数理解析研究所講究録
第 897 巻 1995 年 86-98

LOCAL EXISTENCE THEOREMS FOR NONLINEAR DIFFERENTIAL EQUATIONS AND COMPACTNESS OF INTEGRAL SOLUTIONS IN $L^p(0,T;X)$

玉川大学工学部 塩路 直樹
(Naoki Shioji)

1. Introduction. Let X be a real Banach space, let $A \subset X \times X$ be an m-accretive set, let $u_0 \in \overline{D(A)}$ and let F_T be a mapping from a subset of $L^1(0,T;X)$ into $L^1(0,T;X)$. In this paper, we study the initial value problem

$$\begin{aligned} &\frac{du(t)}{dt} + Au(t) \ni F_T u(t), \quad 0 \le t \le T, \\ &u(0) = u_0. \end{aligned} \tag{1.1}$$

Crandall and Nohel [6], Diaz and Vrabie [7], Gutman [8,9], Hirano [10], Kenmochi and Koyama [11], Liu [12], Mitidieri and Vrabie [13,14], Pazy [15], Vrabie [16–18] and others have studied this kind of problems under several different conditions. Many of these authors used Schauder's fixed point theorem in $C(0,T;X)$ to prove the existence of local solutions of (1.1). So it is essential to study conditions that $\{u^f : f \in B\}$ is relatively compact in $C(0,T;X)$ for $B \subset L^1(0,T;X)$, where u^f is the unique integral solution of

$$\begin{aligned} &\frac{du(t)}{dt} + Au(t) \ni f(t), \quad 0 \le t \le T, \\ &u(0) = u_0 \end{aligned}$$

for $f \in L^1(0,T;X)$. But, in general, we need weaker conditions to prove that $\{u^f : f \in B\}$ is relatively compact in $L^p(0,T;X)$. In this paper, we use Schauder's fixed point in $L^p(0,T;X)$ to prove the existence of local solutions of (1.1). Schauder's fixed point theorem in $L^p(0,T;X)$ requires the continuity of F_T from $L^p(0,T;X)$ into $L^1(0,T;X)$ instead of that of F_T from $C(0,T;X)$ into $L^1(0,T;X)$. But in many applications, it is not a restriction. Concerning relative compactness of $\{u^f : f \in B\}$ in $L^p(0,T;X)$, Baras [1] showed that for every bounded subset B of $L^1(0,T;X)$ and for every $1 \le p < \infty$, $\{u^f : f \in B\}$ is relatively compact in $L^p(0,T;X)$ under the condition that the nonlinear semigroup $\{S(t) : \overline{D(A)} \to \overline{D(A)}, t \ge 0\}$ generated by $-A$ is compact. We use this result to prove the existence of local solutions of (1.1) in the case that $\{S(t)\}$ is compact. When the resolvent $(I + \lambda A)^{-1}$ is compact for every $\lambda > 0$, we show a sufficient condition that for every $1 \le p < \infty$, $\{u^f : f \in B\}$ is relatively compact in $L^p(0,T;X)$ under some hypotheses. We also use this result to prove the existence of local solutions of (1.1).

The next section is devoted to some preliminaries. In section 3, we state our main results and we prove those in section 4. In the final section, we study some examples.

2. Preliminaries. Let X be a real Banach space with norm $\|\cdot\|$. If D is a subset of X, $\overline{D}$ denotes the closure of D. For each $(x,y) \in X \times X$, define

$$\langle x,y\rangle_+ = \lim_{t\downarrow 0} \frac{\|x+ty\| - \|x\|}{t}.$$

Let $A \subset X \times X$. For $x \in X$, we denote by Ax the set $\{y \in X : (x,y) \in A\}$. We define $D(A) = \{x \in X : Ax \neq \emptyset\}$ and $R(A) = \bigcup\{Ax : x \in D(A)\}$. A subset $A \subset X \times X$ is called accretive if

$$\langle x_1 - x_2, y_1 - y_2\rangle_+ \geq 0$$

for every $(x_1,y_1),(x_2,y_2) \in A$. An accretive set A is called m-accretive if $R(I+\lambda A) = X$ for every $\lambda > 0$. Let $T > 0$. $C(0,T;X)$ denotes the space of all continuous functions from $[0,T]$ into X. For $1 \leq p < \infty$, $L^p(0,T;X)$ denotes the space of all strongly measurable, p-integrable, X-valued functions defined almost everywhere on $[0,T]$, and $L^\infty(0,T;X)$ denotes the space of all strongly measurable, essentially bounded, X-valued functions defined almost everywhere on $[0,T]$. Let U be an open subset of X. $C(0,T;U)$ and $L^\infty(0,T;U)$ denote the sets $\{f \in C(0,T;X) : f(t) \in U \text{ on } [0,T]\}$ and $\{f \in L^\infty(0,T;X) : f(t) \in U \text{ a.e. on } [0,T]\}$ respectively.

Let $A \subset X \times X$ be an m-accretive set, $f \in L^1(0,T;X)$ and $u_0 \in \overline{D(A)}$. A function $u : [0,T] \to X$ is called a strong solution of the initial value problem:

$$\begin{aligned} &\frac{du(t)}{dt} + Au(t) \ni f(t), \quad 0 \leq t \leq T, \\ &u(0) = u_0, \end{aligned} \tag{2.1}$$

if u is differentiable almost everywhere on $[0,T]$, u is absolutely continuous, $u(0) = u_0$ and $u'(t) + Au(t) \ni f(t)$ almost everywhere on $[0,T]$. A function $u : [0,T] \to X$ is called an integral solution of the initial value problem (2.1), if u is continuous on $[0,T]$, $u(0) = u_0$, $u(t) \in \overline{D(A)}$ for every $0 \leq t \leq T$ and

$$\|u(t) - x\| \leq \|u(s) - x\| + \int_s^t \langle u(\tau) - x, f(\tau) - y\rangle_+ \, d\tau$$

for every $(x,y) \in A$ and $0 \leq s \leq t \leq T$. If u is a strong solution of (2.1), then u is an integral solution of (2.1). It is known [2,3] that the initial value problem (2.1) has a unique integral solution. If u and v are the integral solutions of (2.1) corresponding to $(f,u_0),(g,v_0) \in L^1(0,T;X) \times \overline{D(A)}$ respectively, then

$$\|u(t) - v(t)\| \leq \|u(s) - v(s)\| + \int_s^t \langle u(\tau) - v(\tau), f(\tau) - g(\tau)\rangle_+ \, d\tau$$

for $0 \leq s \leq t \leq T$. Concerning integral solutions, we also know the following. For its proof, see [19, p.74].

Proposition 1. *Let $T > 0$, $f \in L^1(0,T;X)$ and $u_0 \in \overline{D(A)}$. Let u be the unique integral solution of* (2.1). *Then*

$$\|u(t+s) - u(t)\| \le \int_0^s \|f(\tau)\|\, d\tau + \|S(s)u_0 - u_0\| + \int_0^{T-s} \|f(\tau+s) - f(\tau)\|\, d\tau$$

for $t, s \ge 0$ with $t + s \le T$.

If $A \subset X \times X$ is m-accretive, then

$$S(t)x = \lim_{n\to\infty} \left(I + \frac{t}{n}A\right)^{-n} x$$

exists for each $x \in \overline{D(A)}$ and uniformly for t on every bounded interval in the set of nonnegative real numbers [2, 5]. $\{S(t) : \overline{D(A)} \to \overline{D(A)}, t \ge 0\}$ is called the nonlinear semigroup generated by $-A$. We remark that $t \mapsto S(t)u_0$ is the unique integral solution of (2.1) corresponding to $(0, u_0) \in L^1(0,T;X) \times \overline{D(A)}$. We say $\{S(t) : \overline{D(A)} \to \overline{D(A)}, t \ge 0\}$ is compact if $S(t) : \overline{D(A)} \to \overline{D(A)}$ is compact for every $t > 0$. It is well known [4] that $\{S(t) : \overline{D(A)} \to \overline{D(A)}, t \ge 0\}$ is compact if and only if $J_\lambda = (I + \lambda A)^{-1}$ is compact for every $\lambda > 0$ and for each bounded subset B of X, $\{S(t)x : [0,\infty) \to X, x \in B\}$ is equicontinuous at each $t > 0$. We also know the following.

Proposition 2 (Brézis [4]). *For each $\lambda > 0$ and $x \in \overline{D(A)}$,*

$$\|x - J_\lambda x\| \le \frac{4}{\lambda} \int_0^\lambda \|S(s)x - x\|\, ds.$$

3. Main results. We begin this section with hypotheses (cf. [18]) and notations which we shall use in the sequel.

- (H1) X is a real Banach space and $A \subset X \times X$ is an m-accretive set. J_λ is the resolvent $(I + \lambda A)^{-1}$ for each $\lambda > 0$ and $\{S(t) : \overline{D(A)} \to \overline{D(A)}, t \ge 0\}$ is the nonlinear semigroup generated by $-A$.
- (H2) $1 \le p < \infty$, $T_0 > 0$ and for each $0 < T \le T_0$, $M(0,T;X)$ is a subset of $L^p(0,T;X)$. $\mathcal{F} = \{F_T : M(0,T;X) \to L^1(0,T;X), 0 < T \le T_0\}$ is a family of mappings such that for each $0 < T \le S \le T_0$, $u \in M(0,T;X)$ and $v \in M(0,S;X)$ with $u(t) = v(t)$ a.e. on $[0,T]$, it follows that $F_T u(t) = F_S v(t)$ a.e. on $[0,T]$.
- (H3) For each $0 < T \le T_0$, $M(0,T;X) = L^p(0,T;X)$ and $F_T : L^p(0,T;X) \to L^1(0,T;X)$ is continuous.
- (H4) U is an open subset of X. For each $0 < T \le T_0$, $M(0,T;X) = L^\infty(0,T;U)$ and for every $d > 0$, $F_T : Z_{d,T} \to L^1(0,T;X)$ is continuous, where $Z_{d,T}$ is the topological space $\{u \in L^\infty(0,T;U) : \operatorname{ess\,sup}_{0\le\tau\le T} \|u(\tau)\| \le d\}$ which is endowed with the $L^p(0,T;X)$ topology.
- (H5) For every $d > 0$,

$$\lim_{h\downarrow 0} \int_0^h \|F_{T_0} u(\tau)\|\, d\tau = 0$$

uniformly for $u \in Z_{d,T_0}$.

(H6) There exist $1 \le \eta < \infty$ and $k : (0,\infty) \to [0,\infty)$ such that for each $d > 0$, there exists a function $\alpha_d : (0,T_0] \to [0,\infty)$ which satisfies

(i) $\lim_{h \downarrow 0} \alpha_d(h) = 0$, and

(ii) for every $u \in Z_{d,T_0}$,

$$\int_0^{T-h} \|F_T u(\tau+h) - F_T u(\tau)\| \, d\tau \le \alpha_d(h) + k(d)\Big(\int_0^{T-h} \|u(\tau+h)-u(\tau)\|^\eta \, d\tau\Big)^{\frac{1}{\eta}}$$

for every $0 < T \le T_0$ and for every $0 < h < T$.

Now we state local existence results for nonlinear differential equations.

Theorem 1. *Assume that* (H1), (H2) *and* (H3) *are satisfied and that* $\{S(t) : \overline{D(A)} \to \overline{D(A)}, t \ge 0\}$ *is compact. Then for each* $u_0 \in \overline{D(A)}$, *there exists* $0 < T \le T_0$ *such that* (1.1) *has at least one integral solution* u *belonging to* $C(0,T;X)$.

Theorem 2. *Assume that* (H1), (H2), (H4) *and* (H5) *are satisfied and that* $\{S(t) : \overline{D(A)} \to \overline{D(A)}, t \ge 0\}$ *is compact. Then for each* $u_0 \in \overline{D(A)} \cap U$, *there exists* $0 < T \le T_0$ *such that* (1.1) *has at least one integral solution* u *belonging to* $C(0,T;U)$.

Theorem 3. *Assume that* (H1), (H2), (H4), (H5) *and* (H6) *are satisfied and that* J_λ *is compact for every* $\lambda > 0$. *Then for each* $u_0 \in \overline{D(A)} \cap U$, *there exists* $0 < T \le T_0$ *such that* (1.1) *has at least one integral solution* u *belonging to* $C(0,T;U)$.

Next we show a sufficient condition in order that a set of integral solutions is relatively compact in $L^q(0,T;X)$ for every $1 \le q < \infty$. It will be used in the proof of Theorem 3. For $f \in L^1(0,T;X)$ and $u_0 \in \overline{D(A)}$, we denote by u^f the unique integral solution of (2.1) corresponding to f and u_0.

Theorem 4. *Assume that* (H1) *is satisfied and that* $J_\lambda : X \to X$ *is compact for every* $\lambda > 0$. *Let* $T > 0$ *and let* B *be a bounded subset of* $L^1(0,T;X)$ *such that*

$$\lim_{h \downarrow 0} \int_0^{T-h} \|f(t+h) - f(t)\| \, dt = 0$$

uniformly for $f \in B$ *and*

$$\lim_{h \downarrow 0} \int_0^{h} \|f(t)\| \, dt = 0$$

uniformly for $f \in B$. *Let* $u_0 \in \overline{D(A)}$. *Then* $\{u^f : f \in B\}$ *is relatively compact in* $L^q(0,T;X)$ *for every* $1 \le q < \infty$ *and it is bounded in* $L^\infty(0,T;X)$.

4. Proof of Theorems. First we prove local existence results for nonlinear differential equations under the condition that $\{S(t)\}$ is compact. In the next proof, we use the method employed in [16].

PROOF OF THEOREM 1. Let $u_0 \in \overline{D(A)}$. Choose $0 < T \leq T_0$, $M > 0$ and $r > 0$ such that $T^{\frac{1}{p}} M \leq r$ and

$$\int_0^T \|F_T u(t)\| \, dt \leq M$$

for every $u \in L^p(0,T;X)$ with $\left(\int_0^T \|u(t) - S(t)u_0\|^p \, dt\right)^{\frac{1}{p}} \leq r$. Put

$$K = \{u \in L^p(0,T;X) : \left(\int_0^T \|u(t) - S(t)u_0\|^p \, dt\right)^{\frac{1}{p}} \leq r\}.$$

By the method employed in [16], we define an operator $Q : K \to L^p(0,T;X)$ as follows: for each $u \in K$, let Qu be the unique integral solution $v \in C(0,T;X)$ of

$$\begin{aligned} &\frac{dv(t)}{dt} + Av(t) \ni F_T u(t), \quad 0 \leq t \leq T, \\ &v(0) = u_0. \end{aligned}$$

We shall show that Q is a continuous operator from K into K. Let $u \in K$. Since

$$\|Qu(t) - S(t)u_0\| \leq \int_0^T \|F_T u(s)\| \, ds$$

for $0 \leq t \leq T$, we have

$$\begin{aligned} \left(\int_0^T \|Qu(t) - S(t)u_0\|^p \, dt\right)^{\frac{1}{p}} &\leq \left(\int_0^T \left(\int_0^T \|F_T u(s)\| \, ds\right)^p dt\right)^{\frac{1}{p}} \\ &\leq T^{\frac{1}{p}} \int_0^T \|F_T u(s)\| \, ds \\ &\leq r. \end{aligned}$$

This inequality implies $Q(K) \subset K$. Let $u, v \in K$. Since

$$\|Qu(t) - Qv(t)\| \leq \int_0^T \|F_T u(s) - F_T v(s)\| \, ds$$

for $0 \leq t \leq T$, we have

$$\left(\int_0^T \|Qu(t) - Qv(t)\|^p \, dt\right)^{\frac{1}{p}} \leq T^{\frac{1}{p}} \int_0^T \|F_T u(s) - F_T v(s)\| \, ds. \tag{4.1}$$

This inequality and the continuity of $F_T : L^p(0,T;X) \to L^1(0,T;X)$ imply that Q is continuous. From Théorème 1 in [1], it follows that Q is compact. Hence, by Schauder's fixed point theorem, (1.1) has at least one integral solution. □

PROOF OF THEOREM 2. Let $u_0 \in \overline{D(A)} \cap U$. Choose $r > 0$ and $0 < T_1 \leq T_0$ such that the closed ball with center u_0 and radius $r + \max_{0 \leq \tau \leq T_1} \|S(\tau)u_0 - u_0\|$ is contained in U. Put $d = r + \max_{0 \leq \tau \leq T_0} \|S(\tau)u_0\|$ and choose $0 < T \leq T_1$ such that

$$\int_0^T \|F_{T_0}u(\tau)\| \, d\tau \leq r$$

for every $u \in Z_{d,T_0}$. Set

$$K = \{u \in L^\infty(0,T;U) : \operatorname*{ess\,sup}_{0 \leq t \leq T} \|u(t) - S(t)u_0\| \leq r\}$$

which is endowed with $L^p(0,T;X)$ topology and define $Q : K \to L^p(0,T;X)$ by the same way in the proof of Theorem 1. It is easy to see that K is closed in $L^p(0,T;X)$. We shall show that Q is a continuous operator from K into K. Let $u \in K$. Since

$$\begin{aligned}\|Qu(t) - S(t)u_0\| &\leq \int_0^T \|F_T u(s)\| \, ds \\ &\leq r\end{aligned}$$

for $0 \leq t \leq T$ and

$$\begin{aligned}\|Qu(t) - u_0\| &\leq \|Qu(t) - S(t)u_0\| + \|S(t)u_0 - u_0\| \\ &\leq r + \max_{0 \leq \tau \leq T_1} \|S(\tau)u_0 - u_0\|\end{aligned}$$

for $0 \leq t \leq T$, we have $Q(K) \subset K$. Q is continuous by (4.1). From Théorème 1 in [1], it follows that Q is compact. Hence, by Schauder's fixed point theorem, (1.1) has at least one integral solution. □

Next we prove a sufficient condition in order that a set of integral solutions is relatively compact in $L^p(0,T;X)$.

PROOF OF THEOREM 4. Let $1 \leq q < \infty$ and put $c = \sup_{f \in B} \int_0^T \|f(\tau)\| \, d\tau$. First we remark that $\{u^f(t) : f \in B, 0 \leq t \leq T\}$ is a bounded subset of X. Let $f \in B$ and let $\lambda > 0$. Since, by Proposition 1,

$$\begin{aligned}&\left(\int_0^{T-s} \|J_\lambda u^f(t+s) - J_\lambda u^f(t)\|^q \, dt\right)^{\frac{1}{q}} \\ &\leq \left(\int_0^{T-s} \|u^f(t+s) - u^f(t)\|^q \, dt\right)^{\frac{1}{q}} \\ &\leq T^{\frac{1}{q}}\left(\int_0^s \|f(\tau)\| \, d\tau + \|S(s)u_0 - u_0\| + \int_0^{T-s} \|f(\tau+s) - f(\tau)\| \, d\tau\right),\end{aligned}$$

we have

$$\lim_{s\downarrow 0}\left(\int_0^{T-s} \|J_\lambda u^f(t+s) - J_\lambda u^f(t)\|^q \, dt\right)^{\frac{1}{q}} = 0 \qquad \text{uniformly for } f \in B.$$

Hence $\{J_\lambda u^f : f \in B\}$ is relatively compact in $L^q(0,T;X)$ by the same lines as those in the proof of Theorem A.1 in [8]. Using Proposition 1 and Proposition 2, we have

$$\begin{aligned}
&\|J_\lambda u^f(t) - u^f(t)\| \le \frac{4}{\lambda}\int_0^\lambda \|S(s)u^f(t) - u^f(t)\| \, ds \\
&\le \frac{4}{\lambda}\int_0^\lambda \|S(s)u^f(t) - u^f(t+s)\| \, ds + \frac{4}{\lambda}\int_0^\lambda \|u^f(t+s) - u^f(t)\| \, ds \\
&\le \frac{4}{\lambda}\int_0^\lambda \int_t^{t+s} \|f(\tau)\| \, d\tau \, ds + 4 \sup_{0\le s\le\lambda} \|u^f(t+s) - u^f(t)\| \\
&\le 4\int_t^{t+\lambda} \|f(\tau)\| \, d\tau + 4 \sup_{0\le s\le\lambda}\left\{\int_0^s \|f(\tau)\| \, d\tau + \|S(s)u_0 - u_0\| + \int_0^{T-s} \|f(\tau+s) - f(\tau)\| \, d\tau\right\}
\end{aligned}$$

for $0 \le t \le T$. So we get

$$\begin{aligned}
&\left(\int_0^{T-\lambda} \|J_\lambda u^f(t) - u^f(t)\|^q \, dt\right)^{\frac{1}{q}} \\
&\le 4\left(\int_0^{T-\lambda}\left(\int_t^{t+\lambda} \|f(\tau)\| \, d\tau\right)^q dt\right)^{\frac{1}{q}} \\
&\quad + 4T^{\frac{1}{q}} \sup_{0\le s\le\lambda}\left\{\int_0^s \|f(\tau)\| \, d\tau + \|S(s)u_0 - u_0\| + \int_0^{T-s} \|f(\tau+s) - f(\tau)\| \, d\tau\right\} \\
&\le 4\left(\int_0^{T-\lambda} c^{q-1}\int_t^{t+\lambda} \|f(\tau)\| \, d\tau dt\right)^{\frac{1}{q}} \\
&\quad + 4T^{\frac{1}{q}} \sup_{0\le s\le\lambda}\left\{\int_0^s \|f(\tau)\| \, d\tau + \|S(s)u_0 - u_0\| + \int_0^{T-s} \|f(\tau+s) - f(\tau)\| \, d\tau\right\} \\
&\le 4c\lambda^{\frac{1}{q}} + 4T^{\frac{1}{q}} \sup_{0\le s\le\lambda}\left\{\int_0^s \|f(\tau)\| \, d\tau + \|S(s)u_0 - u_0\| + \int_0^{T-s} \|f(\tau+s) - f(\tau)\| \, d\tau\right\},
\end{aligned}$$

Hence we have

$$\lim_{\lambda\downarrow 0}\left(\int_0^{T-\lambda} \|J_\lambda u^f(t) - u^f(t)\|^q dt\right)^{\frac{1}{q}} = 0 \qquad \text{uniformly for } f \in B,$$

which implies that $\{u^f : f \in B\}$ is relatively compact in $L^q(0,T;X)$. □

Finally we prove a local existence result for nonlinear differential equations under the condition that J_λ is compact for every $\lambda > 0$. In the next proof, we use the method employed in [10,16].

PROOF OF THEOREM 3. Let $u_0 \in \overline{D(A)} \cap U$. Choose r, T_1, d and T by the similar way in the proof of Theorem 2 such that $1 - T^{\frac{1}{\eta}} k(d) > 0$ is also satisfied. Define K and $Q : K \to K$ by the same way in the proof of Theorem 2. Put

$$K_1 = \{u \in K : \left(\int_0^{T-h} \|u(t+h) - u(t)\|^\eta \, dt \right)^{\frac{1}{\eta}} \leq \beta(h) \text{ for every } 0 < h < T\},$$

where

$$\beta(h) = \frac{T^{\frac{1}{\eta}} \left(\sup_{u \in K} \int_0^h \|F_T u(\tau)\| \, d\tau + \|S(h)u_0 - u_0\| + \alpha_d(h) \right)}{1 - T^{\frac{1}{\eta}} k(d)}, \quad 0 < h < T.$$

It is easy to see that K_1 is closed in $L^p(0,T;X)$. We shall prove that $Q(K_1) \subset K_1$. Let $u \in K_1$. Since, by Proposition 1,

$$\begin{aligned} &\|Qu(t+h) - Qu(t)\| \\ &\leq \int_0^h \|F_T u(\tau)\| \, d\tau + \|S(h)u_0 - u_0\| + \int_0^{T-h} \|F_T u(\tau + h) - F_T u(\tau)\| \, d\tau \end{aligned}$$

for $t, h \geq 0$ with $t + h \leq T$, we have

$$\begin{aligned} &\left(\int_0^{T-h} \|Qu(t+h) - Qu(t)\|^\eta \, dt \right)^{\frac{1}{\eta}} \\ &\leq T^{\frac{1}{\eta}} \left(\int_0^h \|F_T u(\tau)\| \, d\tau + \|S(h)u_0 - u_0\| + \int_0^{T-h} \|F_T u(\tau + h) - F_T u(\tau)\| \, d\tau \right) \\ &\leq T^{\frac{1}{\eta}} \left(\sup_{v \in K} \int_0^h \|F_T v(\tau)\| \, d\tau + \|S(h)u_0 - u_0\| + \alpha_d(h) + k(d) \left(\int_0^{T-h} \|u(t+h) - u(t)\|^\eta \, dt \right)^{\frac{1}{\eta}} \right) \\ &\leq T^{\frac{1}{\eta}} \left(\sup_{v \in K} \int_0^h \|F_T v(\tau)\| \, d\tau + \|S(h)u_0 - u_0\| + \alpha_d(h) + k(d)\beta(h) \right) \\ &\leq \beta(h) \end{aligned}$$

for every $0 < h < T$. So we have $Q(K_1) \subset K_1$. By Theorem 4, $Q : K_1 \to K_1$ is compact. Hence, by Schauder's fixed point theorem, (1.1) has at least one integral solution. □

5. Examples. Throughout this section, Ω is a bounded open subset of $\mathbb{R}^n$ $(n \geq 2)$ with sufficiently smooth boundary Γ.

Example 1. We consider the following nonlinear differential equation:

$$\frac{\partial u}{\partial t} - \Delta \rho(u) = f(t, x, u(t,x)) \quad \text{on } [0,T] \times \Omega, \tag{5.1}$$

with a boundary condition

$$\rho(u) = 0 \quad \text{on } [0,T] \times \Gamma \tag{5.2}$$

and an initial condition

$$u(0,x) = u_0(x) \quad \text{on } \Omega. \tag{5.3}$$

Theorem 5. *Let $\rho \in C(\mathbb{R}) \cap C^1(\mathbb{R} \setminus \{0\})$ such that $\rho(0) = 0$ and there exist $C > 0$ and $a > \frac{n-2}{n}$ with*

$$\rho'(r) \geq C|r|^{a-1} \quad \text{for each } r \in \mathbb{R} \setminus \{0\}.$$

Let $T_0 > 0$ and let $f : [0, T_0] \times \Omega \times \mathbb{R} \to \mathbb{R}$ such that $f(t,x,\cdot)$ is continuous for a.e. $(t,x) \in [0,T_0] \times \Omega$ and $f(\cdot,\cdot,u)$ is measurable for every $u \in \mathbb{R}$. Assume that there exist $b \in L^1(0,T_0;\mathbb{R})$ and $c \in L^1(0,T_0;L^1(\Omega))$ such that

$$|f(t,x,u)| \leq b(t)|u| + c(t,x)$$

for a.e. $(t,x) \in [0,T_0] \times \Omega$ and for every $u \in \mathbb{R}$. Then for each $u_0 \in L^1(\Omega)$, there exists $0 < T \leq T_0$ such that (5.1), (5.2) and (5.3) have an integral solution on $[0,T]$.

PROOF. Let $A \subset L^1(\Omega) \times L^1(\Omega)$ be an operator defined by

$$Au = -\Delta\rho(u) \quad \text{for } D(A) = \{u \in L^1(\Omega) : \rho(u) \in W_0^{1,1}(\Omega), \Delta\rho(u) \in L^1(\Omega)\}.$$

It is known [19, Lemma 2.6.2] that A is m-accretive and $-A$ generates a compact semigroup on $\overline{D(A)} = L^1(\Omega)$. For $0 < T \leq T_0$ and $d > 0$, set $Z_{d,T}$ be the space $\{u \in L^\infty(0,T;L^1(\Omega)) : \operatorname{ess\,sup}_{0 \leq \tau \leq T} \|u(\tau)\| \leq d\}$ which is endowed with the $L^1(0,T;L^1(\Omega))$ topology. The operator defined by

$$F_T u(t)(x) = f(t,x,u(t,x)), \quad u \in L^\infty(0,T;L^1(\Omega))$$

is continuous from $Z_{d,T}$ into $L^1(0,T;L^1(\Omega))$. So (H4) is satisfied. For $u \in L^\infty(0,T;L^1(\Omega))$, we write $f(s,u(s))(x)$ instead of $f(s,x,u(s,x))$. Since

$$\int_0^h \|f(s,u(s))\|\, ds \leq d\int_0^h |b(h)|\, ds + \int_0^h \|c(h)\|\, ds$$

for $u \in Z_{d,T_0}$, (H5) is satisfied. Then applying Theorem 2, we can see that for each $u_0 \in L^1(\Omega)$, there exists $0 < T \leq T_0$ such that (5.1), (5.2) and (5.3) have an integral solution on $[0,T]$. □

Example 2. Consider the following differential operator of the form

$$Au = \sum_{|\alpha| \leq m} (-1)^{|\alpha|} D^\alpha A_\alpha(x,u,Du,\cdots,D^m u),$$

where $A_\alpha : \Omega \times \mathbb{R}^N \to \mathbb{R}$. A_α is measurable in x and continuous in the rest of the variables, and there exists $\omega > 0$ such that

$$\sum_{|\alpha| \leq m} \Big(A_\alpha(x,u) - A_\alpha(x,v)\Big)(u_\alpha - v_\alpha) \geq \omega \sum_{|\alpha| \leq m} |u_\alpha - v_\alpha|^2$$

for a.e. $x \in \Omega$ and for every $(u,v) \in \mathbb{R}^N \times \mathbb{R}^N$. Now we consider the following nonlinear integrodifferential equation:

$$\frac{\partial u}{\partial t} + \sum_{|\alpha|\le m} (-1)^{|\alpha|} D^\alpha A_\alpha(x,u,\cdots,D^m u) \tag{5.4}$$
$$+ \int_0^t a(t-s)f(s,x,u(s,x))\,ds = 0 \quad \text{on } [0,T]\times\Omega$$

with Dirichlet boundary conditions

$$D^\alpha u = 0 \qquad \text{on } [0,T]\times\Gamma \text{ for } |\alpha| \le m-1 \tag{5.5}$$

and an initial condition

$$u(0,x) = u_0(x) \qquad \text{on } \Omega. \tag{5.6}$$

We improve Theorem 5.1 in [10].

Theorem 6. *Let $A : H_0^m(\Omega) \to H^{-m}(\Omega)$ be the nonlinear operator defined above. Let $T_0 > 0$, let $a \in L^1(0,T_0)$, and let $f : [0,T_0]\times\Omega\times\mathbb{R} \to \mathbb{R}$ such that $f(t,x,\cdot)$ is continuous for a.e. $(t,x) \in [0,T_0]\times\Omega$ and $f(\cdot,\cdot,u)$ is measurable for every $u \in \mathbb{R}$. Assume that there exist $b \in L^1(0,T_0;\mathbb{R})$ and $c \in L^1(0,T_0;L^2(\Omega))$ such that*

$$|f(t,x,u)| \le b(t)|u| + c(t,x)$$

for a.e. $(t,x) \in [0,T_0]\times\Omega$ and for every $u \in \mathbb{R}$. Then for each $u_0 \in L^2(\Omega)$, there exists $0 < T \le T_0$ such that (5.4), (5.5) and (5.6) have an integral solution on $[0,T]$.

PROOF. Let A_H be an operator defined by

$$A_H u = Au \qquad \text{for } u \in D(A_H) = \{u \in H_0^m(\Omega) : Au \in L^2(\Omega)\}.$$

Then A_H is a maximal monotone operator on $L^2(\Omega)$ and $(I+\lambda A_H)^{-1} : L^2(\Omega) \to L^2(\Omega)$ is compact for every $\lambda > 0$. For $0 < T \le T_0$, define

$$F_T u(t)(x) = -\int_0^t a(t-s)f(s,x,u(s,x))\,ds \qquad \text{for } u \in L^\infty(0,T;L^2(\Omega)).$$

Let $0 < T \le T_0$, let $d > 0$ and let $Z_{d,T}$ be the space $\{u \in L^\infty(0,T;L^2(\Omega)) : \operatorname*{ess\,sup}_{0\le\tau\le T}\|u(\tau)\| \le d\}$ which is endowed with the $L^1(0,T;L^2(\Omega))$ topology. Since

$$\begin{aligned}
&\int_0^T \left\| \int_0^t a(t-s)f(s,u(s))\,ds - \int_0^t a(t-s)f(s,v(s))\,ds \right\| dt \\
&\le \int_0^T\!\!\int_0^t |a(t-s)| \left\| f(s,u(s)) - f(s,v(s)) \right\| dsdt \\
&\le \int_0^T |a(t)|\,dt \int_0^T \left\| f(s,u(s)) - f(s,v(s)) \right\| ds
\end{aligned}$$

for $u, v \in Z_{d,T}$, F_T is continuous from $Z_{d,T}$ into $L^1(0,T;L^2(\Omega))$. So (H4) is satisfied. Since

$$\begin{aligned}\int_0^h \left\| \int_0^t a(t-s)f(s,u(s))\,ds \right\| dt &\le \int_0^h |a(t)|\,dt \int_0^h \left\| f(s,u(s)) \right\| ds \\ &\le \int_0^h |a(t)|\,dt \left(d\int_0^h |b(s)|\,ds + \int_0^h \|c(s)\|\,ds \right)\end{aligned}$$

for $u \in Z_{d,T_0}$, (H5) is satisfied. We shall show (H6) is satisfied. Let $u \in Z_{d,T_0}$. Since

$$\begin{aligned}&\|F_T u(t+h) - F_T u(t)\| \\ &= \left\| \int_0^{t+h} a(t+h-s)f(s,u(s))\,ds + \int_0^t a(t-s)f(s,u(s))\,ds \right\| \\ &\le \int_0^t |a(t+h-s) - a(t-s)|\,\|f(s,u(s))\|\,ds + \int_t^{t+h} |a(t+h-s)|\,\|f(s,u(s))\|\,ds,\end{aligned}$$

we have

$$\begin{aligned}&\int_0^{T-h} \|F_T u(t+h) - F_T u(t)\|\,dt \\ &\le \int_0^{T-h}\int_0^t |a(t+h-s) - a(t-s)|\,\|f(s,u(s))\|\,dsdt \\ &\quad + \int_0^{T-h}\int_t^{t+h} |a(t+h-s)|\,\|f(s,u(s))\|\,dsdt \\ &\le \left(\int_0^{T-h} |a(t+h) - a(t)|\,dt + \int_0^h |a(t)|\,dt \right) \int_0^T \|f(s,u(s))\|\,ds \\ &\le \left(\int_0^{T-h} |a(t+h) - a(t)|\,dt + \int_0^h |a(t)|\,dt \right) \left(d\int_0^T |b(s)|\,ds + \int_0^T \|c(s)\|\,ds \right)\end{aligned}$$

So (H6) is satisfied. Then applying Theorem 3, we can see that for each $u_0 \in L^2(\Omega)$, there exists $0 < T \le T_0$ such that (5.4), (5.5) and (5.6) have an integral solution on $[0,T]$. $\square$

Example 3. Let A be the differential operator defined in Example 2. Consider the following nonlinear differential equation:

$$\frac{\partial u}{\partial t} + \sum_{|\alpha| \le m} (-1)^{|\alpha|} D^\alpha A_\alpha(x, u, \cdots, D^m u) = f(t, x, u(t,x)) \quad \text{on } [0,T] \times \Omega \tag{5.7}$$

with Dirichlet boundary conditions (5.5) and an initial condition (5.6).

Theorem 7. *Let $A : H_0^m(\Omega) \to H^{-m}(\Omega)$ be the nonlinear operator defined in Example 2. Let $T_0 > 0$ and let $f : [0,T_0] \times \Omega \times \mathbb{R} \to \mathbb{R}$ such that $f(t,x,\cdot)$ is continuous for a.e. $(t,x) \in [0,T_0] \times \Omega$ and $f(\cdot,\cdot,u)$ is measurable for every $u \in \mathbb{R}$. Assume that there exist $b \in L^1(0,T_0;\mathbb{R})$, $c \in L^1(0,T_0;L^2(\Omega))$, $\beta \in L^\eta(0,T_0)$ with $1 < \eta < \infty$, $\gamma : [0,2T_0] \to [0,\infty)$ and $\delta \in L^2(\Omega)$ which satisfy*

(i) $|f(t,x,u)| \le b(t)|u| + c(t,x)$ *for a.e.* $(t,x) \in [0,T_0] \times \Omega$ *and for every* $u \in \mathbb{R}$,

(ii) $|f(t,x,u) - f(t,x,v)| \leq \beta(t)|u-v|$ *for a.e.* $(t,x) \in [0,T_0] \times \Omega$ *and for every* $(u,v) \in \mathbb{R} \times \mathbb{R}$,

(iii) $\lim_{h \downarrow 0} \gamma(h) = 0$, *and*

(iv) $|f(t,x,u) - f(s,x,u)| \leq \gamma(|t-s|)\,(|u| + \delta(x))$ *for a.e.* $(t,s,x) \in [0,T_0] \times [0,T_0] \times \Omega$ *and for every* $u \in \mathbb{R}$.

Then for each $u_0 \in L^2(\Omega)$, *there exists* $T > 0$ *such that* (5.7), (5.5) *and* (5.6) *have an integral solution on* $[0,T]$.

PROOF. Define A_H by the same way in the proof of Theorem 6. For $0 < T \leq T_0$ and $d > 0$, set $Z_{d,T}$ be the space $\{u \in L^\infty(0,T;L^2(\Omega)) : \operatorname{ess\,sup}_{0 \leq \tau \leq T} \|u(\tau)\| \leq d\}$ which is endowed with the $L^1(0,T;L^2(\Omega))$ topology. Define

$$F_T u(t)(x) = f(t,x,u(t,x)) \qquad \text{for } u \in L^\infty(0,T;L^2(\Omega)).$$

Let $0 < T \leq T_0$ and let $d > 0$. F_T is continuous from $Z_{d,T}$ into $L^1(0,T;L^2(\Omega))$. For $u \in L^\infty(0,T;L^2(\Omega))$, we write $f(t,u(s))(x)$ instead of $f(t,x,u(s,x))$. Let $u \in Z_{d,T_0}$. We get

$$\int_0^h \|f(t,u(t))\|\,dt \leq d \int_0^h |b(t)|\,dt + \int_0^h \|c(t)\|\,dt.$$

So (H5) is satisfied. Since

$$\begin{aligned}
&\int_0^{T-h} \|f(t+h,u(t+h)) - f(t,u(t))\|\,dt \\
&\leq \int_0^{T-h} \|f(t+h,u(t+h)) - f(t+h,u(t))\|\,dt + \int_0^{T-h} \|f(t+h,u(t)) - f(t,u(t))\|\,dt \\
&\leq \left(\int_0^T |\beta(t)|^\eta\,dt\right)^{\frac{1}{\eta}} \left(\int_0^{T-h} \|u(t+h) - u(t)\|^{\eta'}\,dt\right)^{\frac{1}{\eta'}} + \gamma(h)\,(dT + \|\delta\|),
\end{aligned}$$

(H6) is satisfied. Then applying Theorem 3, we can see that for each $u_0 \in L^2(\Omega)$, there exists $0 < T \leq T_0$ such that (5.7), (5.5) and (5.6) have an integral solution on $[0,T]$. □

References.

1. P. Baras, *Compacité de l'opérateur $f \mapsto u$ solution d'une équation non linéaire $(du/dt) + Au \ni f$*, C. r. hebd. Séanc. Acad. Sci. Paris **286**, 1113-1116 (1978).
2. V. Barbu, *Nonlinear semigroups and differential equations in Banach spaces,* Noordhoff, Leyden (1976).
3. P. Benilan, *Solutions integrales d'équations d'évolution dan un espace de Banach,* C. r. hebd. Séanc. Acad. Sci. Paris **274**, 47-50 (1972).
4. H. Brézis, *New results concerning monotone operators and nonlinear semigroups,* in Lecture Notes of the Research Institute for Mathematical Sciences, Kyoto Univ. **258**, 2-27 (1975).
5. M. G. Crandall & T. M. Liggett, *Generations of semigroups of nonlinear transformations in general Banach spaces,* Am. J. Math. **93**, 265-298 (1971).
6. M. G. Crandall & J. Nohel, *An abstract functional differential equation and a related Volterra equation,* Israel J. Math. **29**, 313-323 (1978).
7. J. I. Diaz & I. I. Vrabie, *Propriétés de compacité de l'opérateur de Green généralisé pour l'équation des milieux poreux,* C. r. hebd. Séanc. Acad. Sci. Paris **309**, 221-223 (1989).

8. S. Gutman, *Compact perturbations of m-accretive operators in general Banach spaces,* SIAM J. Math. Analysis **13**, 789-800 (1982).
9. S. Gutman, *Evolutions governed by m-accretive plus compact operators,* Nonlinear Analysis **7**, 707-715 (1983).
10. N. Hirano, *Local existence theorems for nonlinear differential equations,* SIAM J. Math. Analysis **14**, 117-125 (1983).
11. N. Kenmochi & T. Koyama, *Nonlinear functional variational inequalities governed by time-dependent subdifferentials,* Nonlinear Analysis **17**, 863-883 (1991).
12. S. M. Liu, *The existence of the solutions for a class of semilinear evolution equations,* Chinese Annals Math. **13**, 507-512 (1992).
13. E. Mitidieri & I. I. Vrabie, *A class of strongly nonlinear functional differential equations,* Annali Mat. Pura Appl. **151**, 125-147 (1988).
14. E. Mitidieri & I. I. Vrabie, *Differential inclusions governed by nonconvex perturbations of m-accretive operators,* Diff. Integr. Eqns, **2**, 525-531 (1989).
15. A. Pazy, *A class of semi-linear evolution equations of evolution,* Israel J. Math. **20**, 23-36 (1975).
16. I. I. Vrabie, *The nonlinear version of Pazy's local existence theorem,* Israel J. Math. **32**, 221-235 (1979).
17. Vrabie I. I., *Compactness methods for an abstract Volterra integrodifferential equation,* Nonlinear Analysis **5**, 355-371 (1981).
18. I. I. Vrabie, *An existence result for a class of nonlinear evolution equations in Banach spaces,* Nonlinear Analysis **6**, 711-722 (1982).
19. I. I. Vrabie, *Compactness methods for nonlinear evolutions,* Longman Scientific & Technical, England (1987).

FACULTY OF ENGINEERING, TAMAGAWA UNIVERSITY, TAMAGAWA GAKUEN, MACHIDA, TOKYO 194, JAPAN.