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Sharp characters and their generalizations

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1. Blichfeldt's Theorem

Let G be a finite group and χ a virtual character of G. Let L be the set of values of χ. For $l \in L$, we define the number $B(l)$ as follows:

$$B(l) = \frac{a(l)}{|G|} \prod_{l' \in L-\{l\}} (l-l'),$$

where $a(l)$ denotes the number of elements x in G with $\chi(x) = l$.

Ninety years ago, Blichfeldt [B] proved that $B(l)$ is an algebraic integer for any $l \in L$. Our first aim is to extend this result. We will show that the numbers $B(l)$ $(l \in L)$ are in fact the values of a virtual character $\tilde{\chi}$ of G, constructed from χ in a definite manner. More precisely, we have the following

Theorem 1. Let $\tilde{\chi}$ be a class function on G defined by $\tilde{\chi}(x) = B(\chi(x))$ for $x \in G$. Then $\tilde{\chi}$ is a virtual character of G.

Since the value of a group character is a sum of roots of unity, it is clear that Theorem 1 implies Blichfeldt's Theorem mentioned above.

Proof of Theorem 1. (Outline) For $x \in G$, we let f_x denote the monic polynomial of least degree whose set of roots is $L-\{\chi(x)\}$. Let f be the avarage of f_x over G:

$$f = \frac{1}{|G|} \sum_{x \in G} f_x .$$

Then we have the following

Claim. f is a monic polynomial with integral coefficients of degree $|L|-1$.

In fact, the coefficients of f are expressed by integral linear combinations of $(\chi^i, 1_G)$ $i = 0,1,\ldots$ and symmetric functions of the elements in L. For example, if $L = \{n,l,k\}$ then we have $f(X) = X^2 - ((n+l+k) - (\chi, 1_G))X + ((nl+lk+kn) - (n+l+k)(\chi, 1_G) + (\chi^2, 1_G))$.

Now Theorem 1 follows easily from Claim since $\widetilde{\chi} = f(\chi)$.

Remark. The above f is the polynomial of least degree with $f(l) = B(l)$ for every $l \in L$, that is, the Lagrange interpolation polynomial through the points $\{(l, B(l)) \mid l \in L\}$.

One of the typical properties of $\widetilde{\chi}$ is that it does not take the value 0. So we can define the class function $1/\widetilde{\chi}$. By direct calculation, we obtain

Proposition 2. $(\overline{\chi}^i, 1/\widetilde{\chi}) = 0$ for $i = 0,1,\ldots, |L|-2$.

Using Proposition 2 (i=0), we have the following divisibility conditions.

Proposition 3. For any $l \in L$, $B(l)$ divides $a(l)\prod_{l' \in L-\{l\}} B(l')$ in the ring of algebraic integers. In particular, if χ is a character of degree n, then $B(n)$ divides $\prod_{l \in L-\{n\}} B(l)$.

2. Sharp characters of finite groups

Under the same notation as in Section 1, we will define sharp triples for group characters.

Definitions. The triple (G, χ, l) is called a sharp triple if $B(l)$ is a unit in the ring of algebraic integers. The pair (G, χ) is called a sharp pair if $(G, \chi, \chi(1))$ is a sharp triple.

The concept of sharp pairs was first introduced by Cameron and Kiyota [CK], and their definition of sharp pairs is slightly different from ours. But at least in case χ is a faithful character of G, these two definitions are the same. So the concept of sharp triples is a natural generalization of that of sharp pairs

We will give some examples of sharp triples.

Example 1. Let G be cyclic and χ be a faithful linear character of G. Then (G, χ, l) is sharp for every $l \in \mathrm{Im}\,\chi$.

Example 2. Let G be a sharply t-transitive permutation group and π be the associated permutation character. Then $(G, \pi, t-2)$ is a sharp triple, and (G, π) is a sharp pair.

The following Lemmas are easy to prove. (Use Proposition 3 for Lemma 5.)

Lemma 4. If (G, χ, l) is sharp, then $a(l)$ divides $|G|$.

Lemma 5. Let χ be a character of degree n. If (G, χ, l) is sharp for all $l \in L-\{n\}$, then (G, χ) is a sharp pair.

Question 6. If (G, χ, l) is sharp with χ a faithful character, then is it true that the set $\{x \in G \mid \chi(x) = l\}$ is a single conjugacy class of G?

Problem 7. Determine all finite groups G such that (G, χ, l) is sharp for every non-trivial irreducible character χ and for every $l \in \mathrm{Im}\,\chi$. Note that abelian groups and dihedral groups of twice odd prime order are such examples.

3. Classification of sharp triples for given L

From now on we assume χ is a faithful character of G of degree n. Set

$L = \mathrm{Im}\,\chi$ and $L^* = L - \{n\}$. Cameron and Kiyota [CK] posed the problem of determining all the sharp pairs (G, χ) for a given set L^*. There are many papers on this subject; see the references of [AKN]. In particular Alvis and Nozawa [AN] have given a complete classification of sharp pairs when L^* contains an irrational number.

Now we will consider the analogous problem for sharp triples (G, χ, l). The results known to me are very few. The first one is the simplest case and easy to prove.

<u>Result 1.</u> Let $L^* = \{\alpha_1, \ldots, \alpha_t\}$ with all α_i are algebraically conjugate. If (G, χ, α_1) is sharp, then G is cyclic of prime order.

<u>Proof.</u> Since all α_i are conjugate, (G, χ, α_i) are all sharp, and so (G, χ) is sharp by Lemma 5. If $t \geqq 2$, then the result follows from Theorem 4.1 in [CK]. Now assume $t = 1$. Then by Lemma 4, $a(\alpha_1)$ divides $|G| = 1 + a(\alpha_1)$. Thus $a(\alpha_1) = 1$, and so G is cyclic of order two. This completes the proof.

We will state the other known results without proofs.

<u>Result 2.</u> Let $L^* = \{0, \alpha_1, \ldots, \alpha_t\}$ with all α_i are algebraically conjugate and $t \geqq 2$. If $(\chi, 1_G) = 0$ and $(G, \chi, 0)$ is sharp, then G is cyclic of order 4, dihedral of twice odd prime order, or $E_{2^v} \rtimes Z_p$, where $p = 2^v - 1$ is a Mersenne prime.

<u>Result 3.</u> (Matsuhisa and Yamaki [MY]) Let $L^* = \{0, \varepsilon_1, \ldots, \varepsilon_t\}$ with all ε_i are roots of unity. If $(G, \chi, 0)$ is sharp, then G is a sharply 3-transitive group or a 2-transitive Frobenius group.

<u>Result 4.</u> Let $L^* = \{l, k\}$ with integers l, k. If $(\chi, 1_G) = 0$ and (G, χ, l) is sharp, then one of the following holds:

(i) $k = 0$ and (G, χ) is sharp of type $\{0, l\}$.

(ii) $k=-1$, $l=0$ and G is the symmetric group of degree 3.

(iii) $k=-1$, $l=1$ and G is quaternion or dihedral of order 8.

Problem 8. Determine all sharp triples (G, χ, l) when L^* contains an irrational number.

References

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