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ALGORITHMIC METHODS IN THE BOUNDARY VALUE PROBLEM FOR SYSTEMS OF LINEAR PARTIAL DIFFERENTIAL EQUATIONS WITH REGULAR SINGULARITIES

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Introduction.

Recently, the notion of Fuchsian partial differenital equation of [BG] has been generalized to that of Fuchsian system of linear partial differential equations along a submanifold of arbitrary codimension by Laurent and Monteiro Fernandes [LM]. (See also [Osh2] for a little more restricted class of systems with regular singularities and their boundary value problem.) Especially, it has been proved in [LM] for Fuchsian systems that any power series solution which converges with respect to the variables of Y and formal with respect to the variable(s) normal to Y converges with respect to all the variables. It is also known that the holonomic system with regular singularities in the sense of Kashiwara and Kawai is Fuchsian along any submanifold (cf. [KK], [LM]).

Thus Fuchsian systems constitute a nice and substantially wide class of systems containing many interesting examples (especially as holonomic systems). However, the definition of Fuchsian system is rather abstract and it would be difficult to apply it directly to a given system.

Suppose that a system of linear partial differential equations

$$\mathcal{M}: \quad P_1 u = \cdots = P_s u = 0$$

for an unkown function u in an open subset of $\mathbb{C}^{n+1}$ and a non-singular complex analytic hypersurface Y are given. (For example, if $\mathcal{M}$ is holonomic, then we take as Y an irreducible component of the "loci of singularities" of $\mathcal{M}$.) Then, from the computational point of view, we have the following basic problems about $\mathcal{M}$:

A. Is $\mathcal{M}$ Fuchsian along Y?
B. If so, find the structure of the space of multi-valued analytic (or hyperfunction, etc.) solutions of $\mathcal{M}$ around Y.

If the system $\mathcal{M}$ is Fuchsian, we can define its characteristic exponents as in the case of ordinary differential equations, and the "boundary values" of (multi-valued) analytic solutions of $\mathcal{M}$, which are analytic functions on Y. (Boundary values can be also defined for hyperfunction solutions (cf. [KO],[Osh1],[Osh2],[Oa]). However, in the present paper, we restrict ourselves to analytic solutions for the sake of simplicity.) Then a somewhat vague problem B reduces substantially to the more concrete one:

C. If $\mathcal{M}$ is Fuchsian along Y, compute its characteristic exponents and the system of equations which their boundary values satisfy (i.e. the induced, or the tangential system of $\mathcal{M}$ along Y).

The porpose of the present paper is to present algorithmic methods as partial but effective answers to the problems A and C. More precisely, we first give an algorithmic method, together with its theoretical foundation, that enables us to know whether or not $\mathcal{M}$ is formally Fuchsian in our terminology. Then we describe procedures for answering the problem C with the aid of the first method.

For this porpose, we introduce a new notion of Gröbner basis for the ring of differential operators with respect to a filtration of [K2] attached to the hypersurface Y.

The method of Gröbner basis was first introduced by Buchberger [Bu1] for the polynomial ring, and has been extended to various rings of differential operators by several authors (e.g. [Ga],[C],[N],[Tak1]). In particular, the singular loci and the rank (i.e. the dimension of the solution space) of a holonomic system are efficiently computed by using the Gröbner basis algorithm for the ring of differential operators of polynomial or rational function coefficients (cf. [Tak1], [Tak3]). The Gröbner basis for the ring of differential operators with analytic coefficients, which is more directly related to the analytic theory of systems of differential equations, was studied in [C],[OS].

In the present paper, we introduce Gröbner bases for rings of differential operators with analytic or rational function coefficients. The analytic version, which we call the FD-Gröbner basis (F for filtration, and D for the ring of differential operators with analytic coefficients), has a precise theoretical meaning concerning the local structure of the system, but it would be difficult to carry out actual computation in case of more than two variables. On the other hand, the rational version, which we call the FR-Gröbner basis (R for the ring of differential operators with rational coefficients), has an algebraic and global nature and is more suitable for actual computation by computers. Furthermore, it is shown that an FR-Gröbner basis supplies complete information on the precise local structure at a generic point of Y. (At a non-generic point, however, the FD-Gröbner basis is indispensable.) These Gröbner bases are defined by a new total order among (exponents of) monomials of differential operators, and the fact that this order is not a well-order makes the situation slightly more complicated than in the usual theory of Gröbner basis.

Our methods are efficient enough for the computation (by using computer algebra systems) of holonomic systems of two variables (possibly with additional several parameters) such as those for Appell's hypergeometric functions. We hope methods presented here will serve as new tools for the concrete computation of special systems with regular singularities of real research interest.

After the main part of the present work had been completed the author was informed that Takayama [Tak4] proposed a different method (a kind of Hensel construction) for solving the problem C with the porpose of finding connection formulas of special functions of several variables.

1. Fuchsian system of partial differential equations along a hypersurface.

Let $(t,x) = (t, x_1, \ldots, x_n)$ be a coordinate system of the $(n+1)$-dimensional complex

Euclidean space $X = \mathbb{C}^{n+1}$ (with $n \geq 1$) and we use the notation $\partial_t = \partial/\partial t$ and $\partial_x = (\partial_1, \dots, \partial_n)$ with $\partial_i = \partial/\partial x_i$. We write $x^\alpha = x_1^{\alpha_1} \dots x_n^{\alpha_n}$, $\partial_x{}^\beta = \partial_1^{\beta_1} \dots \partial_n^{\beta_n}$, $|\alpha| = \alpha_1 + \dots + \alpha_n$ for multi-indices $\alpha = (\alpha_1, \dots, \alpha_n)$, $\beta = (\beta_1, \dots, \beta_n) \in \mathbb{N}^n$ with $\mathbb{N} = \{0, 1, 2, \dots\}$.

1.1. Single Fuchsian partial differential equations.

Let us recall the definition of Fuchsian partial differential operator (or equation) following Baouendi and Goulaouic [BG] (in fact, the definition here is slightly more general than that given in [BG]). Let $P = P(t, x, \partial_t, \partial_x)$ be a linear partial differential operator with holomorphic function coefficients defined on an neighborhood of $p = (0, x_0)$. Then P is said to be a *Fuchsian partial differential operator* (and the equation $Pu = 0$ is said to be a *Fuchsian partial differential equation*) with respect to the hypersurface $Y = \{t = 0\}$ at p if there exist non-negative integers k, m such that P is written in the form

$$P = a_0(t, x) t^k \partial_t{}^m + \sum_{j=1}^{m} A_j(t, x, \partial_x) t^{\max\{k-j, 0\}} \partial_t{}^{m-j},$$

where $a_0(t, x)$ is a holomorphic function with $a_0(0, x_0) \neq 0$ and $A_j(t, x, \partial_x)$ is a differential operator of order at most j free from ∂_t satisfying

$$A_j(0, x, \partial_x) = a_j(x) \quad \text{(a holomorphic function of } x\text{)} \quad \text{for } 1 \leq j \leq \min\{k, m\}.$$

Then we call P a *Fuchsian operator of type* (k, m) (or of weight $m - k$ following [BG]). Fuchsian equation is also called equation with regular singularity in a weak sense by Kashiwara and Oshima ([KO], [Osh1]).

The *indicial polynomial* of P at p is a polynomial

$$a_0(0, x_0)\theta^m + \sum_{j=1}^{\min\{k,m\}} a_j(x_0)\theta(\theta - 1)\dots(\theta - m + j + 1)$$

in θ. And its zeros are called the *characteristic exponents* of P at p. Note that if $k < m$, then there are $m - k$ trivial characteristic exponents $0, \dots, m - k - 1$.

These definitions are generalized to general hypersurface instead of $t = 0$. Let $\varphi(t, x)$ be a holomorphic function defined on a neighborhood of $p = (t_0, x_0)$ with $\varphi(p) = 0$ and $d\varphi(p) \neq 0$. Then we can take a holomorphic local coordinate system (t', x') around p such that $t' = \varphi(t, x)$. Then P is said to be a Fuchsian operator with respect to the hypersurface $\varphi = 0$ if P satisfies the conditions above with (t, x) replaced by (t', x'). The characteristic exponents of P are also defined in the same way. It is easy to see that the definition of Fuchsian operator and the characteristic exponents are well-defined, i.e., independent of the choice of the local coordinate system.

The structure of the multi-valued analytic solutions of a Fuchsian equation was determined by Tahara [Tah].

1.2. Fuchsian systems of partial differential equations.

In the present paper we consider the system of linear partial differential equations

$$\mathcal{M}: \quad P_1 u = \dots = P_s u = 0$$

for an unknown function u, where $P_1, \ldots, P_s$ are linear partial differential operators whose coefficients are holomorphic functions on an open subset Ω of $X = \mathbb{C}^{n+1}$. (In the sequel, we assume $0 \in \Omega$.)

To study systems of linear partial differential equations such as $\mathcal{M}$, it is natural to consider the ideal of differential operators generated by $P_1, \ldots, P_s$. For this purpose, let us denote by $\mathcal{D}$ the sheaf of rings of linear partial differential operators with holomorphic coefficients on X. Let $\mathcal{I}$ be the sheaf of left ideals of $\mathcal{D}$ generated by $P_1, \ldots, P_s$; i.e.,

$$\mathcal{I} = \mathcal{D}P_1 + \cdots + \mathcal{D}P_s.$$

Moreover, we can regard the system $\mathcal{M}$ as a coherent sheaf of $\mathcal{D}$-modules $\mathcal{D}/\mathcal{I}$ (cf. [K1]). Let Y be a non-singular complex analytic hypersurface in Ω and p be a point of Y. Then the system $\mathcal{M}$ is called a *Fuchsian system* along Y at p after [LM] if there exists an element (section) P of $\mathcal{I}$ which is a Fuchsian operator with respect to Y at p. We call such P a *Fuchsian generator* of the system $\mathcal{M}$ at p.

1.3. A filtration of $\mathcal{D}_0$ and formally Fuchsian operators.

We denote by $\mathcal{D}_0$ the stalk of the sheaf $\mathcal{D}$ at the origin $0 \in X$. Put $Y = \{(t,x) \in X \mid t = 0\}$. The following arguments apply likewise to the stalk $\mathcal{D}_p$ of $\mathcal{D}$ at $p \in Y$. An element of $\mathcal{D}_0$ is a linear partial differential operator whose coefficients are holomorphic at 0; i.e. convergent power series of (t,x). An operator $P \in \mathcal{D}_0$ is written in the form

$$P = \sum_{\nu \geq 0, \beta \in \mathbf{N}^n} a_{\nu,\beta}(t,x) {\partial_t}^{\nu} {\partial_x}^{\beta} = \sum_{\nu,\mu \geq 0, \beta, \alpha \in \mathbf{N}^n} a_{\mu,\nu,\alpha,\beta} t^{\mu} x^{\alpha} {\partial_t}^{\nu} {\partial_x}^{\beta}, \tag{1.1}$$

where the sum is finite with respect to ν and β. The order of P $\mathrm{ord}(P)$ is defined as the maximum of $\nu + |\beta|$ such that $a_{\nu,\beta}(t,x)$ is non-zero as a power series.

We introduce a filtration $\{\mathcal{F}_m\}_{m \in \mathbf{Z}}$ of $\mathcal{D}_0$ as follows: For each integer m, put

$$\mathcal{F}_m = \{P = \sum_{\mu,\nu,\alpha,\beta} a_{\mu,\nu,\alpha,\beta} t^{\mu} x^{\alpha} {\partial_t}^{\nu} {\partial_x}^{\beta} \in \mathcal{D}_0 \mid a_{\mu,\nu,\alpha,\beta} = 0 \text{ if } \nu - \mu > m\}.$$

Then $\mathcal{F}_m$ is a $\mathbb{C}$-subspace of $\mathcal{D}_0$ and satisfies

$$\ldots \mathcal{F}_{-2} \subset \mathcal{F}_{-1} \subset \mathcal{F}_0 \subset \mathcal{F}_1 \subset \mathcal{F}_2 \ldots, \quad \bigcup_{m \in \mathbf{Z}} \mathcal{F}_m = \mathcal{D}_0.$$

For a nonzero $P \in \mathcal{D}_0$, its *F-order* $\mathrm{ord}_F(P)$ is defined as the minimum integer m satisfying $P \in \mathcal{F}_m$. If the F-order of the operator P written as (1.1) is m, then we put

$$\hat{\sigma}(P) = \hat{\sigma}_m(P) = \sum_{\nu - \mu = m} a_{\mu,\nu,\alpha,\beta} t^{\mu} x^{\alpha} {\partial_t}^{\nu} {\partial_x}^{\beta} \in \mathcal{D}_0$$

and call it the *formal symbol* of P after [LS]. (We put $\hat{\sigma}(0) = 0$.)

The filtration defined above was introduced by Kashiwara [K2] and was used systematically with the formal symbol for the study of induced systems by Laurent and Schapira [LS].

Lemma 1.1. *For $P, Q \in \mathcal{D}_0$, we have $\hat{\sigma}(PQ) = \hat{\sigma}(P)\hat{\sigma}(Q)$.*

Note that in general, $\hat{\sigma}(Q)\hat{\sigma}(P) \neq \hat{\sigma}(P)\hat{\sigma}(Q)$.

Lemma 1.2. *$P \in \mathcal{D}_0$ is a Fuchsian operator with respect to $Y = \{t = 0\}$ at 0 if and only if P satisfies the following two conditions (FC1) and (FC2):*

(FC1) *There exist non-negative integers k, m and holomorphic functions $a_j(x)$ with $a_0(0) \neq 0$ such that*

$$\hat{\sigma}(P) = \sum_{j=0}^{\min\{k,m\}} a_j(x) t^{k-j} \partial_t{}^{m-j}.$$

(FC2) *The order of $\hat{\sigma}(P)$ is equal to the order of P.*

Definition 1.3. We call P a *formally Fuchsian operator* with respect to Y at 0 if P satisfies the condition (FC1). The system $\mathcal{M}$ is said to be *formally Fuchsian* along Y at 0 if there exists a formally Fuchsian operator $P \in \mathcal{I}$ with respect to Y at 0.

The notion of formally Fuchsian system is a special case of that of "système elliptique le long de Y" defined in [LS].

1.4. Characteristic exponents of a Fuchsian system.

Let $\bar{\mathcal{D}}_0$ be the graded ring associated with the filtration $\{\mathcal{F}_m\}$; i.e.,

$$\bar{\mathcal{D}}_0 = \bigoplus_{m \in \mathbf{Z}} \mathcal{F}_m / \mathcal{F}_{m-1}.$$

Note that $\bar{\mathcal{D}}_0$ is a non-commutative ring. Then the formal symbol induces a map

$$\hat{\sigma} = \hat{\sigma}_m : \mathcal{F}_m \longrightarrow \mathcal{F}_m / \mathcal{F}_{m-1} \subset \bar{\mathcal{D}}_0$$

for any integer m.

We shall define an injective ring homomorphism ψ of $\bar{\mathcal{D}}_0$ into the ring

$$\mathcal{D}_0'[\theta, \tau, \tau^{-1}] := \bigoplus_{m \in \mathbf{Z}} \mathcal{D}_0'[\theta] \tau^m,$$

where τ and θ are indeterminates and $\mathcal{D}_0'[\theta]$ denotes the polynomial ring in θ with coefficients in the ring $\mathcal{D}_0' = \mathbb{C}\{x\}\langle \partial_x \rangle$ of differential operators in x with convergent power series coefficients. We give a ring structure to $\mathcal{D}_0'[\theta, \tau, \tau^{-1}]$ by

$$(P(\theta, x, \partial_x)\tau^j) \cdot (Q(\theta, x, \partial_x)\tau^k) := P(\theta - k, x, \partial_x) Q(\theta, x, \partial_x) \tau^{j+k}.$$

Any element P of $\mathcal{T}_m := \mathcal{F}_m \setminus \mathcal{F}_{m-1}$ can be written uniquely in the form

$$P = t^{-m} \hat{P}(t\partial_t, x, \partial_x).$$

Then we put $\psi(P) = \hat{P}(\theta, x, \partial_x)\tau^m$. This defines a map $\psi : \mathcal{F}_m / \mathcal{F}_{m-1} \to \mathcal{D}_0'[\theta]\tau^m$ for each m. Moreover, it is easy to see that this ψ is injective for all m and bijective for $m \leq 0$. Thus ψ defines an injective map of $\bar{\mathcal{D}}_0$ into $\mathcal{D}_0'[\theta, \tau, \tau^{-1}]$.

Lemma 1.4. *$\psi : \bar{\mathcal{D}}_0 \to \mathcal{D}'_0[\theta, \tau, \tau^{-1}]$ is an injective ring homomorphism.*

Now assume the system $\mathcal{M}$ (as in Sect. 1.2) is Fuchsian along $Y = \{t = 0\}$ at 0. (In fact, it suffices to assume $\mathcal{M}$ is formally Fuchsian for the following definitions.) Let $\mathcal{I}_0$ be the stalk at 0 of the sheaf of left ideals $\mathcal{I} = \mathcal{D}P_1 + \cdots + \mathcal{D}P_s$. Let us define a left ideal $\bar{\mathcal{I}}_0$ of $\mathcal{D}'_0[\theta, \tau, \tau^{-1}]$ by

$$\bar{\mathcal{I}}_0 = \bigoplus_{m \in \mathbf{Z}} \hat{\sigma}_m(\mathcal{I}_0 \cap \mathcal{F}_m).$$

Put

$$\mathcal{O}'_0[\theta, \tau, \tau^{-1}] = \bigoplus_{m \in \mathbf{Z}} \mathcal{O}'_0[\theta]\tau^m \subset \mathcal{D}'_0[\theta, \tau, \tau^{-1}]$$

with $\mathcal{O}'_0 = \mathbb{C}\{x\}$ (the ring of convergent power series in x). Let $\mathcal{J}$ be the smallest left ideal of $\mathcal{O}'_0[\theta, \tau, \tau^{-1}]$ that contains $\psi(\bar{\mathcal{I}}_0) \cap \mathcal{O}'_0[\theta, \tau, \tau^{-1}]$ and put $\mathcal{J}_Y(\mathcal{M}, 0) = \mathcal{J} \cap \mathcal{O}'_0[\theta]$, which is an ideal of the commutative ring $\mathcal{O}'_0[\theta]$. Then it is easy to see that $\mathcal{J}$ is generated by $\mathcal{J}_Y(\mathcal{M}, 0)$ over $\mathcal{O}'_0[\theta, \tau, \tau^{-1}]$. Moreover we can easily verify

Lemma 1.5.

$$\mathcal{J}_Y(\mathcal{M}, 0) = \{f(\theta, x) \in \mathcal{O}'_0[\theta] \mid f(\theta, x)\tau^{-m} \in \psi(\bar{\mathcal{I}}_0) \cap \mathcal{O}'_0[\theta]\tau^{-m} \text{ for some } m \geq 0\}.$$

The ideal $\mathcal{J}_Y(\mathcal{M}, p)$ of $\mathcal{O}'_p[\theta]$ is defined likewise with 0 replaced by a point p of Y, where $\mathcal{O}'_p$ denotes the ring of holomorphic functions in x at p.

Definition 1.6. For a point p of Y we call the set

$$e_Y(\mathcal{M}, p) := \{\theta \in \mathbb{C} \mid f(\theta, p) = 0 \text{ for any } f \in \mathcal{J}_Y(\mathcal{M}, p)\}$$

the *set of the characteristic exponents* of $\mathcal{M}$ along Y at p.

Definition 1.7. We define another ideal $\tilde{\mathcal{J}}_Y(\mathcal{M}, p)$ of $\mathcal{O}'_p[\theta]$ by

$$\tilde{\mathcal{J}}_Y(\mathcal{M}, p) = \{f \in \mathcal{O}'_p[\theta] \mid af \in \mathcal{J}_Y(\mathcal{M}, p) \text{ for some } a \in \mathcal{O}'_p\}.$$

Then we define the *set of the strong characteristic exponents* of $\mathcal{M}$ along Y at $p \in Y$ by

$$\tilde{e}_Y(\mathcal{M}, p) = \{\theta \in \mathbb{C} \mid f(\theta, p) = 0 \text{ for any } f \in \tilde{\mathcal{J}}_Y(\mathcal{M}, p)\}.$$

Lemma 1.8. *Suppose that the system $\mathcal{M}$ is formally Fuchsian at 0. Then the ideal $\tilde{\mathcal{J}}_Y(\mathcal{M}, 0)$ is generated by a polynomial $f \in \tilde{\mathcal{J}}_Y(\mathcal{M}, 0)$ monic in θ.*

Example 1.9. Put $n = 1$, $x = x_1$ and let us consider the system

$$\mathcal{N} : \quad (t\partial_t - a)(t\partial_t - b)u = x(t\partial_t - a)u = 0$$

with distinct constants $a, b \in \mathbb{C}$. Then we have

$$\mathcal{J}_Y(\mathcal{N}, 0) = \mathcal{O}'_0[\theta](\theta - a)(\theta - b) + \mathcal{O}'_0[\theta]x(\theta - a),$$
$$\tilde{\mathcal{J}}_Y(\mathcal{N}, 0) = \mathcal{O}'_0[\theta](\theta - a)$$

and hence

$$e_Y(\mathcal{N}, 0) = \{a, b\}, \quad \tilde{e}_Y(\mathcal{N}, 0) = \{a\}.$$

Note that any multi-valued analytic solution of $\mathcal{N}$ is in the form $u = v(x)t^a$ with v holomorphic, whereas, in the real domain $\mathcal{N}$ has a distribution solution $t_+^a + \delta(x)t_+^b$.

1.5. Boundary value problem for Fuchsian systems.

Here we recall some known facts on the structure of analytic solutions of a Fuchsian system. First, let us recall the notion of induced (tangential) system. Let $\mathcal{M}$ and $\mathcal{I}$ be as in Sect. 1.2. Then the induced (tangential) system $\mathcal{M}_Y$ of $\mathcal{M}$ along $Y = \{t = 0\}$ is the sheaf of $\mathcal{D}'$-modules

$$\mathcal{M}_Y := \mathcal{M}/t\mathcal{M} = \mathcal{D}/(t\mathcal{D} + \mathcal{I}),$$

where $\mathcal{D}'$ denotes the sheaf on Y of the ring of linear differential operators with holomorphic functions in x as coefficients. It is shown in [LM] that $\mathcal{M}_Y$ is a coherent $\mathcal{D}'$-module if $\mathcal{M}$ is Fuchsian along Y.

Theorem 1.10 ([LM, Théorème 3.2.2]). *Assume that the system $\mathcal{M}$ is Fuchsian along Y. Then there exists a canonical sheaf isomorphism*

$$\mathcal{H}om_{\mathcal{D}}(\mathcal{M}, \mathcal{O})|_Y \simeq \mathcal{H}om_{\mathcal{D}'}(\mathcal{M}_Y, \mathcal{O}'),$$

where $\mathcal{O}$ and $\mathcal{O}'$ denote the sheaves of holomorphic functions in (t, x) and in x respectively, and $\mathcal{H}om$ the sheaf of homomorphisms.

Theorem 1.11. *Assume that the system $\mathcal{M}$ is Fuchsian along Y at 0 and there exists a Fuchsian operator $P \in \mathcal{I}_0$ whose characteristic exponents $\theta_1, \dots, \theta_m$ are all constant with multiplicity one. Assume also that $\theta_i - \theta_j$ is not an integer for any $i \neq j$. Put $S = \{i \in \{1, \dots, m\} \mid \theta_i \in \tilde{e}_Y(\mathcal{M}, 0)\}$. Then any (multi-valued) analytic solution u of $\mathcal{M}$ on $U \setminus Y$ with U being a neighborhood of $0 \in X$ can be written in the form*

$$u = \sum_{i \in S} v_i(t, x) t^{\theta_i}$$

with holomorphic functions v_i on a neighborhood of $U \cap Y$.

2. FD-Gröbner basis—precise and local algorithmic method.

In this section we develop the theory of FD-Gröbner bases for left ideals of the ring $\mathcal{D}_0$ of differential operators with analytic coefficients. Instead of $\mathcal{D}_0$, the following arguments apply also to the stalk $\mathcal{D}_p$ of the sheaf $\mathcal{D}$ at $p \in Y = \{(t, x) \mid t = 0\}$.

Let $\prec$ be a lexicographic order of $\mathbb{N}^n$ with $\mathbb{N} := \{0, 1, 2, \dots\}$. We define a total order $\prec_{FD}$ of the set $\mathbb{N}^{2n+2}$, which we call the *FD-order*, as follows: For two indices $(\mu, \nu, \alpha, \beta)$ and $(\mu', \nu', \alpha', \beta') \in \mathbb{N} \times \mathbb{N} \times \mathbb{N}^n \times \mathbb{N}^n$,

$$\begin{aligned}(\mu, \nu, \alpha, \beta) \prec_{FD} (\mu', \nu', \alpha', \beta') \quad &\text{if and only if} \quad (\nu - \mu < \nu' - \mu') \\ &\text{or} \quad (\nu - \mu = \nu' - \mu',\ |\beta| < |\beta'|) \\ &\text{or} \quad (\nu - \mu = \nu' - \mu',\ |\beta| = |\beta'|,\ \nu < \nu') \\ &\text{or} \quad (\nu = \nu',\ \mu = \mu',\ \beta \prec \beta') \\ &\text{or} \quad (\nu = \nu',\ \mu = \mu',\ \beta = \beta',\ |\alpha| > |\alpha'|) \\ &\text{or} \quad (\nu = \nu',\ \mu = \mu',\ \beta = \beta',\ |\alpha| = |\alpha'|,\ \alpha \prec \alpha').\end{aligned}$$

Let the *FR-order* $\prec_{FR}$ be the order of $\mathbb{N}^{n+2}$ induced by $\prec_{FD}$; i.e., we define

$$(\mu, \nu, \beta) \prec_{FR} (\mu', \nu', \beta') \quad \text{if and only if} \quad (\mu, \nu, 0, \beta) \prec_{FD} (\mu', \nu', 0, \beta').$$

It is easy to see that any subset of $\{(\mu,\nu,\alpha,\beta)\in\mathbb{N}^{2n+2}\mid \nu+|\beta|\le m\}$ has a maximum element with respect to the FD-order, and any subset of $\{(\mu,\nu,\beta)\in\mathbb{N}^{n+2}\mid \nu-\mu\ge m\}$ has a minimum element with respect to the FR-order for any m. (This definition of the FD-order can be generalized to some extent, but we do not discuss this problem here.) For an element $P\in\mathcal{D}_0$ of the form

$$P=\sum_{\mu,\nu,\alpha,\beta} a_{\mu,\nu,\alpha,\beta}t^{\mu}x^{\alpha}\partial_t{}^{\nu}\partial_x{}^{\beta},$$

we define the set of exponents, leading exponent, leading coefficient, leading term of P with respect to the FD-order by

$$\begin{aligned}
\mathrm{exps}_{FD}(P)&=\{(\mu,\nu,\alpha,\beta)\mid a_{\mu,\nu,\alpha,\beta}\ne 0\},\\
\mathrm{lexp}_{FD}(P)&=\max{}_{FD}(\mathrm{exps}_{FD}(P)),\\
\mathrm{lcoef}_{FD}(P)&=a_{\mu,\nu,\alpha,\beta}\quad\text{with } (\mu,\nu,\alpha,\beta):=\mathrm{lexp}_{FD}(P),\\
\mathrm{lterm}_{FD}(P)&=a_{\mu,\nu,\alpha,\beta}t^{\mu}x^{\alpha}\partial_t{}^{\nu}\partial_x{}^{\beta}\quad\text{with } (\mu,\nu,\alpha,\beta):=\mathrm{lexp}_{FD}(P),
\end{aligned}$$

where $\max{}_{FD}$ denotes the maximum with respect to the FD-order. (If $P=0$, then we put $\mathrm{lexp}_{FD}(P)=(\infty,0,0,0)$, and suppose $(\infty,0,0,0)\prec_{FD}(\mu,\nu,\alpha,\beta)$ for any $(\mu,\nu,\alpha,\beta)\in\mathbb{N}^{2n+2}$.) Let $\pi:\mathbb{N}^{2n+2}\to\mathbb{N}^{n+2}$ be the projection defined by $\pi(\mu,\nu,\alpha,\beta)=(\mu,\nu,\beta)$. Then through this projection, we also define the leading exponent, leading coefficient and leading term of P with respect to the FR-order by

$$\begin{aligned}
\mathrm{lexp}_{FR}(P)&=\pi(\mathrm{lexp}_{FD}(P)),\\
\mathrm{lcoef}_{FR}(P)&=\sum_{\alpha\in\mathbb{N}^n} a_{\mu_0,\nu_0,\alpha,\beta_0}x^{\alpha}\quad\text{with } (\mu_0,\nu_0,\beta_0):=\mathrm{lexp}_{FR}(P),\\
\mathrm{lterm}_{FR}(P)&=\mathrm{lcoef}_{FR}(P)t^{\mu_0}\partial_t{}^{\nu_0}\partial_x{}^{\beta_0}\quad\text{with } (\mu_0,\nu_0,\beta_0):=\mathrm{lexp}_{FR}(P).
\end{aligned}$$

Moreover, for an exponent $(\mu,\nu,\beta)\in\mathbb{N}^{n+2}$, we set

$$\mathrm{coef}_{FR}(P,(\mu,\nu,\beta))=\sum_{\alpha} a_{\mu,\nu,\alpha,\beta}x^{\alpha}.$$

Recall that the principal symbol of P (of order m) is defined by

$$\sigma_m(P)=\sum_{\mu\in\mathbb{N},\alpha\in\mathbb{N}^n,\nu+|\beta|=m} a_{\mu,\nu,\alpha,\beta}t^{\mu}\tau^{\nu}x^{\alpha}\xi^{\beta}$$

regarded as an element of the ring of the convergent power series $\mathbb{C}\{t,\tau,x,\xi\}$ with $\xi=(\xi_1,\dots,\xi_n)$ and $\xi^{\beta}=\xi_1{}^{\beta_1}\cdots\xi_n{}^{\beta_n}$ if P is of order $\le m$. We also write $\sigma(P)=\sigma_m(P)$ if P is precisely of order m.

Lemma 2.1. *For $P,Q\in\mathcal{D}_0$ we have*

$$\begin{aligned}
\mathrm{lexp}_{FD}(PQ)&=\mathrm{lexp}_{FD}(P)+\mathrm{lexp}_{FD}(Q),\\
\mathrm{lcoef}_{FD}(PQ)&=\mathrm{lcoef}_{FD}(P)\mathrm{lcoef}_{FD}(Q),\\
\mathrm{lexp}_{FR}(PQ)&=\mathrm{lexp}_{FR}(P)+\mathrm{lexp}_{FR}(Q),\\
\mathrm{lcoef}_{FR}(PQ)&=\mathrm{lcoef}_{FR}(P)\mathrm{lcoef}_{FR}(Q).
\end{aligned}$$

Lemma 2.2. *$P \in \mathcal{D}_0$ is formally Fuchsian along Y at 0 if and only if* $\mathrm{lexp}_{FD}(P) = (\mu, \nu, 0, 0) \in \mathbb{N} \times \mathbb{N} \times \mathbb{N}^n \times \mathbb{N}^n$ *with some* $\mu, \nu \in \mathbb{N}$.

Lemma 2.3 (A division theorem). *Let P and $P_1, \dots, P_s$ be elements of $\mathcal{D}_0$. Then for any integer m, there exist elements $Q_1, \dots, Q_s$ and R of $\mathcal{D}_0$ such that*

$$P = \sum_{i=1}^{s} Q_i P_i + R,$$

$$\mathrm{exps}_{FD}(R) \cap \bigcup_{i=1}^{s} (\mathrm{lexp}_{FD}(P_i) + \mathbb{N}^{2n+2}) \subset \mathcal{F}_m,$$

$$\mathrm{lexp}_{FD}(Q_i P_i) \preceq_{FD} \mathrm{lexp}_{FD}(P), \quad \mathrm{lexp}_{FD}(R) \preceq_{FD} \mathrm{lexp}_{FD}(P).$$

We denote such R, which is not necessarily unique, by $\mathrm{red}_{FD}(P, \{P_1, \dots, P_s\}, m)$.

Definition 2.4. Let $\mathcal{I}_0$ be a left ideal of $\mathcal{D}_0$. Then a finite subset $\mathbf{G} = \{P_1, \dots, P_s\}$ of $\mathcal{I}_0$ is called an *FD-Gröbner basis* of $\mathcal{I}_0$ (along Y) if it satisfies the following two conditions:

(1) $\mathbf{G}$ generates $\mathcal{I}_0$, i.e., $\mathcal{I}_0 = \mathcal{D}_0 P_1 + \cdots + \mathcal{D}_0 P_s$.
(2) Put $E_{FD}(\mathcal{I}_0) = \{\mathrm{lexp}_{FD}(P) \mid P \in \mathcal{I}_0\}$. Then we have

$$E_{FD}(\mathcal{I}_0) = \bigcup_{P \in \mathbf{G}} (\mathrm{lexp}_{FD}(P) + \mathbb{N}^{2n+2}).$$

Definition 2.5. For $P, Q \in \mathcal{D}_0$ with

$$\mathrm{lexp}_{FD}(P) = (\mu, \nu, \alpha, \beta), \quad \mathrm{lexp}_{FD}(Q) = (\mu', \nu', \alpha', \beta'),$$

the S-polynomial (or S-operator) of P and Q is defined by

$$\begin{aligned}\mathrm{sp}_{FD}(P, Q) = \mathrm{lcoef}_{FD}(Q) t^{\mu \vee \mu' - \mu} \partial_t^{\nu \vee \nu' - \nu} x^{\alpha \vee \alpha' - \alpha} \partial_x^{\beta \vee \beta' - \beta} P \\ - \mathrm{lcoef}_{FD}(P) t^{\mu \vee \mu' - \mu'} \partial_t^{\nu \vee \nu' - \nu'} x^{\alpha \vee \alpha' - \alpha'} \partial_x^{\beta \vee \beta' - \beta'} Q,\end{aligned}$$

where we use the notation

$$\nu \vee \nu' := \max\{\nu, \nu'\}, \quad \alpha \vee \alpha' := (\max\{\alpha_1, \alpha_1'\}, \dots, \max\{\alpha_n, \alpha_n'\})$$

for $\nu, \nu' \in \mathbb{N}$ and $\alpha = (\alpha_1, \dots, \alpha_n)$, $\alpha' = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}^n$.

Theorem 2.6. *Let $\mathcal{I}_0$ be a left ideal of $\mathcal{D}_0$ and $\mathbf{G} = \{P_1, \dots, P_s\}$ be a set of generators of $\mathcal{I}_0$. Then the following two conditions for $\mathbf{G}$ are equivalent:*

(1) *$\mathbf{G}$ is an FD-Gröbner basis of $\mathcal{I}_0$.*
(2) *For any i, j with $1 \le i < j \le s$ and for any $m \in \mathbb{Z}$, there exist $Q_{ij1}, \dots, Q_{ijs} \in \mathcal{D}_0$ and $R_{ij} \in \mathcal{F}_m$ such that*

$$\mathrm{sp}_{FD}(P_i, P_j) = \sum_{k=1}^{s} Q_{ijk} P_k + R_{ij}$$

with $\mathrm{lexp}_{FD}(Q_{ijk} P_k) \prec_{FD} \mathrm{lexp}_{FD}(P_i) \vee \mathrm{lexp}_{FD}(P_j)$ *for any* k.

Theorem 2.6 together with Lemma 2.3 enables us to give an algorithm to compute, at least theoretically, an FD-Gröbner basis of a given left ideal of $\mathcal{D}_0$.

Algorithm 2.7 (FD-Gröbner basis). Given a finite set $\mathbf{G}$ of generators of a left ideal $\mathcal{I}_0$ of $\mathcal{D}_0$ find an FD-Gröbner basis of $\mathcal{I}_0$.

```
m := min{ord_F(P) | P ∈ G};
G_m := G;
REPEAT
    G_{m-1} := G_m;
    m := m - 1;
    REPEAT
        FOR each pair (P, Q) of elements of G_m DO {
            R := red_FD(sp_FD(P, Q), G_m, m);
            IF R ∉ F_m THEN G_m := G_m ∪ {R};
        }
    UNTIL red_FD(sp_FD(P, Q), G_m, m) ∈ F_m for any P, Q ∈ G_m;
UNTIL G_m becomes stationary, i.e. G_m = G_μ for any μ < m;
RETURN G_m;
```

The output of this algorithm is indeed an FD-Gröbner basis by virtue of Theorem 2.6. The termination condition of this algorithm is fulfilled a priori in a finitely many steps because of the Noetherian property of monoideals (or monomial ideals) generated by the leading exponents of elements of $\mathbf{G}_m$ (cf. [CLO, pp. 68–72]). However, at present, we do not have a general criterion for the termination; i.e. we do not know when to stop the algorithm. For a sufficient condition for the termination, see Proposition 3.8.

When $n = 1$, this computation can be actually performed by using a computer algebra system if the given generators are operators with polynomial coefficients. However for $n > 1$, the actual computation would be difficult because of the transcendental nature of the so-called Weierstrass-Hironaka division employed in the proof of Lemma 2.3.

The FD-Gröbner basis solves partially the problem A:

Theorem 2.8. *Let $\mathcal{M}$ and $\mathcal{I}$ be as in Sect. 1.2 and let $\mathcal{I}_0$ be the stalk of the sheaf $\mathcal{I}$ at 0. Assume that $\mathbf{G}$ is an FD-Gröbner basis of the left ideal $\mathcal{I}$ of $\mathcal{D}_0$. Then $\mathcal{M}$ is formally Fuchsian along $Y = \{(t, x) \mid t = 0\}$ at 0 if and only if there exists $P \in \mathbf{G}$ such that* $\mathrm{lexp}_{FD}(P) = (\mu, \nu, 0, 0)$ *with some $\mu, \nu \in \mathbb{N}$.*

3. FR-Gröbner basis—global algorithmic method.

In order to carry out actual computation, we introduce the ring $\mathcal{D}_R$ of differential operators whose coefficients are formal power series of t with rational functions of x as coefficients:

$$\begin{aligned}\mathcal{D}_R &:= \mathbb{C}(x)[[t]]\langle \partial_t, \partial_x \rangle \\ &= \{P = \sum_{\mu,\nu,\beta} a_{\mu,\nu,\beta}(x) t^\mu {\partial_t}^\nu {\partial_x}^\beta \mid a_{\mu,\nu,\beta}(x) \text{ is a rational function of } x\},\end{aligned}$$

where the sum is finite with respect to ν and β.

For theoretical porpose, it is also useful to consider the ring $\mathcal{D}_M$ of differential operators whose coefficients are formal power series of t with meromorphic functions in x as coefficients:

$$\mathcal{D}_M := \mathcal{K}'_0[[t]]\langle\partial_t, \partial_x\rangle = \{P = \sum_{\mu,\nu,\beta} a_{\mu,\nu,\beta}(x)t^\mu {\partial_t}^\nu {\partial_x}^\beta \mid a_{\mu,\nu,\beta}(x) \in \mathcal{K}'_0\},$$

where $\mathcal{K}'_0$ denotes the quotient field of the ring $\mathcal{O}'_0$ of germs of holomorphic functions in x at 0. More generally, we can take any intermediate field lying between $\mathbb{C}(x)$ and $\mathcal{K}'_0$. The following definitions and arguments apply also to such cases instead of $\mathcal{D}_R$.

For an operator $P \in \mathcal{D}_R$ of the form

$$P = \sum_{\mu,\nu,\beta} a_{\mu,\nu,\beta}(x)t^\mu {\partial_t}^\nu {\partial_x}^\beta,$$

we define its leading exponent, leading term, leading coefficient (in the FR-order) by

$$\begin{aligned}
\mathrm{lexp}_{FR}(P) &= \max{}_{FR}\{(\mu,\nu,\beta) \mid a_{\mu,\nu,\beta}(x) \neq 0\},\\
\mathrm{lcoef}_{FR}(P) &= a_{\mu,\nu,\beta}(x) \quad \text{with} \quad (\mu,\nu,\beta) := \mathrm{lexp}_{FR}(P),\\
\mathrm{lterm}_{FR}(P) &= a_{\mu,\nu,\beta}(x)t^\mu {\partial_t}^\nu {\partial_x}^\beta \quad \text{with} \quad (\mu,\nu,\beta) := \mathrm{lexp}_{FR}(P).
\end{aligned}$$

In the same way as Lemma 2.1 we get

Lemma 3.1. *For $P, Q \in \mathcal{D}_R$ we have*

$$\begin{aligned}
\mathrm{lexp}_{FR}(PQ) &= \mathrm{lexp}_{FR}(P) + \mathrm{lexp}_{FR}(Q),\\
\mathrm{lcoef}_{FR}(PQ) &= \mathrm{lcoef}_{FR}(P)\mathrm{lcoef}_{FR}(Q).
\end{aligned}$$

Definition 3.2. Let I be a left ideal of $\mathcal{D}_R$. Then a finite subset $\mathbf{G} = \{P_1, \dots P_s\}$ of $\mathcal{D}_R$ is said to be an *FR-Gröbner basis* of I (along $Y = \{t = 0\}$) if it satisfies the following two conditions:

(1) $\mathbf{G}$ generates I, i.e., $I = \mathcal{D}_R P_1 + \cdots + \mathcal{D}_R P_s$.
(2) Put $E_{FR}(I) := \{\mathrm{lexp}_{FR}(P) \mid P \in I\}$. Then we have

$$E_{FR}(I) = E_{FR}(\mathbf{G}) := \bigcup_{P \in \mathbf{G}} (\mathrm{lexp}_{FR}(P) + \mathbb{N}^{n+2}).$$

Definition 3.3. For $P, Q \in \mathcal{D}_R$ with

$$\mathrm{lexp}_{FR}(P) = (\mu,\nu,\beta), \quad \mathrm{lexp}_{FR}(Q) = (\mu',\nu',\beta'),$$

the S-polynomial (or the S-operator) of P and Q is defined by

$$\begin{aligned}
\mathrm{sp}_{FR}(P,Q) := {}& \mathrm{lcoef}_{FR}(Q)t^{\mu\vee\mu'-\mu}{\partial_t}^{\nu\vee\nu'-\nu}{\partial_x}^{\beta\vee\beta'-\beta}P\\
&- \mathrm{lcoef}_{FR}(P)t^{\mu\vee\mu'-\mu'}{\partial_t}^{\nu\vee\nu'-\nu'}{\partial_x}^{\beta\vee\beta'-\beta'}Q.
\end{aligned}$$

As in the previous section, we define a filtration of $\mathcal{D}_R$ by

$$\mathcal{F}_m = \{P = \sum_{\mu,\nu,\beta} a_{\mu,\nu,\beta}(x)t^\mu {\partial_t}^\nu {\partial_x}^\beta \in \mathcal{D}_R \mid a_{\mu,\nu,\beta}(x) = 0 \text{ if } \nu - \mu > m\}$$

for any integer m (we use the same notation as for the filtration of $\mathcal{D}_0$).

Definition 3.4. Let $\mathbf{G} = \{P_1, \ldots, P_s\}$ be a finite subset of $\mathcal{D}_R$ and m be an arbitrary integer. For an element P of $\mathcal{D}_R$,

(1) P is said to be *$\mathcal{F}_m$-reducible* with respect to $\mathbf{G}$ if and only if

$$\mathrm{lexp}_{FR}(P) \in \left(\bigcup_{i=1}^{s}(\mathrm{lexp}_{FR}(P_i) + \mathbb{N}^{n+2})\right) \setminus \mathcal{F}_m.$$

P is said to be *$\mathcal{F}_m$-irreducible* with respect to $\mathbf{G}$ if it is not $\mathcal{F}_m$-reducible.

(2) Let P be $\mathcal{F}_m$-reducible. Then an *$\mathcal{F}_m$-reduction step* for P by $\mathbf{G}$ is a procedure to replace P by

$$P - \frac{\mathrm{lcoef}_{FR}(P)}{\mathrm{lcoef}_{FR}(P_i)} t^{\mu-\mu_i} \partial_t{}^{\nu-\nu_i} \partial_x{}^{\beta-\beta_i} P_i$$

with an arbitrary $i \in \{1, \ldots, s\}$ such that $\mathrm{lexp}_{FR}(P) \in \mathrm{lexp}_{FR}(P_i) + \mathbb{N}^{n+2}$, where $(\mu, \nu, \beta) = \mathrm{lexp}_{FR}(P)$ and $(\mu_i, \nu_i, \beta_i) = \mathrm{lexp}_{FR}(P_i)$.

(3) An *$\mathcal{F}_m$-reduction procedure* for P by $\mathbf{G}$ is a sequence of $\mathcal{F}_m$-reduction steps so that its final output becomes $\mathcal{F}_m$-irreducible. We denote the output by $\mathrm{red}_{FR}(P, \mathbf{G}, m)$ although it is not uniquely determined by P,$\mathbf{G}$,m.

Note that a sequence of $\mathcal{F}_m$-reduction steps always terminates in a finitely many steps because the FR-order defines a well-order on $\{(\mu, \nu, \beta) \in \mathbb{N}^{n+2} \mid \nu - \mu > m\}$.

Definition 3.5. Let I be a left ideal of $\mathcal{D}_R$ and m be an integer. Then a finite subset $\mathbf{G} = \{P_1, \ldots P_s\}$ of $\mathcal{D}_R$ is said to be a *set of $\mathcal{F}_m$-generators* of I if it satisfies the following two conditions:

(1) $\mathbf{G}$ generates I, i.e.,

$$I = \mathcal{D}_R P_1 + \cdots + \mathcal{D}_R P_s,$$

(2) For any distinct $i, j \in \{1, \ldots, s\}$, the output of *some* $\mathcal{F}_m$-reduction procedure for $\mathrm{sp}(P_i, P_j)$ by $\mathbf{G}$ belongs to $\mathcal{F}_m$.

Theorem 3.6. *Let I be a left ideal of $\mathcal{D}_R$ and $\mathbf{G}$ be a finite set of generators of I. Then the following three conditions are equivalent:*

(1) *$\mathbf{G}$ is an FR-Gröbner basis of I.*
(2) *$\mathbf{G}$ is a set of $\mathcal{F}_m$-generators of I for any integer m.*
(3) *For any $P \in I$ and any integer m, the output of an arbitrary $\mathcal{F}_m$-reduction procedure for P by $\mathbf{G}$ belongs to $\mathcal{F}_m$.*

Algorithm 3.7 (FR-Gröbner basis). Given a finite set $\mathbf{G}$ of generators of a left ideal I

of $\mathcal{D}_R$ find an FR-Gröbner basis of I.

$m := \min\{\mathrm{ord}_F(P) \mid P \in \mathbf{G}\}$;
$\mathbf{G}_m := \mathbf{G}$;
REPEAT
  $\mathbf{G}_{m-1} := \mathbf{G}_m$;
  $m := m - 1$;
  REPEAT
    FOR each pair (P, Q) of elements of $\mathbf{G}_m$ DO {
      $R := \mathrm{red}_{FR}(\mathrm{sp}_{FR}(P, Q), \mathbf{G}_m, m)$;
      IF $R \notin \mathcal{F}_m$ THEN $\mathbf{G}_m := \mathbf{G}_m \cup \{R\}$;
    }
  UNTIL $\mathrm{red}_{FR}(\mathrm{sp}_{FR}(P, Q), \mathbf{G}_m, m) \in \mathcal{F}_m$ for any $P, Q \in \mathbf{G}_m$;
UNTIL $\mathbf{G}_m$ becomes stationary, i.e. $\mathbf{G}_m = \mathbf{G}_\mu$ for any $\mu < m$;
RETURN $\mathbf{G}_m$;

The termination condition of Algorithm 3.7 is satisfied for some m, but we cannot know when it is, in general. It is an open problem to obtain a general criterion for the termination of this algorithm. A sufficient condition will be given in Proposition 3.8.

The output of Algorithm 3.7 is indeed an FR-Gröbner basis in view of Thoerem 3.6. The computation of $\mathcal{F}_m$-reduction procedure can be strictly carried out (e.g. by a computer algebra system) with a general hypersurface Y that can be brought into a hyperplane by a birational transformation of $\mathbb{C}^{n+1}$.

As will turn out in the next sections, it is often enough to find a (formally) Fuchsian operator among the ideal. Hence, in practice, it would be a good policy to stop Algorithm 3.7 when $\mathbf{G}_m$ contains a (formally) Fuchsian operator. This makes much wider the applicability of the algorithm.

I owe the following proposition to T. Shimoyama, which serves as a sufficient condition to terminate the Algorithm 3.7.

Proposition 3.8. *Let $\mathbf{G} = \{P_1, \ldots, P_s\}$ be a finite subset of $\mathcal{D}_R$ and let P be an arbitrary element of $\mathcal{D}_R$. Suppose, for some $m_0 \in \mathbb{Z}$, the output of some $\mathcal{F}_{m_0}$-reduction procedure for P by $\mathbf{G}$ is equal to $a(t, x)P$ with some $a(t, x) \in \mathbb{C}(x)[[t]]$ such that $a(0, x) = 0$. Then for any $m \in \mathbb{Z}$, there exist $Q_1, \ldots, Q_s \in \mathcal{D}_R$ and $R \in \mathcal{F}_m$ such that*

$$P = Q_1P_1 + \cdots + Q_sP_s + R$$

with $\mathrm{lexp}_{FR}(Q_kP_k) \preceq_{FR} \mathrm{lexp}_{FR}(P)$ for any $k = 1, \ldots, s$.

In the same way as was pointed out by Buchberger [Bu2] for the polynomial ring, we can often save computation in Algorithm 3.7 by the following criterion:

Proposition 3.9. *Let $\mathbf{G}$ be a finite subset of $\mathcal{D}_R$ and P, Q be two distinct elements of $\mathbf{G}$. Assume that there exists a sequence $\{P_1, , \ldots, P_k\}$ of elements of $\mathbf{G}$ such that*

(1) $P_1 = P, \quad P_k = Q$,
(2) $\mathrm{lexp}_{FR}(P_1) \vee \cdots \vee \mathrm{lexp}_{FR}(P_k) = \mathrm{lexp}_{FR}(P) \vee \mathrm{lexp}_{FR}(Q)$,
(3) $\mathrm{red}_{FR}(\mathrm{sp}_{FR}(P_j, P_{j+1}), \mathbf{G}, m)$ *belongs to $\mathcal{F}_m$ by an $\mathcal{F}_m$-reduction procedure for any $m \in \mathbb{Z}$ and $j = 0, \ldots, k - 1$.*

Then, for any integer m, the output of some $\mathcal{F}_m$-reduction procedure for $\mathrm{sp}_{FR}(P, Q)$ *by* $\mathbf{G}$ *belongs to* $\mathcal{F}_m$.

Let us denote by $A_{n+1} = \mathbb{C}[t, x]\langle\partial_t, \partial_x\rangle$ the Weyl algebra, or the ring of differential operators with polynomial coefficients (cf. Björk (1979)) and by $\tilde{A}_{n+1} = \mathbb{C}[x][[t]]\langle\partial_t, \partial_x\rangle$ the ring of differential operators whose coefficients are polynomials in x and formal power series in t.

For an operator $P \in \mathcal{D}_R$, there exists a polynomial $b(x)$ of least total degree such that $b(x)P \in A_{n+1}$ and we denote such $b(x)$ by $\mathrm{den}(P)$ and call it the denominator of P. The numerator $\mathrm{num}(P)$ of P is defined as $b(x)P$.

An FR-Gröbner basis provides an FD-Gröbner basis at a generic point of Y as follows:

Theorem 3.10. *Let $P_1, \ldots, P_s$ be elements of A_{n+1}. Assume that* $\mathbf{G} := \{P_1, \ldots, P_s\}$ *is an FR-Gröbner basis of the left ideal*

$$I := \mathcal{D}_R P_1 + \cdots + \mathcal{D}_R P_s$$

of $\mathcal{D}_R$. Put

$$a(x) = \mathrm{lcoef}_{FR}(P_1)(x) \ldots \mathrm{lcoef}_{FR}(P_s)(x)$$

and assume $a(x_0) \neq 0$. Put $p = (0, x_0)$. Then $\mathbf{G}$ *is also an FD-Gröbner basis of the left ideal*

$$\mathcal{I}_p := \mathcal{D}_p P_1 + \cdots + \mathcal{D}_p P_s$$

of $\mathcal{D}_p$.

Corollary 3.11. *Let* $\mathbf{G} = \{P_1, \ldots, P_s\}$ *be a subset of A_{n+1} and let*

$$\mathbf{G}_m = \{P_1, \ldots, P_s, P_{s+1}, \ldots, P_\sigma\}$$

be the output of Algorithm 3.7 with the input $\mathbf{G}$. *Put* $\mathrm{lcoef}_{FR}(P_j) = a_j(x)/b_j(x)$ *with polynomials $a_j(x)$,$b_j(x)$ relatively prime to each other. If a point $(0, x_0)$ of Y satisfies $a_1(x_0) \ldots a_\sigma(x_0) \neq 0$, then* $\mathbf{G}$ *constitutes an FD-Gröbner basis of the left ideal*

$$\mathcal{I}_p = \mathcal{D}_p P_1 + \cdots + \mathcal{D}_p P_s$$

of $\mathcal{D}_p$.

In the following application of FR-Gröbner bases, it is useful to introduce the notion of minimal Gröbner basis as for the polynomial ideals (cf. [CLO]):

Definition 3.12. Let $\mathbf{G} = \{P_1, \ldots, P_s\}$ be a finite subset of $\mathcal{D}_R$ and put $I = \mathcal{D}_R P_1 + \cdots + \mathcal{D}_R P_s$. Then $\mathbf{G}$ is called a *minimal FR-Gröbner basis* of I if $\mathbf{G}$ is an FR-Gröbner basis of I and if, for any $i \in \{1, \ldots, s\}$,

$$\mathrm{lexp}_{FR}(P_i) \notin \bigcup_{j \neq i} (\mathrm{lexp}_{FR}(P_j) + \mathbb{N}^{n+2}).$$

It is easy to construct a minimal FR-Gröbner basis from the output of Algorithm 3.7 by $\mathcal{F}_m$-reduction procedures. From the practical point of view, it would be more efficient to add the $\mathcal{F}_m$-reduction procedure for each $P \in \mathbf{G}_m$ by $\mathbf{G}_m \setminus \{P\}$ in the inner REPEAT–UNTIL loop of Algorithm 3.7.

4. Computation of characteristic exponents.

We use the same notation as in Sect. 1. In particular, let $\mathcal{I}$ be a left ideal of $\mathcal{D}_0$ associated with a Fuchsian system $\mathcal{M}$ as in Sect. 1.2. We assume $Y = \{(t,x) \mid t = 0\}$. In fact, we can treat any non-singlular complex analytic hypersurface Y for the (theoretical) coomputation of Algorithm 2.7. For the (practical) computation of Algorighm 3.6, we can treat any hypersurface Y that can be brought into the hyperplane $t = 0$ by a birational transformation of $\mathbb{C}^{n+1}$.

Theorem 4.1. *Assume that the system $\mathcal{M}$ is formally Fuchsian along Y at 0 with $P_1, \ldots, P_s \in \mathcal{D}_0$. Let $\mathbf{G}$ be an FD-Gröbner basis of $\mathcal{I}_0 := \mathcal{D}_0 P_1 + \cdots + \mathcal{D}_0 P_s$. Put*

$$\mathbf{G}' = \{P \in \mathbf{G} \mid \mathrm{lexp}_{FD}(P) = (\mu, \nu, \alpha, 0) \mid \text{ for some } \mu, \nu \in \mathbb{N} \text{ and some } \alpha \in \mathbb{N}^n\}$$

Then the set of the characteristic exponents of $\mathcal{M}$ at 0 is given by

$$e_Y(\mathcal{M}, 0) = \{\theta \in \mathbb{C} \mid \psi(\hat{\sigma}(P))(\theta, 0) = 0 \text{ for any } P \in \mathbf{G}'\}. \tag{4.1}$$

Moreover, let P be an element of $\mathbf{G}'$ with minimum order with respect to ∂_t. Then there exist a monic polynomial $f(\theta, x) \in \mathcal{O}'_0[\theta]$ and $a(x) \in \mathcal{O}'_0$ such that $\psi(\hat{\sigma}(P)) = a(x) f(\theta, x) \tau^k$ with some $k \in \mathbb{Z}$, and the ideal $\tilde{\mathcal{J}}_Y(\mathcal{M}, 0)$ is generated by f. In particular we have

$$\tilde{e}_Y(\mathcal{M}, 0) = \{\theta \in \mathbb{C} \mid f(\theta, 0) = 0\}.$$

On generic points, we can compute the characteristic exponents from an FR-Gröbner basis. In fact, the following is an immediate consequence of Corollary 3.11 and Theorem 4.1.

Corollary 4.2. *Under the same assumptions as in Corollary 3.11, put*

$$S = \{i \in \{1, \ldots, \sigma\} \mid \mathrm{lexp}_{FR}(P_i) = (\mu_i, \nu_i, 0) \text{ with some } \mu_i, \nu_i \in \mathbb{N}\}.$$

Among the set $\{P_i \mid i \in S\}$, let P_{i_0} have minimum degree with respect to ∂_t and set $\psi(\hat{\sigma}(P_{i_0})) = f_{i_0}(\theta, x) \tau^k$. Then we have

$$\mathcal{J}_Y(\mathcal{M}, p) = \tilde{\mathcal{J}}_Y(\mathcal{M}, p) = \mathcal{O}'_p[\theta] f_{i_0}(\theta, x).$$

5. Computation of the induced system.

Here we use the same notation as above and assume the system $\mathcal{M}$ (as in Sect 1.2.) is formally Fuchsian along $Y = \{(t,x) \mid t = 0\}$ at 0. We study the structure of the induced system $\mathcal{M}_Y = \mathcal{D}/(\mathcal{I} + t\mathcal{D})$ of $\mathcal{M}$ along Y. The induced system is a system which the restriction to Y of the holomorphic solutions of $\mathcal{M}$ satisfy. Our porpose is to determine the structure of the stalk $\mathcal{M}_{Y,0}$ of $\mathcal{M}_Y$ at $0 \in Y$ as a module over $\mathcal{D}'_0 = \mathbb{C}\{x\}\langle \partial_x \rangle$. We denote by u the modulo class of $1 \in \mathcal{D}$ in $\mathcal{M} = \mathcal{D}/\mathcal{I}$, and for $P \in \mathcal{D}$, we denote by $[Pu]$ the modulo class of $P \in \mathcal{D}$ in $\mathcal{M}_Y$.

Let us begin with the following general result:

Theorem 5.1. *Assume $\mathcal{M}$ is formally Fuchsian along Y at 0 and*

$$\{k \in \mathbb{N} \mid k \geq k_0\} \cap e_Y(\mathcal{M}, 0) = \emptyset$$

for some $k_0 \in \mathbb{N}$. Then $\mathcal{M}_{Y,0}$ is generated by $[\partial_t{}^j u]$ with $0 \leq j \leq k_0 - 1$ as a $\mathcal{D}'_0$-module. In particular, we have $\mathcal{M}_{Y,0} = 0$ if $k_0 = 0$.

In view of this theorem, $\mathcal{M}_Y$ represents the relations among the ristrictions

$$u(0,x), \partial_t u(0,x), \ldots, \partial_t{}^{k_0-1}u(0,x)$$

of a holomorphic solution $u(t,x)$ of $\mathcal{M}$ on a neighborhood of Y.

Now let us describe a practical method to compute the induced system $\mathcal{M}_{Y,0}$ under some moderate condition, which is always satisfied at a generic point of Y. (See [Tak2] for a different general method not based on Theorem 5.1.)

Assume that the system $\mathcal{M}$ satisfies the same assumptions as in Theorem 5.1. Let $\mathbf{G}$ be a finite set of generators of the left ideal $\mathcal{I}_0$ of $\mathcal{D}_0$. We assume that there exists an element P_0 of $\mathbf{G}$ such that $\psi(\hat{\sigma}(P_0)) = f(\theta, x)\tau^{-j_0}$ and that $f(k, 0) \neq 0$ for any integer $k \geq k_0$. (We may assume $j_0 \geq 0$.) In view of Corollary 4.2, this assumption is satisfied if $\mathbf{G}$ satisfies the conditions of Theorem 3.10 at 0; i.e., if $\mathbf{G}$ consists of elements of A_{n+1} with $\mathrm{lcoef}_{FR}(P)(0) \neq 0$ for any $P \in \mathbf{G}$, and if $\mathbf{G}$ is an FR-Gröbner basis of the ideal which it generates over $\mathcal{D}_R$.

We define a $\mathcal{D}'_0$-homomorphism $\rho : \mathcal{D}_0 \longrightarrow \mathcal{D}'_0[\partial_t]$ as follows: Write $P \in \mathcal{D}_0$ explicitly as (1.1). Then we put

$$\rho(P) = \sum_{\nu,\alpha,\beta} a_{0,\nu,\alpha,\beta} x^\alpha \partial_x{}^\beta \partial_t{}^\nu \in \mathcal{D}'_0[\partial_t].$$

For an element P of $\mathcal{D}'_0[\partial_t]$, its F-order $\nu = \mathrm{ord}_F(P)$ denotes the order of P with respect to ∂_t and its formal symbol is of the form $\hat{\sigma}(P) = A(x, \partial_x)\partial_t{}^\nu$ with some $A \in \mathcal{D}'_0$. Let us denote this A by $\mathrm{coef}(P, \partial_t, \nu)$.

By the proof of Theorem 5.1, we have, for any $k \geq k_0$,

$$\hat{\sigma}(\rho(\partial_t{}^{j_0+k} P_0)) = p_k(x)\partial_t{}^k$$

with some $p_k(x) \in \mathbb{C}\{x\}$ such that $p_k(0) \neq 0$.

Now for an arbitrary element P of $\mathcal{D}'_0[\partial_t]$, let us define another element $\mathrm{ind}(P, P_0)$ of $\mathcal{D}'_0[\partial_t]$ by the following algorithm:

Algorithm 5.2.

```
INPUT P ∈ D'_0[∂_t];
WHILE ν := ord_F(P) ≥ k_0 DO
    P := P − (coef(P, ∂_t, ν)/p_ν) ρ(∂_t^{j_0+ν} P_0);
RETURN P;
```

Put

$$\mathcal{D}'_0{}^{(k_0)} = \bigoplus_{0 \leq k \leq k_0-1} \mathcal{D}'_0 \partial_t{}^k \subset \mathcal{D}'_0[\partial_t].$$

Then $\mathrm{ind}(\cdot, P_0)$ defines a $\mathcal{D}'_0$-homomorphism of $\mathcal{D}'_0[\partial_t]$ to $\mathcal{D}'_0{}^{(k_0)}$. For an element $Q = \sum_{k=0}^{k_0-1} Q_k(x, \partial_x)\partial_t{}^k$ of $\mathcal{D}'_0{}^{(k_0)}$, we write

$$[Qu] = \sum_{k=0}^{k_0-1} Q_k(x, \partial_x)[\partial_t{}^k u] \in \mathcal{D}'_0{}^{(k_0)}.$$

Theorem 5.3. *Under the assumptions above, there exists an integer $j_0 \geq 0$ such that the induced system $\mathcal{M}_{Y,0}$ is explicitly given by the system of equations for unknowns* $[u], \ldots, [\partial_t{}^{k_0-1}u]$

$$[\mathrm{ind}(\rho(\partial_t{}^j P), P_0)u] = 0 \quad \textit{for any } P \in \mathbf{G} \textit{ and any } j = 0, 1, \ldots, j_0.$$

6. Examples of actual computation.

In the sequel we put $n = 1$ and use the notation $\partial_x = \partial/\partial x$, $\partial_y = \partial/\partial y$ with $(x, y) \in \mathbb{C}^2$ as well as $(t, x) \in \mathbb{C}^2$ as in the preceding sections.

As examples, we treat the systems for Appell's hypergeometric functions of two variables. We can verify that these systems are in fact Fuchsian along all the irreducible components of their singular loci and can compute their characteristic exponents and induced systems completely by using Algorithms 2.7, 3.7, 5.2.

Let us describe briefly the computation for the systems for Appell's F_3 and for F_4. Maybe such facts have been known (at least implicitly) by using concrete expression of their solutions (see e.g., [Tak3] for the systems for F_1, F_2, F_3). Note that in the following computation we do not use any information on the concrete expression of the solutions (power series or integral representation) in advance.

The following computation was carried out by using our implementation of Algorithms 2.7, 3.7, 5.2 on a computer algebra system Risa/asir (cf. [NT]).

Example 6.1 (System for Appell's F_3). Let us consider the system $\mathcal{M}_3$ for Appell's hypergeometric function F_3 defined by

$$\mathcal{M}_3 : \quad P_{31}u = P_{32}u = 0,$$

where

$$\begin{aligned} P_{31} &:= x(1-x)\partial_x^2 + y\partial_x\partial_y + \{\gamma - (\alpha + \beta + 1)x\}\partial_x - \alpha\beta, \\ P_{32} &:= y(1-y)\partial_y^2 + x\partial_x\partial_y + \{\gamma - (\alpha' + \beta' + 1)y\}\partial_y - \alpha'\beta' \end{aligned}$$

with parameters $\alpha, \alpha', \beta, \beta', \gamma \in \mathbb{C}$. (We assume these parameters take generic values.) By the Gröbner basis algorithm for the ring of differential operators with polynomial (or rational function) coefficients, we know that $\mathcal{M}_3$ is a holonomic system of rank 4 and its singular loci are defined by $xy(x-1)(y-1)(xy - x - y - 1) = 0$. (See [OS] for the precise computation of the characteristic variety.)

Put $Y = \{(x, y) \mid x = 0\}$ and $I = \mathcal{D}_R P_{31} + \mathcal{D}_R P_{41}$. Then Algorithm 3.7 with the aid of Propositions 3.8, 3.9 returns $\mathbf{G} := \{P_{31}, P_{32}, P_{33}\}$ as a minimal FR-Gröbner basis for I_3 along Y; here

$$\begin{aligned} P_{33} = {} & (1-x)yx^2\partial_x{}^3 + (y-1)yx^2\partial_y\partial_x{}^2 \\ & + \{(-\alpha + \alpha' - \beta + \beta' - \gamma - 3)x + (-\alpha' - \beta' + 2\gamma + 1)\}yx\partial_x{}^2 \\ & + (\alpha + \beta + 1)(y-1)yx\partial_y\partial_x \\ & + [\{(\alpha' - \beta + \beta' - \gamma - 1)\alpha + (\beta + 1)\alpha' + (\beta' - \gamma - 1)\beta + \beta' - \gamma - 1\}x \\ & + (\beta' - \gamma)\alpha' - \gamma\beta' + \gamma^2]y\partial_x \\ & + \alpha\beta(y-1)y\partial_y + \alpha\beta(\alpha' + \beta' - \gamma)y. \end{aligned}$$

Their leading terms are

$$\mathrm{lterm}_{FR}(P_{31}) = y\partial_x\partial_y, \quad \mathrm{lterm}_{FR}(P_{32}) = y(1-y){\partial_y}^2, \quad \mathrm{lterm}_{FR}(P_{33}) = yx^2{\partial_x}^3.$$

This implies that $\mathcal{M}_3$ is Fuchsian along Y on $\{(0,y) \in Y \mid y \neq 0,1\}$. (We can also verify that $\mathcal{M}_3$ is also Fuchsian along Y at $(0,0)$ and $(0,1)$ by Algorithm 2.7.) We get

$$e_Y(\mathcal{M}_3, p) = \tilde{e}_Y(\mathcal{M}_3, p) = \{0, \alpha' - \gamma + 1, \beta' - \gamma + 1\}$$

for any $p \in Y \setminus \{(0,0),(0,1)\}$.

Any multi-valued analytic solution u of $\mathcal{M}_3$ around Y is written in the form

$$u = v_1(x,y) + v_2(x,y)x^{\alpha'-\gamma+1} + v_3(x,y)x^{\beta'-\gamma+1}$$

with v_1, v_2, v_3 holomorphic on a neighborhood of $Y \setminus \{(0,0),(0,1)\}$. Moroever, the computation of the induced systems shows that $v_1(0,y), v_2(0,y), v_3(0,y)$ satisfy the equations

$$\begin{aligned}
&\{y(1-y){\partial_y}^2 + (\gamma - (\alpha' + \beta' + 1)y)\partial_y - \alpha'\beta'\}v_1(0,y) = 0,\\
&(y\partial_y + \alpha')v_2(0,y) = 0, \qquad (y\partial_y + \beta')v_3(0,y) = 0.
\end{aligned}$$

We know that these systems coincide precisely with the induced systems because the sum of the rank of these systems equals 4, which is the rank of the system $\mathcal{M}_3$.

Example 6.2 (System for Appell's F_4). The system $\mathcal{M}_4$ for Appell's F_4 is defined by

$$P_{41}u = P_{42}u = 0,$$

where

$$\begin{aligned}
P_{41} &:= x(1-x)\partial_x^2 - 2xy\partial_x\partial_y - y^2\partial_y^2 + \{\gamma - (\alpha+\beta+1)x\}\partial_x - (\alpha+\beta+1)y\partial_y - \alpha\beta,\\
P_{42} &:= y(1-y)\partial_y^2 - 2xy\partial_x\partial_y - x^2\partial_x^2 + \{\gamma' - (\alpha+\beta+1)y\}\partial_y - (\alpha+\beta+1)x\partial_x - \alpha\beta
\end{aligned}$$

with parameters $\alpha, \beta, \gamma, \gamma' \in \mathbb{C}$. This is a holonomic system of rank 4 with sigular loci $xy(x^2 + y^2 - 2xy - 2x - 2y + 1) = 0$. Put $I = \mathcal{D}_R P_{41} + \mathcal{D}_R P_{42}$ and

$$Y = \{(x,y) \mid x^2 + y^2 - 2xy - 2x - 2y + 1 = 0\}.$$

We make a birational coordinate transformation

$$t = x^2 + y^2 - 2xy - 2x - 2y + 1, \quad x = x - y$$

and rewrite P_{41}, P_{42} in the new coordinate system (t, x).

Inputting $\{P_{41}, P_{42}\}$ to Algorithm 3.7, we get, as the output of the algorithm stopped when $m = -1$, $\mathbf{G} = \{P_{41}, P_{42}, P_{43}, P_{44}\}$ with leading terms

$$\begin{aligned}
\mathrm{lterm}_{FR}(P_{41}) &= (x+1)(x-1)^2\partial_t\partial_x, & \mathrm{lterm}_{FR}(P_{42}) &= (x+1)^2(x-1)\partial_t\partial_x,\\
\mathrm{lterm}_{FR}(P_{43}) &= 2(x+1)(x-1)t{\partial_t}^2, & \mathrm{lterm}_{FR}(P_{44}) &= \frac{1}{2}(x+1)^3(x-1)^2{\partial_x}^3.
\end{aligned}$$

Moreover P_{43}, and hence $\mathcal{M}_4$, is Fuchsian along Y on $Y \setminus \{(0,1),(0,-1)\}$. (By using Algorithm 2.7 we can verify that $\mathcal{M}_4$ is also Fuchsian along Y at $(0,\pm 1)$). We do not know if $\mathbf{G}$ is indeed an FR-Gröbner basis of I along Y. In any case, we get from this set of generators

$$e_Y(\mathcal{M}_4, p) \subset \{0, \gamma + \gamma' - \alpha - \beta - \frac{1}{2}\}$$

for any $p \in Y \setminus \{(0,1),(0,-1)\}$. Hence any multi-valued analytic solution u of $\mathcal{M}_4$ around Y is written in the form

$$u = v_1(t,x) + v_2(t,x)t^{\gamma+\gamma'-\alpha-\beta-1/2}$$

with v_1, v_2 holomorphic on a neighborhood of $Y \setminus \{(0,1),(0,-1)\}$. Moroever, the computation of the induced systems shows that $v_1(0,x), v_2(0,x)$ satisfy the equations

$$R_1 v_1(0,x) = 0, \quad R_2 v_2(0,x) = 0$$

with

$$\begin{aligned}
R_1 &= (x-1)^2(x+1)^2\partial_x{}^3 \\
&\quad + (x-1)(x+1)\{(2\alpha+2\beta+\gamma+\gamma'+2)x - 3\gamma + 3\gamma'\}\partial_x{}^2 \\
&\quad + [\{(4\beta+2\gamma+2\gamma')\alpha + (2\gamma+2\gamma')\beta + \gamma + \gamma'\}x^2 - 2(\gamma-\gamma')(2\alpha+2\beta+1)x \\
&\quad + (-4\beta+2\gamma+2\gamma'-4)\alpha + (2\gamma+2\gamma'-4)\beta + (-8\gamma'+5)\gamma + 5\gamma' - 4]\partial_x \\
&\quad + 4\alpha\beta\{(\gamma+\gamma'-1)x - \gamma + \gamma'\}, \\
R_2 &= (x-1)(x+1)\partial_x + \{(3\gamma+3\gamma'-2\alpha-2\beta-2)x - \gamma + \gamma'\}.
\end{aligned}$$

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