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不等式相条件を持つ変分問題に現われる包絡線

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An envelope in a variational problem with inequality phase constraints

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Various types of extremal problems are formulated as an abstract optimization problem in Banach spaces:

$$\begin{array}{ll} \text{Minimize} & f(x) \\ \text{subject to} & g(x) \in K, \quad h(x) = 0. \end{array}$$

where X, V, W are Banach spaces, K is a convex cone in V with non-empty interior, $f : X \to R$, $g : X \to V$ and $h : X \to W$ are of C^2-class.

One of the authors has been studying second-order necessary optimality conditions for the abstract problem, and clarified that the generalized inequality constraint $g(x) \in K$ often form an envelope and that we have to take into account of the envelope when we consider second-order optimality conditions.

There are two families of extremal problems which form envelopes. One is a family of Tchebycheff approximation problems and the other is a family of variational problems with inequality phase constraints:

$$\begin{array}{lll} (P) & \text{Minimize} & f(x) = \int_0^1 F(t, x(t), \dot{x}(t))dt \\ & \text{subject to} & x(0) = x_0, \quad x(1) = x_1, \quad x \in X, \\ & & G(t, x(t)) \leq 0 \quad {}^\forall t \in [0,1]. \end{array}$$

where x_0 and x_1 are given points in R^n, $F : R^{2n+1} \to R$ is of C^2-class w.r.t. x and $\dot{x}$, $G : [0,1] \times R^n \to R^m$ is of C^2-class w.r.t. x and $\dot{x}$. We take

$$X = \{x = (x_1, x_2, \cdots, x_n) \mid x_i \text{ ; absolutely conti. } ||x|| < \infty\}$$

equipped with the norm:

$$||x|| = \max_{t \in [0,1]} ||x(t)|| + \operatorname*{esssup}_{t \in [0,1]} ||\dot{x}(t)|| < \infty.$$

We assume that the weak minimal solution $\bar{x}(t)$ is piecewise smooth. We use the abbreviation:

$$\hat{F}(t) = F(t, \bar{x}, \dot{\bar{x}}(t)), \quad \hat{G}(t) = G(t, \bar{x}(t)), \text{ e.t.c.}$$

The aim of this paper is to clarify the effect of the envelope formed by the phase constraints on second-order necessary optimality condition (Legendre condition).

Definition The feasible region M of the abstract optimization problem is said to satisfy the Mangasarian-Fromovitz condition at $\bar{x}$ if

$$\begin{array}{ll} (i) & h'(\bar{x}) \ : \ X \to W \ \text{ is onto} \\ (ii) & {}^{\exists}x_0 \in X, \ \ h'(\bar{x})x_0 = 0, \ \ g(\bar{x}) + g'(\bar{x})x_0 \in \text{int}K. \end{array}$$

The following theorem can be found in many literatures, e.g. Ben-Tal and Zowe [1] and Kawasaki [12].

Theorem (First-order necessary optimality condition) Let x be a weak minimal solution of the abstract optimization problem. Assume that the feasible region satisfies the Mangasarian-Fromovitz condition at x. Then there exist $v^* \in K^o$ and $w^* \in W^*$ such that $L(x) := f(x)+ < v^*, g(x) > + < w^*, h(x) >$

$$L'(x) = 0,$$

$$< v^*, g(x) >= 0,$$

where $K^o := \{v^*; \ < v^*, v >\leq 0 \ {}^{\forall}v \in K\}$

Definition A direction $y \in X$ is called a critical direction if

$$f'(x)y = 0, \ \ g'(x)y \in \text{clcone}(K - g(x)), \ \ h'(x)y = 0.$$

where clcone$(K - g(x))$ denotes the closure of the conical hull o f $K - g(x)$.
Definition For any $u, \ v \in V$, we define

$$K(u, v) := \{w \in V; \ \theta^2 u + \theta v + w + o(1) \in K \ \ {}^{\forall}\theta > 0\},$$

$$K(y) := K(g(x), g'(x)y).$$

Theorem (Second-order necessary optimality condition) (Kawasaki [12]) Let x be a minimal solution of the abstract optimization problem. Assume that the feasible region satisfies the Mangasarian-Fromovitz condition at x. Then, for each critical direction $y \in X$ satisfying $K(y) \neq \phi$, there exist $v^* \in K^o$ and $w^* \in W^*$ such that

$$L'(x) = 0,$$

$$L''(x)(y, y) - 2\delta^*(v^*|K(y)) \geq 0$$

$$< v^*, g(x) >= 0, \quad < v^*, g'(x)y >= 0.$$

where $\delta^*(v^*|K(y)) := \sup\{< v^*, v >;\ v \in K(y)\}$.

For the variational problem (P), the extra term $\delta^*(v^*|K(y))$ is represented as an integration, see Kawasaki[13]:

$$\delta^*(v^*|K(y)) = -\int_0^1 d\psi^T E,$$

where ψ is a n-dimensional vector-valued nondecreasing function defined on $[0,1]$ and $E(t)$ is defined by

$$u(t) := -G(t, x(t)), \quad v(t) := -G_x(t, x(t))y(t)$$

$$E(t) := \begin{cases} \sup\left\{\limsup \frac{v(t_n)^2}{4u(t_n)};\ \{t_n\} \text{ satisfies } (1)\right\}, & \text{if } t \in T_0, \\ 0 \qquad \text{if } u(t) = v(t) = 0 \text{ and } t \notin T_0, & \\ -\infty & \text{otherwise,} \end{cases}$$

$$T_0 := \left\{ t \in T;\ \exists t_n \to t \ s.t.\ u(t_n) > 0,\ -\frac{v(t_n)}{u(t_n)} \to +\infty \right\}. \tag{1}$$

Let us now apply the above theorem to the variational problem (P). For this aim, we need the following notation.

$$I(t) := \{j \in \{1, 2, \cdots, m\} |\ \hat{G}_j(t) = 0\ \}.$$

$$J_L(t) := \{j|^\exists \delta > 0,\ \ \hat{G}_j < 0 \text{ on } (t - \delta, t)\}$$

$$J_R(t) := \{j|^\exists \delta > 0,\ \ \hat{G}_j < 0 \text{ on } (t, t + \delta)\}$$

In (P), the Mangasarian-Fromovitz condition is guaranteed by the following conditions.

(A_1) The matrix $(\hat{G}_{jx}(t))_{j\in I(t)}$ has full rank for all $t \in [0,1]$
(A_2) $\hat{G}(0) < 0,\ \hat{G}(1) < 0$

Theorem 1 Let $\bar{x}(t)$ be a weak minimal solution for (P). Assume that (A_1) and (A_2) are satisfied at $\bar{x}$, then

(i) $\xi^T \hat{F}_{\dot{x}\dot{x}}(t-0)\xi \geq 0$ $\ ^\forall \xi \in R^n$ satisfying $(\hat{G}_{jx}(t))_{j \in I(t)\setminus J_L(t)}\xi = 0$
(ii) $\xi^T \hat{F}_{\dot{x}\dot{x}}(t+0)\xi \geq 0$ $\ ^\forall \xi \in R^n$ satisfying $(\hat{G}_{jx}(t))_{j \in I(t)\setminus J_R(t)}\xi = 0.$

When we consider the one-sided phase constraint:

$$s(t) \leq x(t) \ \ \forall t,$$

we get Corollary 1 from Theorem 1.

Corollary 1 (One-sided phase constraint) Let $\bar{x}(t)$ be a weak minimal solution for (P). Assume that $s(0) < x(0),\ s(1) < x(1)$, then

$$\begin{array}{ll} (i) & \xi^T \hat{F}_{\dot{x}\dot{x}}(t-0)\xi \geq 0 \quad {}^\forall \xi \in R^n \ \text{s.t.}\ \xi_j = 0 \quad {}^\forall j \notin J_L(t) \\ (ii) & \xi^T \hat{F}_{\dot{x}\dot{x}}(t+0)\xi \geq 0 \quad {}^\forall \xi \in R^n \ \text{s.t.}\ \xi_j = 0 \quad {}^\forall j \notin J_R(t). \end{array}$$

Neither Theorem 1 nor Corollary 1 does touch on any interval where some phase constraint is active. The following theorem and corollary touch on such intervals.

Theorem 2 Under the assumption of Theorem 1, let $E_L(t)$ denote the set of indices $i \notin J_L(t)$ such that the Euler equation w.r.t. x_i:

$$\frac{d}{dt}\hat{F}_{\dot{x}_i}(t) - \hat{F}_{x_i}(t) = 0$$

holds a.e. on $(t-\delta, t)$ for some $\delta > 0$. Then we may replace $(\hat{G}_{jx}(t))_{j\in I(t)\setminus J_L(t)}\xi = 0$, in (i) of Theorem 1, by

$$(\hat{G}_{jx_i}(t))_{j\in I(t)\setminus J_L(t),\ i\in E_L(t)}(\xi_i)_{i\in E_L(t)} \leq 0$$

$$(\hat{G}_{jx_i}(t))_{j\in I(t)\setminus J_L(t),\ i\notin E_L(t)}(\xi_i)_{i\notin E_L(t)} = 0.$$

If the Euler equation w.r.t. x_i holds a.e. on $(t, t+\delta)$ for some $\delta > 0$, then we may similarly replace ξ in (ii) of Theorem 1

Corollary 2 (One-sided phase constraint) Under the assumption of Corollary 1, let $E_L(t)$ denote the set of indices $i \notin J_L(t)$ such that the Euler equation w.r.t. x_i:

$$\frac{d}{dt}\hat{F}_{\dot{x}_i}(t) - \hat{F}_{x_i}(t) = 0$$

holds a.e. on $(t-\delta, t)$ for some $\delta > 0$. Then we may replace $\xi_j = 0$, in (i) of Corollary 1, by

$$\xi_j \geq 0 \text{ for } j \in E_L(t), \quad \xi_j = 0 \text{ for } j \in J_L(t)\setminus E_L(t).$$

If the Euler equation w.r.t. x_i holds a.e. on $(t, t+\delta)$ for some $\delta > 0$, then we may similarly replace $\xi_j = 0$, in (ii) of Corollary 1, by $\xi_j \geq 0$.

Example 1 In this example, an non-optimal solution is excluded by Corollary 2, though Corollary 1 can not exclude it.

$$\begin{array}{ll} \text{minimize} & \int_{-2}^{2}(t^2-1)\dot{x}^2(t)dt \\ \text{subject to} & x(t) \geq s(t), x(-2) = 1, x(2) = 1, \end{array}$$

where

$$s(t) = \begin{cases} -t(t+2) & -2 \le t \le -1 \\ 1 & -1 \le t \le 1 \\ -t(t-2) & 1 \le t \le 2 \end{cases}$$

Take $\bar{x}(t) = 1$. Then, from the Euler-Lagrange equation, we get

$$\psi(t) = 2\dot{\bar{x}}(t)(1-t^2) + C = C.$$

Hence $\bar{x}$ satisfies the Euler equation on $[-2, 2]$. Since $\hat{f}_x = 0$ and $\hat{f}_{\dot{x}} = 2\dot{\bar{x}}(t)(t^2-1)$, we have

$$\int_t^1 \hat{f}_x(s)ds + \hat{f}_{\dot{x}}(t) = \int_t^1 0ds + (t^2-1)\dot{\bar{x}}(t) = 0 \text{ on } [-2, 2].$$

Since

$$\hat{f}_{\dot{x}\dot{x}}(t) \ge 0 \quad \text{on} \quad [-2, -1] \cup [1, 2],$$

$\bar{x}$ satisfies all the conditions in Corollary 1. However, since

$$\hat{f}_{\dot{x}\dot{x}}(0) = -2 < 0,$$

we see from Corollary 2 that $\bar{x}$ is not a weak minimal solution.

By the way, no extra term appear in Theorem 1, Theorem 2, Corollary 1 and Corollary 2. As was shown in Kawasaki [12] [13], the extra term appears only when an envelope is formed by the constraints. Hence the authors once guessed that no envelope was formed in the variational problem (P). But it was not correct. In the following example, an envelope is formed by the one-sided phase constraint.

Example 2

$$\begin{aligned} &\text{minimize} && \int_{-1}^1 \{x(t) + \dot{x}^2(t)\}dt \\ &\text{subject to} && x(-1) = x(1) = \frac{1}{4}, \quad x(t) \ge 1 - |t| \text{ on } [-1, 1] \end{aligned}$$

Take

$$\bar{x}(t) = \begin{cases} \frac{t^2}{4} + t + 1 & -1 \le t \le 0 \\ \frac{t^2}{4} - t + 1 & 0 \le t \le 1 \end{cases}$$

For sufficiently small $r < 0$, put

$$y(t) = \begin{cases} rt & \frac{r}{2} \le t \le 0 \\ r(r-t) & r \le t \le \frac{r}{2} \\ 0 & \text{otherwise} \end{cases}$$

Then it is easily seen that y is a critical direction. Computing $E(t)$, we get

$$E(t) = \begin{cases} r^2 & t = 0 \\ -\infty & t \neq 0 \end{cases}$$

Hence

$$\delta^*(v^*|K(y)) = -\int_{-1}^{1} d\psi(t)E(t) = 4r^2 > 0,$$

which implies that an envelope is formed.

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