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On Clarkson-Boas-type inequalities

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Introduction

In the context of uniform convexity, Clarkson's inequalities for L_p were proved in [2]. Boas [1] considered their generalization in parameters, which was 'completed' by Koskela [10]; we call these inequalities given by Boas and Koskela 'of Clarkson-Boas-type' (see also Kato [6] for their high-dimensional versions).

In this expository note, some recent results on Clarkson's and Clarkson-Boas-type inequalities are given especially in connection with type, cotype properties:

(i) By applying vector-valued interpolation to the Littlewood matrices as operators between $L_p(L_q)$-valued $l_r^{2^n}$-spaces, 'Clarkson's inequality' for $L_p(L_q)$ (and for some other Banach spaces as corollaries) is obtained in the high-dimensional setting (Kato & Miyazaki [7]): This might provide, in particular, one of the most concise proofs of classical Clarkson's inequalities (cf. Miyazaki & Kato [14]). The same argument, applied to the '*Rademacher matrices*', yields type inequalities with the best 'type constant' 1 for $L_p(L_q)$ (Kato, Miyazaki & Takahashi [8]). Our idea comes from Pietsch's work [15] (cf. [14], [11], [12]).

(ii) Further application of interpolation with decomposition argument of operators yields 'generalized Clarkson's inequalities' (high-dimensional Clarkson-Boas-type inequalities) for $L_p(L_q)$. This 'completes' and generalizes Boas' another inequality ([1]). Such high-dimensional versions of Clarkson-Boas-type inequalities are closely related with the Grothendieck inequality (see Tonge [16]). As a straightforward application the von Neumann-Jordan constant ([3]; cf. also [5]) for the spaces considered here is determined ([7]).

(iii) In general, Banach spaces with 'type or cotype constant' 1 are characterized as those satisfying Clarkson-Boas-type inequalities (Kato & Takahashi [9]).

1. Preliminaries. p', q', ... denote the conjugate exponents of p, q, ...

1. 1. Clarkson's inequalities (Clarkson [2]). (i) Let $1 < p \leqq 2$. Then, for all f and g in L_p,

$$\text{(CI-1)} \quad (\| f+g \|_p^{p'} + \| f-g \|_p^{p'})^{1/p'} \leqq 2^{1/p'} (\| f \|_p^p + \| g \|_p^p)^{1/p},$$

$$\text{(CI-2)} \quad (\| f+g \|_p^p + \| f-g \|_p^p)^{1/p} \leqq 2^{1/p} (\| f \|_p^p + \| g \|_p^p)^{1/p}.$$

(ii) Let $2 \leqq p < \infty$. Then, for all f and g in L_p,

$$\text{(CI-3)} \quad (\| f+g \|_p^p + \| f-g \|_p^p)^{1/p} \leqq 2^{1/p} (\| f \|_p^{p'} + \| g \|_p^{p'})^{1/p'},$$

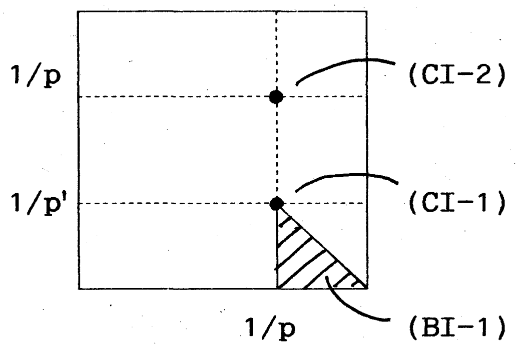
(CI-4) $\quad (\|f+g\|_p^p + \|f-g\|_p^p)^{1/p} \leqq 2^{1/p'}(\|f\|_p^p + \|g\|_p^p)^{1/p}.$

1.2. **Boas' inequality** (Boas [1], Theorem 1). (i) Let $1 < r \leqq p \leqq s < \infty$ and $s' \leqq r$. Then, for all f and g in L_p,

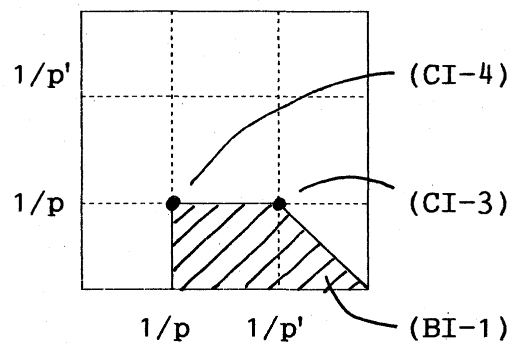
(BI-1) $\quad (\|f+g\|_p^s + \|f-g\|_p^s)^{1/s} \leqq 2^{1/r'}(\|f\|_p^r + \|g\|_p^r)^{1/r}.$

(BI-1) includes (CI-1), (CI-3) and (CI-4). The situation is well expressed in the following unit squares with the coordinates $1/r$ (horizontal) and $1/s$ (vertical):

(i) the case $1 < p \leqq 2$:



(ii) the case $2 < p < \infty$:



Let $A_n = (\varepsilon_{ij})$ be the Littlewood matrices, that is,

$$A_1 = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}, \quad A_{n+1} = \begin{pmatrix} A_n & A_n \\ A_n & -A_n \end{pmatrix} \qquad (n = 1, 2, \ldots).$$

1.3. **Generalized Clarkson's inequalities** (Kato [6]; cf. [10], [16], [11], [12], [14]). Let $1 < p < \infty$ and $1 \leqq r, s \leqq \infty$. Then, for an arbitrary positive integer n and all $f_1, f_2, \ldots, f_{2^n} \in L_p$,

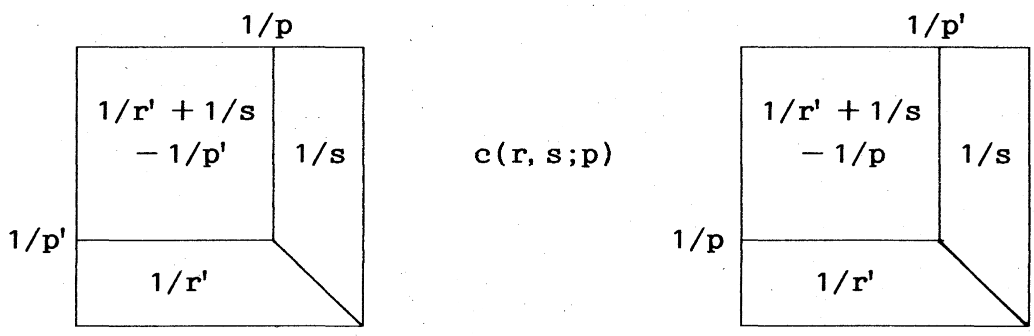
$$\text{(GCI)} \qquad \left\{ \sum_{i=1}^{2^n} \left\| \sum_{j=1}^{2^n} \varepsilon_{ij} f_j \right\|_p^s \right\}^{1/s} \leqq 2^{nc(r,s;p)} \left\{ \sum_{j=1}^{2^n} \| f_j \|_p^r \right\}^{1/r},$$

where

$$c(r,s;p) = \begin{cases} \dfrac{1}{r'} + \dfrac{1}{s} - \min\left(\dfrac{1}{p}, \dfrac{1}{p'}\right) & \text{if (i) } \min(p, p') \leqq r \leqq \infty, \\ & \quad 1 \leqq s \leqq \max(p, p'), \\ \dfrac{1}{s} & \text{if (ii) } 1 \leqq r \leqq \min(p, p'), \\ & \quad 1 \leqq s \leqq r', \\ \dfrac{1}{r'} & \text{if (iii) } s' \leqq r \leqq \infty, \\ & \quad \max(p, p') \leqq s \leqq \infty. \end{cases}$$

Equality is attained in (GCI) for all $1 \leqq r, s \leqq \infty$. In other words,

$$\| A_n : l_r^{2^n}(L_p)) \to l_s^{2^n}(L_p)) \| = 2^{nc(r,s;p)} :$$



$1 < p \leqq 2$ $\qquad$ $2 < p < \infty$

1. 4. **Remark.** A more generalized inequality including (GCI) is considered in Maligranda and Persson [11] (see also [12]).

1. 5. **Definition.** A Banach space X is said to be *of (Rademacher)*

type p $(1 \leqq p \leqq 2)$ resp. *of cotype q* $(2 \leqq q \leqq \infty)$ if X satisfies with some $M > 0$ and $1 \leqq s < \infty$

$$\text{(TpI)} \qquad \left\{ \int_0^1 \left\| \sum_{j=1}^n r_j(t) x_j \right\|^s dt \right\}^{1/s} \leqq M \left\{ \sum_{j=1}^n \| x_j \|^p \right\}^{1/p}$$

resp.

$$\text{(CqI)} \qquad \left\{ \sum_{j=1}^n \| x_j \|^q \right\}^{1/q} \leqq M \left\{ \int_0^1 \left\| \sum_{j=1}^n r_j(t) x_j \right\|^s dt \right\}^{1/s}$$

for all finite system $\{x_j\}$ in X, where $r_j(t)$ are the Rademacher functions, i. e., $r_j(t) = \mathrm{sgn}(\sin 2^j \pi t)$. (Usually, s is taken to be 1, 2; or p resp. q: Recall Khinchin-Kahane's inequality.) The smallest constant M in (TpI) resp. (CqI) are denoted by $T_{p(s)}(X)$ resp. $C_{q(s)}(X)$.

Now, we define *Rademacher matrices* $R_n = (r_{ij}^{(n)})$ inductively by

$$R_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}, \quad R_{n+1} = \left(\begin{array}{c|c} 1 & \\ \vdots & R_n \\ 1 & \\ \hline -1 & \\ \vdots & R_n \\ -1 & \end{array} \right) \qquad (n = 1, 2, \ldots).$$

Then, since

$$\left\{ \int_0^1 \left\| \sum_{j=1}^n r_j(t) f_j \right\|^s dt \right\}^{1/s} = \left\{ \frac{1}{2^n} \sum_{\theta_j = \pm 1} \left\| \sum_{j=1}^n \theta_j f_j \right\|^s \right\}^{1/s}$$

$$= \left\{ \frac{1}{2^n} \sum_{i=1}^{2^n} \left\| \sum_{j=1}^n r_{ij}^{(n)} f_j \right\|^s \right\}^{1/s},$$

type and cotype inequalities are described by the norms of the Rademacher matrices (see [8], Proposition 2. 3).

2. Clarkson's inequality of 2^n-dimension, type, cotype constants for $L_p(L_q)$ and interpolation.

2. 1. Theorem (Clarkson's inequality (CI−1) of 2^n-dimension for L_p and interpolation; Kato [6], Miyazaki & Kato [14]). Let $1 \leqq p \leqq 2$. Then, for all $f_1, f_2, \ldots, f_{2^n} \in L_p$,

$$\text{(CI-1')} \qquad \left\{ \sum_{i=1}^{2^n} \left\| \sum_{j=1}^{2^n} \varepsilon_{ij} f_j \right\|_p^{p'} \right\}^{1/p'} \leqq 2^{n/p'} \left\{ \sum_{j=1}^{2^n} \| f_j \|_p^p \right\}^{1/p}$$

or equivalently,

$$\| A_n : l_p^{2^n}(L_p)) \to l_{p'}^{2^n}(L_p)) \| \leqq 2^{n/p'}.$$

Indeed, it is immediate to see that

$$M_1 = \| A_n : l_1^{2^n}(L_1) \to l_\infty^{2^n}(L_1) \| = 1 \qquad \text{(the case } p = 1\text{)},$$

$$M_2 = \| A_n : l_2^{2^n}(L_2) \to l_2^{2^n}(L_2) \| = 2^{n/2} \qquad \text{(the case } p = 2\text{)}.$$

Put $\theta = 2/p'$ $(0 < \theta < 1)$, where $1 < p < 2$. Then, by interpolation,

$$\| A_n : l_p^{2^n}(L_p) \to l_{p'}^{2^n}(L_p) \| \leqq M_1^{1-\theta} M_2^{\theta} \leqq 2^{n/p'},$$

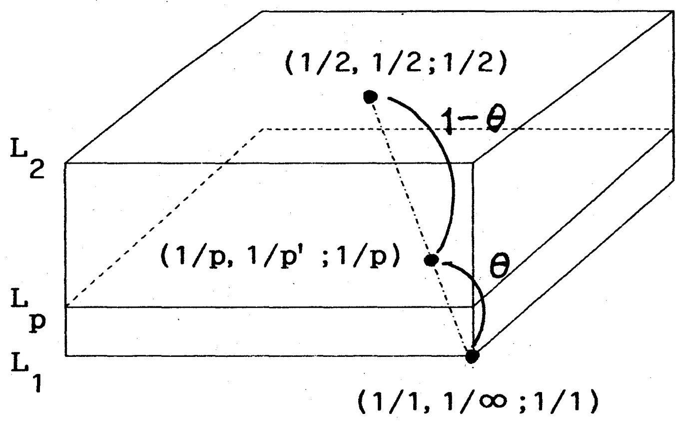
as is desired ([14]). (Note that

$$(l_1^{2^n}(L_1),\ l_2^{2^n}(L_2))_{[\theta]} = l_p^{2^n}(L_p) \quad \text{with equal norms,}$$

$$(l_\infty^{2^n}(L_1),\ l_2^{2^n}(L_2))_{[\theta]} = l_{p'}^{2^n}(L_p) \quad \text{with equal norms}$$

since $(1-\theta)/1 + \theta/2 = 1/p$ and $(1-\theta)/\infty + \theta/2 = 1/p'$.)

Note. The following figure may visually explain what we have done just above:



Note $(1/p, 1/p'; 1/p) = (1-\theta)(1/1, 1/\infty; 1/1) + \theta(1/2, 1/2; 1/2)$, or figuratively, $(1/p, 1/p'; L_p) = (1-\theta)(1/1, 1/\infty; L_1) + \theta(1/2, 1/2; L_2)$, where $\theta = 2/p'$.

2. 2. Theorem (Clarkson's inequality (CI-1) of 2^n-dimension for $L_p(L_q)$ and interpolation; Kato & Miyazaki [7]). Let $1 < p,\ q < \infty$ and $t = \min\{p, q, p', q'\}$. Then, for an arbitrary positive integer n and all $f_1, f_2, \ldots, f_{2^n}$ in $L_p(L_q) = L_p(\mu; L_q(\nu))$,

$$\text{(CI-1'')} \quad \left\{ \sum_{i=1}^{2^n} \left\| \sum_{j=1}^{2^n} \varepsilon_{ij} f_j \right\|_{p(q)}^{t'} \right\}^{1/t'} \leq 2^{n/t'} \left\{ \sum_{j=1}^{2^n} \| f_j \|_{p(q)}^{t} \right\}^{1/t},$$

or equivalently

$$\| A_n : l_t^{2^n}(L_p(L_q)) \to l_{t'}^{2^n}(L_p(L_q)) \| \leqq 2^{n/t'} .$$

In this case, more skillful use of interpolation is required (see [7]). By applying the same argument to 'Rademacher matrices', 'the type inequality' for $L_p(L_q)$ is obtained:

2. 3. Theorem (Type inequality for $L_p(L_q)$; Kato, Miyazaki & Takahashi [8]). Let $1 < p, q < \infty$ and $t = \min\{p, q, p', q'\}$. Then,

$$\| R_n : l_t^n(L_p(L_q)) \to l_{t'}^{2^n}(L_p(L_q)) \| \leqq 2^{n/t'} :$$

In other words, for all $f_1, f_2, \ldots, f_n$ in $L_p(L_q)$,

$$\left\{ \frac{1}{2^n} \sum_{i=1}^{2^n} \left\| \sum_{j=1}^{n} r_{ij}^{(n)} f_j \right\|_{p(q)}^{t'} \right\}^{1/t'} \leqq \left\{ \sum_{j=1}^{n} \| f_j \|_{p(q)}^{t} \right\}^{1/t} ;$$

that is, <u>$L_p(L_q)$ is of type t</u> and <u>its 'type t constant', $T_{t(t')}((L_p(L_q))$, is 1.</u> Here, t' in the left side may be replaced by any s with $1 \leqq s \leqq t'$, i. e., $T_{t(s)}(X) = 1$.)

Here, the constants t and t' are optimal as far as 'the type constant' is 1.

2. 4. Remark. Theorem 2. 3 with duality argument yields analogous results on cotype inequalities for $L_p(L_q)$ ([8]). Type and cotype inequalities for Sobolev spaces given in Milman [13] and Cobos [4] are immediately derived as its corollaries.

3. Generalized Clarkson's inequalities for $L_p(L_q)$

3.1. Theorem (Generalized Clarkson's inequalities for $L_p(L_q)$; Kato & Miyazaki [7]). Let $1 < p,\ q < \infty$ and $1 \leqq r,\ s \leqq \infty$. Then, for an arbitrary positive integer n and all $f_1, f_2, \ldots, f_{2^n}$ in in $L_p(L_q)$,

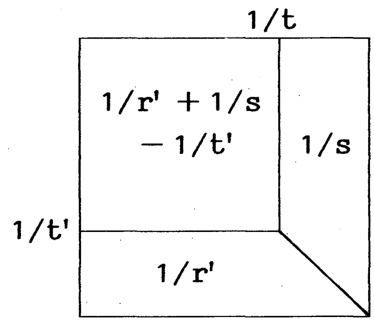
$$\text{(GCI}'\text{)} \quad \left\{ \sum_{i=1}^{2^n} \left\| \sum_{j=1}^{2^n} \varepsilon_{ij} f_j \right\|_{p(q)}^{s} \right\}^{1/s} \leqq 2^{nc(r,s;p,q)} \left\{ \sum_{j=1}^{2^n} \| f_j \|_{p(q)}^{r} \right\}^{1/r},$$

where, letting $t = \min\{p, q, p', q'\}$ and $1/t + 1/t' = 1$,

$$c(r,s;p,q) = \begin{cases} \dfrac{1}{r'} + \dfrac{1}{s} - \dfrac{1}{t'} & \text{if (i) } t \leqq r \leqq \infty, \\ & \quad\ 1 \leqq s \leqq t', \\ \dfrac{1}{s} & \text{if (ii) } 1 \leqq r \leqq t, \\ & \quad\ 1 \leqq s \leqq r', \\ \dfrac{1}{r'} & \text{if (iii) } s' \leqq r \leqq \infty, \\ & \quad\ t' \leqq s \leqq \infty. \end{cases}$$

Equality is attained in (GCI') for all $1 \leqq r,\ s \leqq \infty$; and hence

$$\| A_n : l_r^{2^n}(L_p(L_q)) \to l_s^{2^n}(L_p(L_q)) \| = 2^{nc(r,s;p,q)}:$$



c(r, s;p, q)

3. 2. Remarks. (i) (GCI') covers all the 'rest cases' of Boas' inequality for $L_p(L_q)$ ([1], Theorem 2): Let $1 < p, q < \infty$. Let $1 < r \leqq p \leqq s < \infty$ and let $s' \leqq r \leqq s\{\min(q, q') - 1\}$. Then, for all f and g in $L_p(L_q)$,

(BI-2) $$(\| f+g \|^s_{p(q)} + \| f-g \|^s_{p(q)})^{1/s} \leqq 2^{1/r'}(\| f \|^r_{p(q)} + \| g \|^r_{p(q)})^{1/r}$$

(ii) (GCI) for L_p, $l_p(L_p)$, and $W^k_p(\Omega)$ are corollaries of (GCI'). The last one includes Milman's ([13]) and Cobos' result ([4]).

The von Neumann-Jordan constant for a Banach space X ([3]), $C_{NJ}(X)$, is the smallest constant C satisfying

$$\frac{1}{C} \leqq \frac{\| x + y \|^2 + \| x - y \|^2}{2(\| x \|^2 + \| y \|^2)} \leqq C$$

for all x and y in X with $\| x \|^2 + \| y \|^2 \neq 0$. For any Banach space X, $1 \leqq C_{NJ}(X) \leqq 2$; and it is a Hilbert space if and only if $C_{NJ}(X) = 1$ ([5]). For L_p, $C_{NJ}(L_p) = 2^{2\max(1/p, 1/p') - 1}$ ([3]).

3. 3. Corollary ([7]). Let $1 \leqq p, q \leqq \infty$ and let $t = \min\{p, q, p', q'\}$. Then,

$$C_{NJ}(L_p(L_q)) = C_{NJ}(l_p(L_q)) = 2^{2\max(1/t, 1/t') - 1}$$

and

$$C_{NJ}(l_p(L_p)) = C_{NJ}(W^k_p(\Omega)) = 2^{2\max(1/p, 1/p') - 1}.$$

4. Banach spaces satisfying Clarkson's inequalities

4. 1. Theorem (Kato & Takahashi [9]). Let X be a Banach space.

(i) Let $1 < p \leqq 2$ and $p \leqq s \leqq p'$. Then, X satisfies the Clarkson-Boas-type inequality

$$\text{(CBI-1)} \quad (\|x+y\|^s + \|x-y\|^s)^{1/s} \leqq 2^{1/s}(\|x\|^p + \|y\|^p)^{1/p}$$

if and only if X is <u>of type p</u> and <u>$T_{p(s)}(X) = 1$.</u> In particular, X satisfies Clarkson's inequalities

$$\text{(CI-1*)} \quad (\|x+y\|^{p'} + \|x-y\|^{p'})^{1/p'} \leqq 2^{1/p'}(\|x\|^p + \|y\|^p)^{1/p}$$

resp.

$$\text{(CI-2*)} \quad (\|x+y\|^p + \|x-y\|^p)^{1/p} \leqq 2^{1/p}(\|x\|^p + \|y\|^p)^{1/p}$$

if and only if X is <u>of type p,</u> and <u>$T_{p(p')}(X) = 1$</u> resp. <u>$T_{p(p)}(X) = 1$.</u>

(ii) Let $2 \leqq q < \infty$ and $q' \leqq s \leqq q$. Then, X satisfies the Clarkson-Boas-type inequality

$$\text{(CBI-2)} \quad (\|x+y\|^q + \|x-y\|^q)^{1/q} \leqq 2^{1/s'}(\|x\|^s + \|y\|^s)^{1/s}$$

if and only if X is <u>of cotype q</u> and <u>$C_{q(s)}(X) = 1$.</u> In particular, X satisfies Clarkson's inequalities

$$\text{(CI-3*)} \quad (\|x+y\|^q + \|x-y\|^q)^{1/q} \leqq 2^{1/q}(\|x\|^{q'} + \|y\|^{q'})^{1/q'}$$

resp.

(CI-4*) $(\| x+y \|^q + \| x-y \|^q)^{1/q} \leq 2^{1/q'} (\| x \|^q + \| y \|^q)^{1/q}$

if and only if X is <u>of cotype q,</u> and $\underline{C_{q(q')}(X) = 1}$ resp. $\underline{C_{q(q)}(X) = 1.}$

Note. The above theorem implies in particular that the notions of G_0- and G_n-Fourier type for a Banach space in Milman [13] are equivalent. (See [9] for some other related results.)

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