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REPRESENTATIONS OF MAPPING CLASS GROUPS OBTAINED FROM THE MODULAR INVARIANCE PROPERTY OF CYCLIC GROUP ASSOCIATION SCHEMES

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§0. Introduction

The concept of the modular invariance property of fusion algebra is very important in conformal field theory. Since Eiichi Bannai found relations between Bose-Mesner algebras of association schemes and the fusion algebras([1]), many people started to pay attention to the modular invariance property of association schemes also. It is known that the modular invariance property of association schemes and generalized spin models have some kind of relations([3],[5],[9]) and many examples of generalized spin models are constructed using the modular invariance property of association schemes([2],[4], etc.). It is also known that the spin models give invariants of links (see [10]). In this paper we consider another use of the modular invariance property.

T. Kohno constructed topological invariant of 3-manifold using the modular invariance property of the truncated representations of $sl(2,C)$ (see [11], [12]). He first constructed projective linear representations of the mapping class groups of Riemann surfaces and then using Heegaard decompositions of 3-manifolds obtained invariants of 3-manifolds. Motivated by his work, in this paper we construct some kind of representations of the mapping class groups using the modular invariance property of association schemes on finite cyclic groups. Using the Kohno's technique we can construct invariants of 3-manifold from those representations. The invariants seem to coincide with the ones obtained by Gocho ([8]), however I am not ready to discuss in detail right now so I just give the representations of mapping class groups only(see Theorem 2 in §4).

§1. Modular invariance property of association schemes

First we introduce the definition of the modular invariance property of association schemes. Let $\mathfrak{X} = (X, \{R_i\}_{0\leq i\leq d})$ be a self-dual association scheme and let P and Q be the first and second eigenmatrices of $\mathfrak{X}$ respectively (note that $P = \overline{Q}$). For any $i \in \{0, 1, \dots, d\}$ we define i' by ${}^tA_i = A_{i'}$, where A_i is the adjacency matrix with respect to the relation R_i.

Definition 1 (see [3], [5]) Let $\mathfrak{X}$ be a self-dual association scheme. We say that $\mathfrak{X}$ has the modular invariance property if there exists a diagonal matrix $T = \mathrm{diag}(t_0, t_1, \dots, t_d)$ with $t_0 \neq 0$ which satisfies one of the following mutually equivalent conditions:

(1) $PTQTPT' = t_0 D^3 I$,
(2) $PTQTQT = t_0 D P^2$,
(3) $(\overline{P}T)^3 = t_0 D^3 I$,
(4) $(QT)^3 = t_0 D^3 I$,
(5) $(PU)^3 = t_0{}^{-1} D^3 I$,
where $D^2 = |X|$, $T' = \mathrm{diag}(t_0, t_{1'}, \ldots, t_{d'})$ and $U = \mathrm{diag}(t_0{}^{-1}, t_1{}^{-1}, \ldots, t_d{}^{-1})$.

Let $\mathfrak{X}(G_n) = (G_n, \{R_i\}_{0 \le i \le n-1})$ be the association scheme defined on the finite cyclic group G_n of order n. It is known that $\mathfrak{X}(G_n)$ is self-dual and the character table (first eigen matrix) is given by $P = (\zeta^{ij})$ with a primitive n-th root of unity. The modular invariance property of $\mathfrak{X}(G_n)$ is completely determined (see [2], [6]).

Theorem 1(see [2]) $(\frac{1}{\sqrt{n}}PT)^3 = I$ with a diagonal matrix $T = \mathrm{diag}(\lambda_0, \lambda_1, \ldots, \lambda_{n-1})$ if and only if the following conditions are satisfied:

$$\lambda_0{}^3 = \sqrt{n}\, \eta^{u^2} / \sum_{l=0}^{n-1} \eta^{l^2} ,$$

$$\lambda_i = \eta^{i(i+2u)} \lambda_0 \quad \text{for } i \in \{0, 1, \ldots, n-1\} ,$$

where $\eta = \zeta^{\frac{n+1}{2}}$ for n odd and $\eta^2 = \zeta$ for n even and $u \in \{0, 1, \ldots, n-1\}$.

§3. Mapping class groups

Let Σ_g be an orientable surface of genus g and $\mathfrak{H}_g$ be the set of all the orientation preserving homeomorphisms of Σ_g. The mapping class group $\mathfrak{M}_g$ of Σ_g is the group consists of all the isotopy classes in $\mathfrak{H}_g$. As for the following fact the reader is referred to [7], [11], [13]. It is known that $\mathfrak{M}_g$ is generated by the isotopy classes $\alpha_i, \beta_i, \delta_i, \epsilon_i, 1 \le i \le g$, of the Dehn twists about the circles $a_i, b_i, d_i, e_i, 1 \le i \le g$, respectively as shown in Fig. 1.

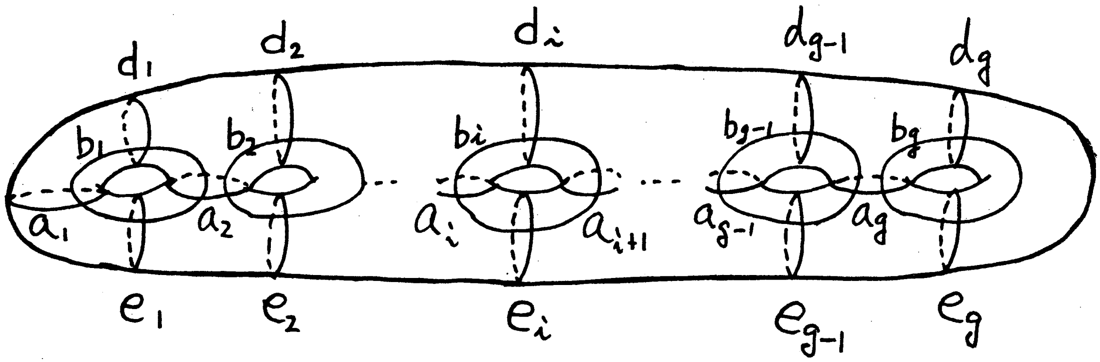

Fig. 1.

The generating relations are given by
1) $\delta_1 = \epsilon_1 = \alpha_1, \delta_g = \epsilon_g$,
2) Let α and β in the generating set.
If the corresponding circles do not intersect, then

$$\alpha\beta = \beta\alpha$$

If the corresponding circles intersect at one point, then

$$\alpha\beta\alpha = \beta\alpha\beta.$$

3) If $g = 2$, then $(\delta_2\beta_2\alpha_2\beta_1{\alpha_1}^2\beta_1\alpha_2\beta_2\delta_2)^2 = 1$.

4) $(\alpha_1\beta_1\alpha_2)^4 = \epsilon_2\delta_2$,

5) $b_2\delta_2b_1 = \alpha_1\alpha_2\alpha_3\delta_3$,

where $b_1 = (\beta_2\alpha_2\alpha_3\beta_2)^{-1}\delta_2(\beta_2\alpha_2\alpha_3\beta_2)$ and $b_2 = (\beta_1\alpha_1\alpha_2\beta_1)^{-1}b_1(\beta_1\alpha_1\alpha_2\beta_1)$,

6) $\delta_{k+1} = J_k\delta_{k-1}{J_k}^{-1}, 2 \leq k \leq g-1$ and $\epsilon_k = G_k\delta_kG_k{}^{-1}, 1 \leq k \leq g$,

where $J_k = \beta_{k+1}\alpha_{k+1}(\beta_k\delta_k\alpha_k\beta_k)\alpha_{k+1}\beta_{k+1}\beta_{k-1}\alpha_k(\beta_k\delta_k\alpha_{k+1}\beta_k)\alpha_k\beta_{k-1}$

and $G_k = \beta_k\alpha_k\cdots\beta_1{\alpha_1}^2\beta_1\cdots\alpha_k\beta_k$.

§4. Representations of the mapping class group

Let V be a vector space over the complex number field $\mathbf{C}$ with a basis $\{v_0, v_1, \dots, v_{n-1}\}$. We define linear isomorphisms S, Λ and $T_u, 0 \leq u \leq n-1$ of V and $W_u, 0 \leq u \leq n-1$, of $V \otimes V$ in the following manner:

$$S(v_i) = \frac{1}{\sqrt{n}}\sum_{l=0}^{n-1}\zeta^{il}v_l,$$
$$\Lambda(v_i) = v_{i'} = v_{n-i}$$
$$T_u(v_i) = \lambda_0\eta^{i(i+2u)},$$
$$W_u(v_i \otimes v_j) = {\lambda_0}^{-1}\eta^{-(i-j)(i-j+2u)}v_i \otimes v_j,$$

for $i, j = 0, 1, \dots, n-1$, where ζ and η are given in Theorem 1. Then we have

$$(ST_u)^3 = I \quad \text{for any } u \in \{0, 1, \dots, n-1\}.$$

Now we define the linear isomorphisms $\overline{\alpha_i}$, $\overline{\beta_i}$, $\overline{\delta_i}$, $\overline{\epsilon_i}$, $1 \leq i \leq g$, on $\underbrace{V \otimes \cdots \otimes V}_{g \text{ times}}$ by the following equations:

$$\overline{\alpha}_1 = {T_u}^{-1} \otimes \underbrace{I \otimes \cdots \otimes I}_{g-1 \text{ times}}$$

$$\overline{\alpha}_x = \underbrace{I \otimes \cdots \otimes I}_{x-2 \text{ times}} \otimes W_{(-1)^x u} \otimes \underbrace{I \otimes \cdots \otimes I}_{g-x \text{ times}} \quad \text{for } x = 2, 3, \ldots, g,$$

$$\overline{\beta}_x = \underbrace{\Lambda \otimes \cdots \otimes \Lambda}_{x-1 \text{ times}} \otimes T_{(-1)^{x-1} u}\, S\, T_{(-1)^{x-1} u} \otimes \underbrace{\Lambda \otimes \cdots \otimes \Lambda}_{g-x \text{ times}} \quad \text{for } x = 1, 2, \ldots, g,$$

$$\overline{\delta}_x = \underbrace{I \otimes \cdots \otimes I}_{x-1 \text{ times}} \otimes \{T_{(-1)^{x-1} u}\}^{-1} \otimes \underbrace{I \otimes \cdots \otimes I}_{g-x \text{ times}} \quad \text{for } x = 1, 2, \ldots, g, \text{ (note that } \overline{\delta}_1 = \overline{\alpha}_1),$$

$$\overline{\epsilon}_x = \underbrace{I \otimes \cdots \otimes I}_{x-1 \text{ times}} \otimes \{T_{(-1)^{x} u}\}^{-1} \otimes \underbrace{I \otimes \cdots \otimes I}_{g-x \text{ times}} \quad \text{for } x = 2, 3, \ldots, g-1,$$

$$\overline{\epsilon}_1 = \overline{\delta}_1 = \overline{\alpha}_1, \quad \overline{\epsilon}_g = \overline{\delta}_g$$

$$\overline{\Lambda} = \underbrace{\Lambda \otimes \cdots \otimes \Lambda}_{g \text{ times}}$$

$$K_x(v_{i_1} \otimes \cdots \otimes v_{i_x} \otimes \cdots \otimes v_{i_g}) = \zeta^{2ui_x} v_{i_1} \otimes \cdots \otimes v_{i_x} \otimes \cdots \otimes v_{i_g} \quad \text{for } x = 1, 2, \ldots, g,$$

$$P_x(v_{i_1} \otimes \cdots \otimes v_{i_x} \otimes \cdots \otimes v_{i_g}) = v_{i_1} \otimes \cdots \otimes v_{i_x - 2u} \otimes \cdots \otimes v_{i_g} \quad \text{for } x = 1, 2, \ldots, g.$$

We have the following theorem.

Theorem 2 (1) Let G and Γ be the groups generated by $\{\overline{\alpha}_x, \overline{\beta}_x, \overline{\delta}_x, \overline{\epsilon}_x \mid x = 1, 2, \ldots, g\}$ and $\{\lambda_0 id, \overline{\Lambda}, K_x, P_x \mid x = 1, 2, \ldots, g\}$ respectively, where id is the identity map of $\underbrace{V \otimes \cdots \otimes V}_{g \text{ times}}$. Then Γ is a normal subgroup of G.

(2) If we define a map ρ from $\mathfrak{M}_g$ to the group G/Γ by $\rho(\alpha_x) = \overline{\alpha}_x\Gamma$, $\rho(\beta_x) = \overline{\beta}_x\Gamma$, $\rho(\delta_x) = \overline{\delta}_x\Gamma$, $\rho(\epsilon_x) = \overline{\epsilon}_x\Gamma$ and homomorphically for other elements in $\mathfrak{M}_g$, then ρ is a well defined group homomorphism.

These representations are neither usual representations of groups nor projective linear representations. Actually ρ is a homomorphism from the mapping class group $\mathfrak{M}_g$ onto a factor group of a subgroup in a linear group of finite dimension. It would be interesting if we could find out the relation with the symmplectic group $Sp(2g, \boldsymbol{Z})$.

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