数理解析研究所講究録
第 894 巻 1995 年 29-42

THE ACOUSTIC WAVE PROPAGATION IN TWO UNBOUNDED MEDIA

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1. Introduction.

In the present paper we study the limiting absorption and amplitude principle for the acoustic wave operators in two unbounded media. We assume that the propagation speed is discontinuous at the interface and the equilibrium density is 1.

Let $n\geq 2$ and $x=(y,z)\in \mathbf{R}^{n-1}\times \mathbf{R}$. The following equation decribes the wave propagation here :

$$\partial_t^2u(t,x)-a(x)^2\triangle u(t,x)=0,\quad (t,x)\in \mathbf{R}\times \mathbf{R}^n, \tag{1.1}$$

where $a(x)$ is a propagation speed. When we consider the limiting amplitude principle, we deal with the asymptotic behaviour (as $t\rightarrow+\infty$) of the solutions of the following Cauchy problem

$$\left\{\begin{array}{ll}\partial_t^2u(t,x)-a(x)^2\triangle u(t,x)=exp(-it\sqrt{\omega})f(x) & (t,x)\in \mathbf{R}_+\times \mathbf{R}^n,\\ u(0,x)=\partial_tu(0,x)=0, & \end{array}\right. \tag{1.2}$$

where $\omega>0$. But we can not deal with $n=2$ (see (3.13)).

We make the assumptions for the interface separating two media and $a(x)$.

Let $\varphi_0(y)=b|y|$ and $\varphi(y)\in C^1(\mathbf{R}^{n-1}\backslash\{0\})$, where $b\geq 0$. We assume that $\varphi(y)$ describes the interface and satisfies

$$\sum_{|\alpha|\leq 1}|y|^{|\alpha|}|\partial^\alpha(\varphi(y)-\varphi_0(y))|=O(|y|^{-\theta})\quad (|y|\longrightarrow\infty), \tag{A.0}$$

for some $\theta>0$, and

$$\sum_{|\alpha|\leq 1}|y|^{|\alpha|}|\partial^\alpha\varphi(y)|=O(|y|^{-\sigma})\quad (|y|\longrightarrow 0). \tag{A.1}$$

where $0<\sigma<1/2$. For $\varphi(y)$, we use the following notation :

$$\begin{aligned}\Omega_+&=\{x=(y,z):z>\varphi(y)\},\\ \Omega_-&=\{x=(y,z):z<\varphi(y)\},\\ S&=\{x=(y,z):z=\varphi(y)\}.\end{aligned}$$

We denote the unit normal vector at the point $x \in S$ by $\nu = (\nu_1, \nu_2, \ldots, \nu_z)$ with $\nu_z > 0$.

The propagation speed $a(x) > 0$ is assumed to satisfy the following : for some $c > 1$,

$$\text{(A.2)} \qquad 1/c < a(x) < c$$

and there exist $a_\pm > 0, a_L^\pm(x) \in \mathbf{B}^1(\Omega_\pm)$ and $a_S(x) \in L^\infty(\mathbf{R}^n)$ such that $a(x)$ is decomposed as

$$\text{(A.3)} \qquad \begin{cases} a(x) & = a_\pm + a_L^\pm(x) + a_S(x) \quad (x \in \Omega_\pm), \\ \sum_{|\alpha|\leq 1} |x|^{|\alpha|} & |\partial^\alpha a_L^\pm(x)| = O(|x|^{-\theta}) \quad (|x| \longrightarrow \infty, x \in \Omega_\pm), \\ a_S(x) & = O(|x|^{-\theta-1}) \quad (|x| \longrightarrow \infty) \end{cases}$$

for some $\theta > 0$.

Under $(A.0) \sim (A.3)$, we show the nonexistence of eigenvalues and the limiting absorption and amplitude principle of the acoustic operator $-a(x)^2\Delta$ for (1.1).

There are many works dealing with the acoustic wave propagation problem with the discontinuous propagation speed at the interface separating media. Eidus [**6**] proved the limiting absorption and amplitude principle for two unbounded media problem with the interface satisfying the following conditions : for any $x \in S$

$$\text{(1.3)} \qquad \nu_z \geq C_1 > 0,$$

$$\text{(1.4)} \qquad |x \cdot \nu| \leq C_2,$$

where $C_j > 0, (j = 1, 2)$, are independent of $x \in S$. For example,

$$\varphi(y) \in C^1(\mathbf{R}^{n-1}), \quad \varphi(y) = \frac{\sin|y|}{|y|} \quad (|y| >> 1), \quad \varphi_0(y) = 0$$

satisfies (1.3) and (1.4), but not satisfies (A.0). We can also deal with the following interface not satisfying (1.3),

$$\varphi(y) = |y|^{-\sigma}, \quad \varphi_0(y) = 0$$

where, $0 < \sigma < 1/2$.

The propagation speed considered in Eidus [**6**] is a piecewise constant function while we can perturb the propagation speed.Wilcox [**18**] considered two stratified fluilds in a half space and established the eigenfunction expansion theorem. Ben-Artzi [**1**], Weder [**15**]~[**17**] and Dermenjian and Guillot [**3**] considered perturbed stratified fluilds problems. They showed the limiting absorption principle by the approach of Wilcox [**18**]. Kikuchi and Tamura [**9**] also proved the limiting amplitude principle for perturbed stratified fluilds. On the other hand there are some works dealing with the case where the equilibrium density is discontinuous at the interface separating media, for example, Debièvre and Pravica [2] and Wilcox [18].

In order to show the limiting absorption and amplitude principle for our operator, we use Mourre's method. This method was first developed by Mourre [10] to prove the limiting absorption principle for 3-body Schrödinger operators (see also Perry, Sigal and Simon [11] and Tamura [13]). In Froese and Herbst [8], they showed by Mourre's method that N-body Schrödinger operators have no positive eigenvalues. Iwashita [8] and Weder [14] showed the limiting absorption principle for first order symmetric systems. For the acoustic wave operators in perturbed stratified fluilds, Debièrve and Pravica [2] obtained the similar results as in [8] and [14]. Tamura [12] used Mourre's method in order to prove the limiting amplitude principle for the acoustic wave operators (see also Kikuchi and Tamura [9]).

We now define the acoustic operator L as

$$L = -a(x)^2\triangle \tag{1.5}$$

Under the above assumptions, (A.0)~(A.3), L is a symmetric operator in the Hilbert space $L^2(\mathbf{R}^n; a^{-2}(x)dx)$ and admits a unique self-adjoint realization . We denote by the same notation L this self-adjoint realization. Then L is a positive operator (zero is not an eigenvalue) and the domain $D(L)$ is given by $D(L) = H^2(\mathbf{R}^n_x), H^s(\mathbf{R}^n_x)$ being the Sobolev space of order s over $\mathbf{R}^n_x$. We also denote by $R(z; L)$ the resolvent $(L-z)^{-1}$ of L for $Imz \neq 0$.

We need several notations to describe our results. Let L^2 be the usual L^2space defined on $\mathbf{R}^n$, with the inner product

$$< u, v >= \int_{\mathbf{R}^n} u(x)\overline{v(x)}dx$$

and the corresponding norm $|\cdot|_0$. For $\alpha \in \mathbf{R}$ let L^2_α be the weighted L^2space defined by

$$L^2_\alpha = \{u(x) :< x >^\alpha u(x) \in L^2(\mathbf{R}^n_x)\}, < x >= (1+|x|^2)^{1/2},$$

with the norm

$$|u|^2_\alpha = \int_{\mathbf{R}^n} < x >^{2\alpha} |u(x)|^2 dx.$$

Let $A : L^2_\alpha \longrightarrow L^2_\beta$ be a bounded operator. We denote by $||A||_{\alpha\to\beta}$ the operator when norm considered as an operator from L^2_α to L^2_β. If, in particular, $A : L^2 \longrightarrow L^2$ is considered as an operator from L^2 into itself, then its norm is denoted by the simplified notation $||A||$.

The main results are

THEOREM 1.1. *Assume that* $(A.0) \sim (A.3)$. *Then*
(i) L has no eigenvalues.

(ii) Let $\lambda_0 > 0$ and $\alpha > 1/2$. Then for any compact interval $I \subset \mathbf{R}_+$ containing λ_0, there exists a positive constant $C = C(I, \alpha)$ such that

$$\| < x >^{-\alpha} R(\lambda \pm i\kappa; L) < x >^{-\alpha} \| \leq C,$$

for $\lambda \in I$, $0 < \kappa < 1$.
(iii) For every $\lambda > 0$ and $\alpha > 1/2$, following two limits

$$R(\lambda \pm i0; L) = \lim_{\kappa \downarrow 0} R(\lambda \pm i\kappa; L),$$

exist in the uniform operator topology of $\mathfrak{B}(L^2_\alpha, L^2_{-\alpha})$. Moreover $R(\lambda \pm i0; L)$ are locally Hölder continuous.

THEOREM 1.2. *Assume that $(A.0) \sim (A.3)$ and $n \geq 3$, Let $\alpha > 1$ and $\beta > 1/2$. Then there exists $d, 0 < d < 1/2$, such that*

$$\|R(\lambda \pm i0; L)\|_{\beta \to -\alpha} = O(\lambda^{-d}), \quad (\lambda \to 0).$$

By Theorem 1.1 and 1.2, we have the following theorem (see Eidus [**5**] or Tamura [**8**]).

THEOREM 1.3(LIMITING AMPLITUDE PRINCIPLE). *Assume that $(A.0) \sim (A.3)$ and $n \geq 3$. Let $u = u(t, x)$ be the solution of (1.2) with $f \in L^2_\beta, \beta > 1/2$. Then $u(t, x)$ behaves like*

$$u = exp(-it\sqrt{\omega})R(\omega + i0; L)f + o(1), \quad (t \to \infty)$$

strongly in $L^2_{-\alpha}, \alpha > 1$.

2. The limting absorption principle.

We consider only the case $1 = a_-^{-2} < a_+^{-2}$. The other cases can be proved similarly. We define the self-adjoint operator $H(\lambda)$ on L^2 by

$$\begin{cases} H(\lambda) = -\Delta - \lambda(a^{-2}(x) - 1) \\ D(H(\lambda)) = H^2(\mathbf{R}^n). \end{cases}$$

Then we have

$$R(\lambda \pm i\kappa; L) = Q(\lambda, \pm i\kappa; H(\lambda))a^{-2}(x),$$

where $Q(\lambda, \pm i\kappa; H(\lambda)) = (H(\lambda) - \lambda \mp i\kappa a^{-2}(x))^{-1}$.
We apply Mourre's commutator method to $H(\lambda)$ on L^2 (see Tamura [**12**] or Kikuchi and Tamura [**9**]).

By (A.3), we can decompose $a^{-2}(x) = E_L^{\pm}(x) + E_S(x)(x \in \Omega_\pm)$ in such a way that

$$\sum_{|\alpha| \leq 1} |x|^{|\alpha|} |\partial^\alpha (E_L^{\pm}(x) - a_\pm^{-2})| = O(|x|^{-\theta}), (|x| \longrightarrow \infty, x \in \Omega_\pm),$$
$$E_S(x) = O(|x|^{-1-\theta}), (|x| \longrightarrow \infty).$$

Let A be the generator of the dilation unitary group;

$$A = \frac{1}{2i}(x \cdot \nabla + \nabla \cdot x).$$

We define the commutator $i[H(\lambda), A]$ as a form on $H^2(\mathbf{R}^n) \cap D(A)$ as follows; For $u, v \in H^2(\mathbf{R}^n) \cap D(A)$

$$\begin{aligned} &< i[H(\lambda), A]u, v > \\ &= i(< Au, H(\lambda)v > - < H(\lambda)u, Av >). \end{aligned}$$

Our main calculation is the integration by parts. Then, an integration on S appears because of the discontinuity of $a(x)$. In order to estimate this integration, we use the following lemma

LEMMA 2.1. *Let $s > 1/2$. For $u \in \mathbf{S}(\mathbf{R}^n)$ (Schwartz space), we define*

$$(T_\varphi u)(y) = u(y, \varphi(y)).$$

T_φ has an extension to a bounded operator from $H^s(\mathbf{R}^n)$ to $L^2(\mathbf{R}^{n-1})$.

PROOF: we show that :

$$|T_\varphi u|_{L^2(\mathbf{R}^{n-1})} \leq C|u|_{H^s(\mathbf{R}^n)} \tag{2.1}$$

for $u \in \mathbf{S}(\mathbf{R}^n)$. Let $u \in \mathbf{S}(\mathbf{R}^n)$. Then $u(y, \varphi(y))$ is represented as

$$u(y, \varphi(y)) = (2\pi)^{-1/2} \int_{-\infty}^{+\infty} e^{i\varphi(y)\cdot\zeta}(\mathfrak{F}_z u)(y, \zeta)d\zeta, \tag{2.2}$$

where $\xi = (\eta, \zeta) \in \mathbf{R}_\eta^{n-1} \times \mathbf{R}_\zeta = \mathbf{R}_\xi^n$ *and* $\mathfrak{F}_z$ is the Fourier transform in $\mathbf{R}_z$. By Schwartz's inequality, we have

$$\begin{aligned} &|(T_\varphi u)(y)|^2 \\ &\leq (2\pi)^{-1} \int_{-\infty}^{+\infty} (1 + \zeta^2)^{-s} d\zeta \int_{-\infty}^{+\infty} |(1 + \zeta^2)^{s/2} (\mathfrak{F}_z u)(y, \zeta)|^2 d\zeta. \end{aligned} \tag{2.3}$$

By integrating both sides of the above inequality over $\mathbf{R}^{n-1}$, we obtain (2.1). ∎

Moreover, a singularity at $y = 0$ for $\varphi(y)$ (see (A.2)) appears. We analyze this singularity by the argument using Hölder inequality and the following Sobolev's theorem

LEMMA 2.2 (SOBOLEV'S THEOREM). *Suppose that*

$$1/2 - l/m = 1/q, 2 < q < \infty.$$

Then we have the embedding

$$H^l(\mathbf{R}^m) \hookrightarrow L^q(\mathbf{R}^m).$$

The following lemmas play an important role in the proof of the theorem 1.1 (see Mourre **[10]** and Perry, Sigal and Simon **[11]**).

LEMMA 2.3. *the form* $i[H(\lambda), A]$ *defined on* $H^2(\mathbf{R}^n) \cap D(A)$ *is extended to a bounded operator from* $H^1(\mathbf{R}^n)$ *to* $H^{-1}(\mathbf{R}^n)$ *which is denoted by* $i[H(\lambda), A]^0$. *Moreover we have*

$$\begin{aligned} &i[H(\lambda), A]^0 \\ &= -2\Delta + \lambda((x\cdot\nabla E_L) - E_S x\cdot\nabla + \nabla^*\cdot E_S x - nE_S \\ &- T_\varphi^*(E_L^{+0} - E_L^{-0})(y\cdot\nabla_y\varphi(y) - \varphi(y))T_\varphi), \end{aligned}$$

where $E_L^{\pm} = E_L(y, \varphi(y) \pm 0)$.

LEMMA2.4. *Let* $\lambda_0 > 0$ *and* $0 < \delta < min(1, \lambda_0/4)$ *and take* $f_\delta(p) \in C_0^\infty(\mathbf{R}), 0 \le f_\delta \le 1$ *such that* f_δ *has support in* $(\lambda_0 - 3\delta, \lambda_0 + 3\delta)$ *and* $f_\delta = 1$ *on* $[\lambda_0 - 2\delta, \lambda + 2\delta]$. *Then, there exist a positive constant* α *and a compact operator* K *on* L^2 *which depend on only* λ_0 *such that*

$$\begin{aligned} &f_\delta(H(\lambda))i[H(\lambda), A]^0 f_\delta(H(\lambda)) \\ &\ge \alpha f_\delta(H(\lambda))^2 + f_\delta(H(\lambda))K(\lambda)f_\delta(H(\lambda)) \end{aligned} \tag{2.4}$$

for $\lambda \in (\lambda_0 - \delta, \lambda_0 + \delta)$, *in the form sense.*

SKETCH OF THE PROOF OF LEMMA 2.3: By integration by parts, we have

$$\begin{aligned} &< i[H(\lambda), A]u, u > \\ &= 2 < \nabla u, \nabla u > +\lambda < (x\cdot\nabla E_L)u, u > \\ &- \lambda(< T_\varphi^*(\chi_{|y|<r}(y\cdot\nabla_y\varphi - \varphi)(E_L^{+0} - E_L^{-0}))T_\varphi u, u > \\ &+ < T_\varphi^*(\chi_{|y|>r}(y\cdot\nabla_y\varphi - \varphi)(E_L^{+0} - E_L^{-0}))T_\varphi u, u >) \\ &- n\lambda < E_S u, u > -\lambda < E_S u, x\cdot\nabla u > -\lambda < x\cdot\nabla u, E_S u >, \end{aligned} \tag{2.5}$$

for any $r > 0$.

We define an operator R_φ^r as

$$R_\varphi^r u = \chi_{|y|<r}(y)(y\cdot\nabla_y\varphi(y) - \varphi(y))^{1/2}T_\varphi u.$$

We show that R_φ^r is a bounded operator from $H^1(\mathbf{R}^n)$ to $L^2(\mathbf{R}^{n-1})$.

If $0 < \sigma < 1/2$, there exist some $s > 1/2$ and $p > n - 1$ such that

$$n - 1 - \sigma p > 0, \tag{2.6}$$

$$\frac{n-1}{2p} + s = 1. \tag{2.7}$$

By Hölder inequality, we have

$$\begin{aligned} &|R_\varphi^r u|^2_{L^2(\mathbf{R}^{n-1})} \\ &\le C\int_{-\infty}^{+\infty}(1+\zeta^2)^s\int_{|y|<r}|y\cdot\nabla_y\varphi - \varphi||(\mathfrak{F}_z u)(y,\zeta)|^2 dy d\zeta \\ &\le Cr^{\frac{n-1-\sigma p}{p}}\int_{-\infty}^{+\infty}(1+\zeta^2)^s|(\mathfrak{F}_z u)(\cdot,\zeta)|^2_{L^{\frac{2p}{p-1}}(\mathbf{R}^{n-1})}d\zeta, \end{aligned} \tag{2.8}$$

where $C>0$ is independent of r.
Lemma 2.2 implies that

$$|\mathfrak{F}_z u(\cdot,\zeta)|_{L^{\frac{2p}{p-1}}(\mathbf{R}^{n-1})} \le C|\mathfrak{F}_z u(\cdot,\zeta)|_{H^{\frac{n-1}{2p}}(\mathbf{R}^{n-1})}. \tag{2.9}$$

It follows from (2.7),(2.8) and (2.9) that

$$|R^r_\varphi u|_{L^2(\mathbf{R}^{n-1})} \le Cr^{\frac{n-1-\sigma p}{2p}}|u|_{H^1(\mathbf{R}^n)}, \tag{2.10}$$

where C is independent of r.

It follows from Lemma 2.1 that T^*_φ is a bounded operator from $L^2(\mathbf{R}^{n-1})$ to $H^{-1}(\mathbf{R}^n)$. Moreover, since $E_S x\cdot\nabla$ is a bounded operator from $H^1(\mathbf{R}^n)$ to $L^2(\mathbf{R}^n)$, the adjoint operator $(\nabla)^*\cdot xE_S$ is a bounded operator from $L^2(\mathbf{R}^n)$ to $H^{-1}(\mathbf{R}^n)$.

Hence, from (2.5) and (2.10), the form $i[H(\lambda),A]$ has an extension to a bounded operator $i[H(\lambda),A]^0$ on from $H^1(\mathbf{R}^n)$ to $H^{-1}(\mathbf{R}^n)$. ∎

SKETCH OF THE PROOF OF LEMMA 2.4: We simply write f_δ instead of $f_\delta(H(\lambda))$. By Lemma 2.3, we have for $u\in L^2$,

$$\begin{aligned}
&< f_\delta i[H(\lambda),A]^0 f_\delta u,u>\\
&=2<-\Delta f_\delta u,f_\delta u>+\lambda(<((x\cdot\nabla E_L)-nE_S)f_\delta u,f_\delta u>\\
&-<x\cdot\nabla f_\delta u,E_S f_\delta u>-<E_S f_\delta u,x\cdot\nabla f_\delta u>\\
&-<(T_\varphi)^*\chi_{|y|>r}(y\cdot\nabla_y\varphi-\varphi)(E_L^{+0}-E_L^{-0})T_\varphi f_\delta u,f_\delta u>\\
&-<(R^r_\varphi)^*(E_L^{+0}-E_L^{-0})R^r_\varphi f_\delta u,f_\delta u>).
\end{aligned}$$

Let $0<r<<1$. By (2.9), there exists a positive number C independent of r such that,

$$\begin{aligned}
&|\lambda<(R^r_\varphi)^*(E_L^{+0}-E_L^{-0})R^r_\varphi f_\delta u,f_\delta u>|\\
&=|\lambda<(E_L^{+0}-E_L^{-0})R^r_\varphi f_\delta u,R^r_\varphi f_\delta u>_{L^2(\mathbf{R}^{n-1})}|\\
&\le Cr^{\frac{n-1-\sigma p}{p}}(|\nabla f_\delta u|_0^2+|f_\delta|_0^2).
\end{aligned}$$

Let $E_0(x)=a_\pm^{-2}(x\in\Omega_\pm)$. Then we have

$$\begin{aligned}
&2<-\Delta f_\delta u,f_\delta u>-\lambda<(R^r_\varphi)^*(E_L^{+0}-E_L^{-0})R^r_\varphi f_\delta u,f_\delta u>\\
&\ge\Big(\frac{(2-Cr^{\frac{n-1-\sigma p}{p}})\lambda_0}{4}-Cr^{\frac{n-1-\sigma p}{p}}\Big)|f_\delta u|_0^2\\
&+(2-Cr^{\frac{n-1-\sigma p}{p}})\lambda<(E-E_0)f_\delta u,f_\delta u>.
\end{aligned}$$

Take $h\in C_0^\infty(\mathbf{R}), 0\le h\le 1$ such that $h=1$ on $(\lambda_0/4,7\lambda_0/4)$. Using h, we define an operator $K(\lambda)$ as

$$K(\lambda)=\sum_{j=1}^4 K_j(\lambda)$$

where,

$$K_1(\lambda) = \lambda(h(H(\lambda))((x \cdot \nabla E_L) - nE_S + (2 - Cr^{\frac{n-1-\sigma p}{p}})(E - E_0))h(H(\lambda)),$$
$$K_2(\lambda) = -\lambda h(H(\lambda))E_S x \cdot \nabla h(H(\lambda)),$$
$$K_3(\lambda) = -\lambda h(H(\lambda))\nabla^* \cdot xE_S h(H(\lambda)),$$
$$K_4(\lambda) = -\lambda h(H(\lambda))T_\varphi^*(E_L^{+0} - E_L^{-0})\chi_{|y|>r}(y \cdot \nabla\varphi - \varphi)T_\varphi h(H(\lambda)).$$

For each $\lambda \in (\lambda_0 - \delta, \lambda_0 + \delta)$, $K_j(\lambda)(j = 1, 2, 3, 4)$ is a compact operator on L^2. ∎

Using Lemma 2.3 and Lemma 2.4, we can prove theorem 1.1(i) in the same way as in Froese and Herbst [**7**] and Appendix I of Tamura [**12**].

Next, we consider the proof of Theorem 1.1(ii) and (iii).

Following Tamura [**12**], we consider cut off functions, $\chi_n(x) \in C_0^\infty(\mathbf{R}^n)$ such that $\chi_n(x)$ has support in $\{x \in \mathbf{R}^n; |x| < 2\}$ and $\chi_n = 1$ for $|x| \leq 1$. For $\epsilon > 0$ small enough, we define

$$E_{L,\epsilon}(x) = E_0(x) + \chi_n(\epsilon x)(E_L(x) - E_0(x)),$$
$$E_{S,\epsilon}(x) = \chi_n(\epsilon x)E_S(x),$$

and

$$V_\epsilon^r(y) = \chi_{|y|>r}(y)\chi_{n-1}(\epsilon y)(y \cdot \nabla_y\varphi(y) - \varphi(y)).$$

We further define an operator $B(\epsilon; \lambda)$ as

$$\begin{aligned} &B(\epsilon; \lambda) \\ &= -2\Delta + \lambda((x \cdot \nabla E_{L,\epsilon}) - nE_{S,\epsilon} + \nabla^* \cdot xE_{S,\epsilon} - E_{s,\epsilon}x \cdot \nabla \\ &- T_\varphi^* V_\epsilon^r(y)(E_L^+ - E_L^-)T_\varphi - (R_\varphi^r)^*(E_L^+ - E_L^-)R_\varphi^r). \end{aligned}$$

We can consider that $B(\epsilon; \lambda)$ to be a bounded operator from $H^1(\mathbf{R}^n)$ to $H^{-1}(\mathbf{R}^n)$ (see Lemma2.1). Moreover, for $u \in H^2(\mathbf{R}^n) \cap D(A)$ satisfing $Au \in H^1(\mathbf{R}^n)$, we define the commutator $[B(\epsilon; \lambda), A]$ as follows

$$\begin{aligned} &< i[B(\epsilon; \lambda), A]u, u > \\ &= \sum_{j=1}^{4} < i[B_j(\epsilon; \lambda), A]u, u > \\ &= \sum_{j=1}^{4} i(< Au, B_j(\epsilon; \lambda)u > - < B_j(\epsilon; \lambda)u, Au >), \end{aligned}$$

where

$$\begin{aligned} B_1(\epsilon; \lambda) &= -2\Delta + \lambda(x \cdot \nabla E_{L,\epsilon}), \\ B_2(\epsilon; \lambda) &= -\lambda((E_{S,\epsilon}x \cdot \nabla)^* + E_{S,\epsilon}x \cdot \nabla + nE_{S,\epsilon}) \\ &= -\lambda(\nabla^* \cdot xE_{S,\epsilon} + E_{S,\epsilon}x \cdot \nabla + nE_{S,\epsilon}), \\ B_3(\epsilon; \lambda) &= -\lambda T_\varphi^* V_\epsilon^r(E_L^{+0} - E_L^{-0})T_\varphi, \\ B_4(\epsilon; \lambda) &= -\lambda(R_\varphi^r)^*(E_L^{+0} - E_L^{-0})R_\varphi^r. \end{aligned}$$

Then the $i[B(\epsilon;\lambda),A]$ has an extension $[B(\epsilon;\lambda),A]^0$ to be a bounded operator from $H^2(\mathbf{R}^n)$ to $H^{-2}(\mathbf{R}^n)$. Then we can prove the following lemma

LEMMA2.5. *Let $M(\epsilon;\lambda)=f_\delta(H(\lambda))B(\epsilon;\lambda)f_\delta(H(\lambda))$. Then $[M(\epsilon;\lambda),A]$ defined as a form on $D(A)$ is extented to a bounded operator on L^2 which is denoted by $[M(\epsilon;\lambda),A]^0$.*

We can also prove the following lemma by a straightforward calculation.

LEMMA2.6. *Let $\lambda_0-\delta<\lambda<\lambda_0+\delta, 0<\epsilon<1$. There exists a positive constant C independent of λ and ϵ such that*

(i) $$\|(-\Delta+1)^{-1/2}(B(\lambda)-B(\epsilon;\lambda))(-\Delta+1)^{-1/2}\|\le C\epsilon^\theta,$$

(ii) $$\|(-\Delta+1)^{-1/2}(d/d\epsilon)B(\epsilon;\lambda)(-\Delta+1)^{-1/2}\|\le C\epsilon^{\theta-1},$$

(iii) $$\|(-\Delta+1)^{-1}[B(\epsilon;\lambda),A]^0(-\Delta+1)^{-1}\|\le C\epsilon^{\theta-1},$$

where $B(\lambda)=[H(\lambda),A]^0$.

SKETCH OF PROOF OF THEOREM 1.1(II) AND (III): By Theorem 1.1(i) and Lemma 2.3, we can take δ so small that

(2.11) $$M(\lambda)\equiv f_\delta(H(\lambda))i[H(\lambda),A]^0f_\delta(H(\lambda)) \ge(\alpha/2)f_\delta(H(\lambda))^2$$

in the form sense.

Moreover,(2.11) together with Lemma 2.6(i), implies that

(2.12) $$M(\epsilon;\lambda)\ge\gamma f_\delta(H(\lambda))^2$$

for $\epsilon>0$ small enough, where $\gamma>0$.

It follows from (2.12) that $M(\epsilon;\lambda)$ is non-negative and hence we define an operator, $G_\kappa(\epsilon;\lambda)$, on L^2 by

$$G_\kappa(\epsilon;\lambda)=(H(\lambda)-\lambda-i\kappa E(x)-i\epsilon M(\epsilon;\lambda))^{-1}$$

for $0<\kappa<1$ and $0<\epsilon<<1$.

For $1/2<\alpha\le1$, we write

$$F_\kappa(\epsilon;\lambda)=X_\alpha(\epsilon)G_\kappa(\epsilon;\lambda)X_\alpha(\epsilon),$$

where

$$X_\alpha(\epsilon)=(1+|x|)^{-\alpha}(1+\epsilon|x|)^{\alpha-1}.$$

Differentiating $F_\kappa(\epsilon;\lambda)$ in ϵ and using Lemma 2.3~Lemma2.6, we have (see Tamura [**13**])

(2.13) $$\|(d/d\epsilon)F_\kappa(\epsilon;\lambda)\|\le C(\epsilon^{\alpha-1}+\epsilon^{\alpha-3/2}\|F_\kappa\|^{1/2}+\epsilon^{\theta-1}\|F_\kappa\|).$$

Let $\epsilon_0, 0<\epsilon_0<<1$. Then by (2.12), we have

(2.14) $$\|F_\kappa(\epsilon_0;\lambda)\|\le C\epsilon_0^{-1}.$$

By (2.13) and (2.14), we immediately obtain

(2.15) $$\|F_\kappa(\epsilon;\lambda)\|\le C,$$

where $C>0$ independent of $0<\kappa<1, 0<\epsilon<\epsilon_0$ and $\lambda_0-\delta<\lambda<\lambda_0+\delta$. (2.13) and (2.15)imply Theorem 1.1(ii) and (iii). ∎

3. The limiting amplitude principle.

We consider only the case $1 = a_-^{-2} < a_+^{-2}$. The other cases can be proved similarly. Theorem 1.2 is obtained as an immediate consequence of the following

LEMMA 3.1. *Let the pair (α, β) be as in Theorem 1.2. Then*

$$||Q(\lambda, \pm i0; H(\lambda))^{-1}||_{\beta \to -\alpha} = O(\lambda^{-d}), \quad (\lambda \to 0),$$

for some $d, 0 < d < 1/2$.

By the assumption (A.3), we can decompose $a^{-2}(x)$ as $a^{-2}(x) = E_1^{\pm}(x) + E_2(x)(x \in \Omega_\pm)$ in such a way :

$$\sum_{|\alpha| \le 1} |x|^{|\alpha|} |\partial_x^\alpha (E_1^{\pm}(x) - a_\pm^{-2})| = O(|x|^{-\theta}) \quad (|x| \to \infty, x \in \Omega_\pm), \tag{3.1}$$

$$E_2(x) = O(|x|^{-1-\theta}) \quad (|x| \to \infty), \tag{3.2}$$

$$\sum_{|\alpha| \le 1} < x >^{|\alpha|} |\partial_x^\alpha (E_1^{\pm}(x) - a_\pm^{-2})| \le \delta_0 \quad (x \in \Omega_\pm), \tag{3.3}$$

for $\delta_0 > 0$ small enough, δ_0 being fixed throughout.

LEMMA 3.2. *Let $H_1(\lambda) = -\Delta - \lambda(E_1(x) - 1)$, and $\alpha > 1$. Then, we have*

$$||(H_1(\lambda) - \lambda \mp i\kappa a^{-2}(x))^{-1}||_{\alpha \to -\alpha} = O(1), \quad (\lambda \to 0), \tag{3.4}$$

uniformly in $\kappa > 0$ small enough, where $E_1(x) = E_1^{\pm}(x)(x \in \Omega_\pm)$

By interplation, We can show that Lemma 3.1 follows from Lemma 3.2.

we also prove Lemma 3.2 by making use of commutator method. We define the commutator $i[H_1(\lambda), A]$ as a form on $H^2(\mathbf{R}^n) \cap D(A)$ as follows : For $u, v \in H^2(\mathbf{R}^n) \cap D(A)$,

$$\begin{aligned} &< i[H_1(\lambda), A]u, v > \\ &= i(< Au, H_1(\lambda)v > - < H_1(\lambda)u, Av >). \end{aligned}$$

The key lemmas in this section are the following lemmas (Lemma 3.3 and 3.4).

LEMMA 3.3. *The form $i[H_1(\lambda), A]$ defined on $H^2(\mathbf{R}^n) \cap D(A)$ is extended to a bounded operator from $H^1(\mathbf{R}^n)$ to $H^{-1}(\mathbf{R}^n)$ which is denoted by $i[H_1(\lambda), A]^0$. Moreover we have*

$$\begin{aligned} &i[H_1(\lambda), A] \\ &= -2\Delta + \lambda(F_1 - T_\varphi^*(E_1^{+0} - E_1^{-0})\chi_{|y|>r}(y)(y \cdot \nabla_y \varphi(y) - \varphi(y))T_\varphi \\ &- (R_\varphi^r)^*(E_1^{+0} - E_1^{-0})R_\varphi^r), \end{aligned}$$

where $0 < r << 1$ *and* $F_1 = x \cdot \nabla E_1(x)(x \in \Omega_\pm)$.

LEMMA 3.4. *Let* $0 < \lambda << 1$, *take* $f_\lambda(p) \in C_0^\infty(\mathbf{R}), 0 \leq f_\lambda \leq 1$ *such that* f_λ *has support in* $(\lambda/3, 3\lambda)$ *and* $f_\lambda = 1$ *on* $[\lambda/2, 2\lambda]$. *Then, there exists a positive constant* C *which is independent of* λ *such that*

$$f_\lambda(H_1(\lambda))i[H_1(\lambda), A]f_\lambda(H_1(\lambda)) \geq C\lambda f_\lambda(H_1(\lambda))^2. \tag{3.5}$$

in the form sence.

We can prove Lemma 3.3 in the same way as in the proof of Lemma 2.3. Here, we give only a sketch of the proof of Lemma 3.4. In order to show Lemma 3.4, we need the following lemma.

LEMMA 3.5. *Let* $u \in H^1(\mathbf{R}^n)$. *Then, we have*

$$|T_\varphi u|^2_{L^2(\mathbf{R}^{n-1})} = \mp 2Re \int_{\Omega_\pm} u\overline{\partial_z u}dx.$$

PROOF: Let $u \in \mathbf{S}(\mathbf{R}^n)$. Noting that

$$\int_{\Omega_\pm} u\overline{\partial_z u}dx = \pm \int_{\mathbf{R}^{n-1}} \int_{\varphi(y)}^{\pm\infty} u\overline{\partial_z u}dzdy,$$

we have by integrating by parts

$$\int_{\Omega_\pm} u\overline{\partial_z u}dx = \mp \int_{\mathbf{R}^{n-1}} |T_\varphi u|^2 dy - \int_{\Omega_\pm} \partial_z u\overline{u}dx$$

Thus we have

$$\int_{\mathbf{R}^{n-1}} |T_\varphi u|^2 dy = \mp 2Re \int_{\Omega_\pm} u\overline{\partial_z u}dx$$

This implies lemma because $\mathbf{S}(\mathbf{R}^n)$ is dense in $H^1(\mathbf{R}^n)$. ∎

SKECH OF THE PROOF OF LEMMA 3.4: We simply write f_λ instead of $f_\lambda(H_1(\lambda))$.For $u \in L^2$, we have (for detail, see the proof of Lemma 2.3)

$$\begin{aligned} &< f_\lambda i[H_1(\lambda), A]f_\lambda u, u > \\ &= 2 < \nabla f_\lambda u, \nabla f_\lambda u > +\lambda < F_1 f_\lambda u, f_\lambda u > \\ &- \lambda \int_{\mathbf{R}^{n-1}} (y \cdot \nabla_y \varphi - \varphi)(E_1^{+0} - E_1^{-0})|T_\varphi f_\lambda u|^2 dy. \end{aligned} \tag{3.6}$$

We estimate the second and the third term respectively. (3.3) implies that

$$| < F_1 f_\lambda u, f_\lambda u > | \leq \delta_0 |f_\lambda u|_0^2. \tag{3.7}$$

Let $r > 0$ small enough. We decompose the third term as the following form

$$\begin{aligned}&\int_{\mathbf{R}^{n-1}}(y\cdot\nabla_y\varphi-\varphi)(E_1^{+0}-E_1^{-0})|T_\varphi f_\lambda u|^2dy\\&=\int_{|y|<r}(y\cdot\nabla_y\varphi-\varphi)(E_1^{+0}-E_1^{-0})|T_\varphi f_\lambda u|^2dy\\&+\int_{|y|>r}(y\cdot\nabla_y\varphi-\varphi)(E_1^{+0}-E_1^{-0})|T_\varphi f_\lambda u|^2dy\end{aligned}$$

Repeating the argument in the proof of (2.10), we have

$$(3.8)\quad |\int_{|y|<r}(y\cdot\nabla\varphi-\varphi)(E_1^{+0}-E_1^{-0})|T_\varphi f_\lambda u|^2dy|\le Cr^{\frac{n-1-\sigma p}{p}}(|\nabla f_\lambda u|_0^2+|f_\lambda|_0^2),$$

for some $p, n-1 < p < \frac{n-1}{\sigma}$.

Noting that $\chi_{|y|>r}(y)(y\cdot\nabla\varphi-\varphi)$ is bounded, by Lemma 3.5, we have

$$|\int_{|y|>r}(y\cdot\nabla\varphi-\varphi)(E_1^{+0}-E_1^{-0})|T_\varphi f_\lambda u|^2dy\le C<|f_\lambda u|,|\partial_z f_\lambda u|>.$$

Moreover, let $\delta > 0$, small enough. Then we have

$$(3.9)\qquad |\int_{|y|>r}(y\cdot\nabla\varphi-\varphi)(E_1^{+0}-E_1^{-0})|T_\varphi f_\lambda u|^2dy|\le\delta|f_\lambda u|_0^2+1/\delta|\nabla f_\lambda u|_0^2.$$

Noting $0<\lambda, r, \delta, \delta_0 << 1$, by (3.6) $\sim$ (3.9), we have (3.5). ∎

We define

$$E_{1,\epsilon}(x)=E_0(x)+\chi_n(\epsilon x)(E_1(x)-E_0(x)),$$

where $\epsilon > 0$ small enough. We further define an operator $C(\epsilon;\lambda)$ as

$$\begin{aligned}C(\epsilon;\lambda)=-2\Delta+\lambda(F_{1,\epsilon}&-T_\varphi^*(E_1^{+0}-E_1^{-0})V_\epsilon^r(y)T_\varphi\\&-(R_\varphi^r)^*(E_1^{+0}-E_1^{-0})R_\varphi^r),\end{aligned}$$

where $F_{1,\epsilon}=x\cdot\nabla E_{1,\epsilon}(x)(x\in\Omega_\pm)$. We can consider $C(\epsilon;\lambda)$ to be a bounded operator from $H^1(\mathbf{R}^n)$ to $H^{-1}(\mathbf{R}^n)$. $u\in H^2(\mathbf{R}^n)\cap D(A)$ satisfing $Au\in H^1(\mathbf{R}^n)$, we define the commutator $i[C(\epsilon;\lambda),A]$ as follows

$$\begin{aligned}&<i[C(\epsilon;\lambda),A]u,u>\\&=\sum_{j=1}^{3}<i[C_j(\epsilon;\lambda),A]u,u>\\&=\sum_{j=1}^{3}(<Au,C_j(\epsilon;\lambda)u>-<C_j(\epsilon;\lambda)u,Au>),\end{aligned}$$

where

$$C_1(\epsilon;\lambda) = -2\Delta + \lambda F_{1,\epsilon},$$
$$C_2(\epsilon;\lambda) = -\lambda T_\varphi^*(E_1^{+0} - E_1^{-0})V_\epsilon^r T_\varphi,$$
$$C_3(\epsilon;\lambda) = -\lambda(R_\varphi^r)^*(E_1^{+0} - E_1^{-0})R_\varphi^r.$$

We can also show that $i[C(\epsilon;\lambda),A]$ is extented to a bounded oparator $H^2(\mathbf{R}^n)$ to $H^{-2}(\mathbf{R}^n)$. Moreover, we have the following (see Lemma 2.5)

LEMMA 3.6. *Let $N(\epsilon;\lambda) = f_\lambda(H(\lambda))C(\epsilon;\lambda)f_\lambda(H(\lambda))$. Then $[N(\epsilon;\lambda),A]$ defined as a from on $D(A)$ is extended to a bounded operator on L^2 which is denoted by $[N(\epsilon;\lambda),A]^0$.*

We can prove the following lemma by a straightforward calculation.

LEMMA 3.7. *As $\lambda \to 0$, one has*
(i) $\|(-\Delta+\lambda)^{-1/2}(C(\epsilon;\lambda) - C(\lambda)(-\Delta+\lambda)^{-1/2}\| = \epsilon^\theta O(1)$,
(ii) $\|(-\Delta+\lambda)^{-1/2}(\frac{d}{d\epsilon}C(\epsilon;\lambda))(-\Delta+\lambda)^{-1/2}\| = \epsilon^{\theta-1}O(1)$,
(iii) $\|(-\Delta+\lambda)^{-1}[C(\epsilon;\lambda),A]^0(-\Delta+\lambda)^{-1}\| = \epsilon^{\theta-1}O(\lambda^{-1}) + O(\lambda^{-1})$,
where $C(\lambda) = i[H_1(\lambda),A]^0$.

SKETCH OF PROOF OF LEMMA 3.2: Lemma 3.4 and 3.7 (i) imply that

$$N(\epsilon;\lambda) \geq \gamma\lambda f_\lambda^2 \tag{3.10}$$

for $\epsilon > 0$ small enough, where $\gamma > 0$ is independent of ϵ, λ. It follows from (3.10) that $C(\epsilon;\lambda)$ is non-negative and hence we define an operator, $J_\kappa(\epsilon;\lambda)$, on L^2 by

$$J_\kappa(\epsilon;\lambda) = (H_1(\lambda) - \lambda - i\kappa a^{-2}(x) - i\epsilon N(\epsilon;\lambda))^{-1}$$

for $\kappa, 0 < \kappa < 1$ and $\epsilon > 0$ small enough.

We write

$$E_\kappa(\epsilon;\lambda) = X_1 J_\kappa(\epsilon;\lambda)X_1,$$

where $X_1 = (1+|x|^2)^{-1/2}$.
Differentiating $E_\kappa(\epsilon;\lambda)$ in ϵ and using Lemma 3.3~ 3.7 , we have

$$\|(d/d\epsilon)E_\kappa(\epsilon;\lambda)\| \leq C(1 + \epsilon^{-1/2}\|E_\kappa\|^{1/2} + \epsilon^{\theta-1}\|E_\kappa\|). \tag{3.11}$$

Let $\epsilon_0, 0 < \epsilon_0 << 1$. and

$$\|E_\kappa(\epsilon_0;\lambda)\| = \epsilon_0^{-1}O(1), \quad (\lambda \to 0). \tag{3.12}$$

(3.11) and (3.12) imply

$$\|E_\kappa(0;\lambda)\| = O(1), \quad (\lambda \to 0),$$

uniformly $\kappa, 0 < \kappa < 1$. Thus, we obtain Lemma 3.2. ∎

Remark. In order to show (3.12), we need the well-known inequality

$$\int_{\mathbf{R}^n} <x>^{-2} |u(x)|^2 dx \leq C\int_{\mathbf{R}^n} |\nabla u(x)|^2 dx, \tag{3.13}$$

where $n \geq 3$.

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