decreases exponentialy) , see [P1]. This integral is a generalization of Airy integral (defined by oscillatory integral having the phase being an universal deformation of isolated critical point) and Weber integral (defined by oscillatory integral having the phase being an universal deformation of function on manifold with boundary). Following Sato's idea, we define two C-algebras of microdifferential operators on a symplectic manifold and a contact manifold. In this talk we give the following results:

1.1: Every lagrangian (legendrian) regular r-cubic configuration is associated to one class of reticular phase function satisfying a "reticular nondegeneracy condition".

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*A joint work with F. Pham (see reference [DP]).

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1.2: Generalized Airy-Weber function is a solution of system of n microdifferential equations with involution on a symplectic manifold and it is a solution of system of $n+1$ microdifferential equations with involution on a contact manifold.

Now instead of s as in the definiton, we consider s depending on x. The following result shows that a generalized Airy-Weber function with monodromy exponent depending on the parameter x is not a solution of holonomic system in variables (q, ξ), (q = variables of space, ξ = variables of phase).

1.3: Solutions of microdifferential system with the the characteristic variety being a regular r-cubic configuration on a symplectic manifold is a generalized Airy-Weber function up to an invertible microdifferential operator, such that monodromy exponents $s_1 = s_1(x), \ldots, s_r = s_r(x)$ are symbols of the Gevrey's class.

2 Oscillatory Integrals and Phase Functions

In the last twenty years the theory of singularities has been exceptionally closely linked with the investigation of oscillatory integrals (i.e. integrals of the form:

$$\int_{R^m} e^{ik\varphi(q,y)} a(q,y) dy_1 \ldots dy_m \tag{1}$$

for large values of the real parameter k, here q belongs to parameter space, φ and a are smooth functions, the function φ is called the phase and the function a is called the amplitude). On the one hand a great many resonable problems of the theory of singularities arose from attempts to understand the nature of the behaviors of integrals. On the other hand much of the study of critical points has found direct application in the study of asymptotics.

In [A], [DU], [M] they studied systematically the asymptotic behaviors of oscillatory integrals by associating almost every lgerm of Lagrangian (or legendrian) smooth variety to one class of the phase germ satisfying a "nondegeneracy condition " in the sense of R. Thom. Hence the study of asymptotic behavior of oscillatory integrals with nondegenerated phases reduces to the theory of singularities of germs of functions or to the theory of singularities of germs of lagrangian (or legendrian) submanifolds.

We remark that in [P1] Pham gave a meaning to integrals (1) by " Steepest Descent method of many variables". From now on we are interested in the complex analytic case.

We consider a germ of lagrangian or legendrian variety not to be smooth; for example, a legendrian (lagrangian) analytic germ consisting of two components:

$$\Lambda = \Lambda_1 \cup \Lambda_2$$

where Λ_1 and Λ_2 are lagrangian (resp. legendrian) submanifolds, they intersect along a submanifold of codimention 1 and

$$T(\Lambda_1 \cap \Lambda_2) = T(\Lambda_1) \cap T(\Lambda_2)$$

The pair (Λ_1, Λ_2) as above is called (by Kashiwara) a geometric regular interaction.

More generally, Let M (resp. W) be a symplectic manifold (resp. contact manifold) a lagrangian (resp. legendrian) regular r-cubic configuration of M (resp. W) is a germ Λ (resp. V) such that in suitable symplectic (resp. Darboux) coordinate system it can be of the form:

$$\Lambda = \{(q,p) \in M : p_1 q_1 = \ldots = p_r q_r = p_{r+1} = \ldots = p_n = 0\} \tag{2}$$

resp.

$$V = \{(q,p,z) \in W : z = p_1 q_1 = \ldots = p_r q_r = p_{r+1} = \ldots = p_n = 0\} \tag{2'}$$

Our problem is local. We can consider W (resp. M) as $J^1(Q,C)$ a bundle of 1-jets of holomorphic functions from a manifold Q to C (resp. $T^*(Q)$ a cotangent bundle of Q) , where Q is a complex analytic manifold of the dimension n.

The main result of this paragraph is to associate every lagrangian (legendrian) regular r-cubic configuration to one class of the phase satisfying " a reticular non-degeneracy condition " (the generalization of R. Thom's notion one).

Now we are going to give a definition of "reticular non-degeneracy condition". Let Z be a germ of a smooth analytic variety of dimension $m \geq r$ with a divisor in normal crossing consisting of r components

$$Z_1, \ldots, Z_r.$$

For each $\sigma \subset \{1, \ldots, r\}$ we denote

$$Z_\sigma = \cap_{i \in \sigma} Z_i$$

and $Z_\emptyset = Z$. A germ Z with

$$\underline{Z} = (Z_\sigma)$$

is called a germ of r-reticular variety.

Let $\varphi : Z \to C$ be a germ of a holomorphic function, we say that φ is r-reticular if Z is a germ of r-reticular variety and we denote by

$$\varphi : \underline{Z} \to C$$

An automorphism

$$\phi : Z \to Z$$

is called an reticular automorphism if

$$\phi(Z_\sigma) = Z_\sigma$$

for every $\sigma \subset \{1, \ldots, r\}$

Two r-reticular functions are $\mathcal{R}-$equivalent if there is a reticular automorphism, which sends first to second one.

Let

$$\Psi : Q \times Z \to C$$

be a deformation of

$$\varphi : \underline{Z} \to C$$

(i. e. $\Psi(0, z) = \varphi(z)$).

A critical set of this deformation is defined as follows:

$$\Sigma = \cup \Sigma_\sigma,$$

$$\Sigma_\sigma = \{(q, z) \in Q \times Z : \frac{\partial \Psi}{\partial z} |_{Z_\sigma} = 0\}.$$

We say that a deformation Ψ satisfies "the reticular nondegeneracy condition" if $\partial_z \Psi$ is transversal to S_σ^1 for all σ , where

$$\partial_z \Psi : Q \times Z \ni (q, z) \to \partial_z \Psi \in T^* Z$$

and

$$S_\sigma^1 = \{\zeta \in T^* Z : \zeta \mid Z_\sigma = 0\}.$$

We see that

$$\Sigma_\sigma = (\partial_z \Psi)^{-1} (S_\sigma^1).$$

Let χ be a characteric map defined as

$$\chi : \Sigma \longrightarrow J^{\,1}(Q, C)$$

which sends (q, z) to 1-jet of Ψ with z constant and

$$\chi_\sigma = \chi \mid \Sigma_\sigma$$

We can prove that $\chi_{\sigma\alpha}$ is a legendrian immersion and

$$V(\Psi) = \cup \{Im(\chi_\sigma)\}$$

is a legendrian regular r-cubic configuration. We can prove that there is an one-to-one correspondence between legendrian regular r-cubic configuration and lagrangian one by the projection

$$J^1(Q, C) \to T^*(Q)$$

The $V(\Psi)$ (resp. $\Lambda(\Psi)$ the image of $V(\Psi)$) is called a characteristic variety of Ψ on the contact (resp. symplectic) manifold.

In [DDP], [DP] we proved that

Theorem 2.1: ([DDP], [DP])

a) Let V (resp. Λ) be a regular r-cubic configuration, such that $\Pi \mid_V$ is finite, where Π is the projection of the bundle $J^1(Q, C)$, then there exists a reticular phase function Ψ such that $V(\Psi) = V$ (resp. $\Lambda(\Psi) = \Lambda$).

b) If $\Psi_i : Q \times Z_i \mapsto C$ are reticular phases, then

$$V(\Psi_1) = V(\Psi_2) \Longleftrightarrow \Psi_1 \cong \Psi_2$$

Where $\cong$ means a $\mathcal{R}-$ equivalence.

c) $V(\Psi)$ is stable $\Longleftrightarrow$ Ψ is $\mathcal{R}^+$-versal.

The notion of stable of Lagrangian (Legendrian) variety and versality of Ψ can be found in [DDP], [DP].

In next paragraphs, we try to characterize generalized Airy-Weber functions by system of microdifferential equations that they satisfy.

3 Microdifferential operators

Following Sato, we introduce two C-algebras of microdifferential operators $\underline{\mathcal{E}} \subset \mathcal{E}$ such that their symbols of order 0 are germs of holomorphic functions on M resp. on W and by above reason we call these operators the M-differential operators resp. W-differential operators.

Let $\mathcal{O}$ be a ring of germs of holomorphic functions (for example: $\mathcal{O}_M$, $\mathcal{O}_W$, $\mathcal{O}_Q$). A x-symbol on $\mathcal{O}$ is a formal series

$$\sum_{k=-\infty}^{m} a_k x^k$$

where $a_k \in \mathcal{O}$ with common radius of convergence ρ and satisfies the Gevrey's condition of order 1:

$$\sum_{l=1}^{\infty} \|a_{-l}\|_\rho \frac{\xi^{l-1}}{(l-1)!} \in C\{\xi\} \tag{3}$$

We denote the $\mathcal{O}$ -algebra of x-symbols by $\mathcal{O}((x^{-1}))_1$ and $\mathcal{O}[[x^{-1}]]_1$ its subalgebra of x-symbols of order ≤ 0.

***M*-differential operators.**

Let $\mathcal{O} = \mathcal{O}_M$, we can write $a \in \mathcal{O}((x^{-1}))_1$ in the form:

$$a(q,p;x) = \sum_k \sum_{\alpha \in N^n} a_{k,\alpha}(q) p^\alpha x^k \tag{4}$$

Instead of p_i we put $x^{-1}\partial q_i$ and p^α becomes $x^{-|\alpha|}\partial q^\alpha$ with remarking that x and ∂q are of order 1. This expression is called a M-differential operator and a is its total symbol. We denote by $\underline{\mathcal{E}}$ a ring of M-differential operators.

Remark: A multiplicity rule of two M-differential operators is a extension of multiplicity rule of two differential operators.

***W*-differential operators.**

Let now $\mathcal{O} = \mathcal{O}_W$ we can write x-symbol in the form

$$a(q,p,\xi;x) = \sum a_k(q,p,\xi) x^k \tag{5}$$

as above, we put $-\partial_x$ instead of ξ with remarking that the operator ∂_x is of order 0. This expression is called W-differential operator and its total symbol is a. We denote by $\mathcal{E}$ a ring of W-differential operators.

Remark: A formal Laplace transformation by $x \mapsto \partial_\xi$ and $\partial x \mapsto -\xi$ sends $\mathcal{E}$ into the algebra of Sato's microdifferential operators in codirection $d\xi$ of $C^n \times C$,

$$\sum a_\alpha(q,\xi)\partial q^\alpha \partial_\xi^{k-|\alpha|} \tag{6}$$

with $\partial q, \partial_\xi$ of order 1 (cf. [SKK]).

The main result of this paragraph is following:

Theorem 3.1: ([DP])

Generalized Airy-Weber function is a solution of system of n M-differential equations with involution on a symplectic manifold , i. e:

$$P_1 u = \ldots = P_n u = 0, \quad [P_i, P_j]u = 0 \tag{7}$$

where principal symbols $\sigma(P_1), \ldots, \sigma(P_n)$ form one system of equations of lagrangian variety $\Lambda(\varphi)$ (a characteristic variety of φ on the symplectic).

Moreover, it is also a solution of system of n+1 W-differential equations with involution on a contact manifold, i. e:

$$P_0 u = P_1 u = \ldots = P_n u = 0, \quad [P_i, P_j]u = 0 \tag{8}$$

where principal symbols $\sigma(P_i)$ form one system of equations of legendrian variety $V(\varphi)$.

Idea of the proof: Consider a microdifferential system

$$(z_1\partial_{z_1} - s_1)u = 0, \ldots, (z_r\partial_{z_r} - s_r)u = 0, \partial_{z_{r+1}}u = 0, \ldots, \partial_{z_m}u = 0,$$

resp.

$$(z_1\partial_{z_1} - s_1)u = 0, \ldots, (z_r\partial_{z_r} - s_r)u = 0, \partial_{z_{r+1}}u = 0, \ldots, \partial_{z_m}u = 0, \partial_x u = 0$$

Let $u = z_1^{s_1} \ldots z_r^{s_r}$ be a solution of this system and we denote $e^{-x\varphi}dz \otimes u$ by u^φ. It is enough to prove that there are epimorphisms of right $\underline{\mathcal{E}}$ (resp. $\mathcal{E}$) -modules:

$$\underline{\mathcal{E}}u^\varphi \to \underline{\mathcal{E}}\mathcal{A}^\varphi_{s,\Gamma}$$

$$\mathcal{E}u^\varphi \to \mathcal{E}\mathcal{A}^\varphi_{s,\Gamma}$$

We can prove that $\underline{\mathcal{E}}u^\varphi$ (resp. $\mathcal{E}u^\varphi$) coincides with the direct image of $\underline{\mathcal{E}}_z$ (resp. $\mathcal{E}_z$) -modulle $\underline{\mathcal{E}}_z u$ (resp. $\mathcal{E}_z u$) along φ (we donote it by $\underline{\mathcal{G}}^\varphi$ (resp. $\mathcal{G}^\varphi$)).

We can prove the following, too:

Theorem 3.4: ([DP])

$\mathcal{G}^\varphi$ is stable if and only if φ is $\mathcal{R}^+$-versal.

4 Solution of M-differential system

In this paragraph we solve a M-differential system

$$P_1\Psi = 0, \ldots, P_n\Psi = 0, P_1, \ldots, P_n \in \underline{\mathcal{E}} \tag{9}$$

such that: a characteristic variety of this system is a lagrangian regular r-cubic configuration Λ and principal symbols of the ideal generated by $P_1, \ldots, P_n$ in $\underline{\mathcal{E}}$ form a reduced ideal in $\mathcal{O}$ defining Λ.

The main result of this paragraph is following:

Theorem 4.1 ([DP])

Solutions of the system (9) are of the following form:

$$\Psi = C\mathcal{A}_{s,\Gamma}$$

where C is an invertible M-differential operator, $s_1 = s_1(x), \ldots, s_r = s_r(x)$ are elements of $\mathcal{C}[[x^{-1}]]_1$ and

$$\mathcal{A}_{s,\Gamma}(q, x) = \int_\Gamma e^{-xS(q,\hat{q})}\hat{q}_1^{s_1} \ldots \hat{q}_r^{s_r}\, d\hat{q}_1 \ldots d\hat{q}_n$$

Idea of the proof: First we introduce a notion of meromorphic microdifferential opreators on M (see [P2], [P3]) and by using a Division Lemma for meromorphic microdifferential operators, we can prove that:

The system (9) is equivalent, by a change of an unknown function of the form $\Psi = B\Phi$ (B is inverstible in $\underline{\mathcal{E}}$) , to the following

$$(q_1\partial_1 - s_1)\Phi = 0, \ldots, (q_r\partial_r - s_r)\Phi = 0, \partial_{r+1}\Phi = 0, \ldots, \partial_n\Phi = 0 \tag{10}$$

Where (q,p) is a local symplectic coordinate system such that Λ is of the form:

$$p_1q_1 = \ldots = p_rq_r = 0, p_{r+1} = \ldots = p_n = 0 \tag{11}$$

$\partial_i := \partial_{q_i}$ and $s_1, \ldots, s_r \in C[[x^{-1}]]_1$ are uniquely determined by system (9) and we call them monodromy exponents of system (9).

Now let S be a generating function of canonical transformation which sends Λ to the set of the form (11) , this S is a nondegenerated deformation of a reticular phase uniquely determined up to the $\mathcal{R}$-equivalence (by the theorem 2. 1). Using a quantized canonical transformation induced by S we can prove that

$$\Psi(q, x) = \int_\Gamma e^{-xS(q,\hat{q})}\hat{\Psi}(\hat{q}, x) d\hat{q}_1 \ldots d\hat{q}_n$$

is the solution of (9) if and only if $\hat{\Psi}$ is the solution of (10) , hence by (10) we have:

$$\hat{\Psi} = \hat{C}\hat{q}_1^{s_1} \ldots \hat{q}_r^{s_r}.$$

Therefore $\Psi = C\mathcal{A}_{s,\Gamma}$.

Now we assume that Λ is stable, hence $\hat{q} \mapsto S(q, \hat{q})$ is a miniversal of $\hat{q} \mapsto S(0, \hat{q})$ therefore $\mathcal{O}_\Lambda$ is free over $\mathcal{O}_Q$ with generators $1, p_1, \ldots, p_n$ and we conclude that $\underline{\mathcal{E}}A$ is generated by $\mathcal{A}, \partial_{q_1}\mathcal{A}, \ldots, \partial_{q_n}\mathcal{A}$ in $\mathcal{O}((x^{-1}))_1$, where $\mathcal{A} = \mathcal{A}_{s,\Gamma}$. We get the following:

Theorem 4.2 ([DP])
If Λ is stable then the solution of (9) has an unique decomposition

$$\Psi = c_0\mathcal{A}_{s,\Gamma} + c_1\partial_{q_1}\mathcal{A}_{s,\Gamma} + \ldots + c_n\partial_{q_n}\mathcal{A}_{s,\Gamma}, c_i \in \mathcal{O}_Q((x^{-1}))_1$$

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