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Hausdorff Dimensions of Almost Periodic Attractors

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1. Introduction and notations

Landau proposed, as a possible mechanism for the onset of turbulance, the successive bifurcations along with increasing of a stress parameters of the system; the infinitely many bifurcated frequencies yield the destabilization of the fluid modes. Ruelle and Takens challanged Landau's conjecture in [4] by showing that there exist arbitrarily small changes of the system which convert the flow from a quasiperiodic three-frequency flow to chaotic one. They proved that only a few bifurcations can create a chaotic flow. After their scenario, on the problem of characterizing possible routes to chaos in nonlinear dynamical systems there have been reported several ways; infinite period-doubling cascades [3], intermittency and crises. On the other hand, R-T results have been questioned by Yorke et al., since the small perturbations necessary to make the flow chaotic may have to be very delicately chosen and hence may be unlikely to occur in practice. In [7] they showed numerically that one can typically observe three-frequency quasiperiodicity in experiments.

Anyway, since the specific mechanism from quasiperiodicity to chaos remains in doubt, one can admit some posibilities of finding an open or an outstanding route to chaos by detecting or analyzing properties of quasiperiodic states. In this paper we estimate fractal dimensions of equi-almost periodic attractors by using the order of ε-almost period and the coefficient of Hölder's continuity. Furthermore, we estimate the ε-almost period of quasiperiodic functions by using Diophantine approximations. In n-frequency case we can show that the fractal dimension of its orbit is majorized by the value n/δ where $\delta, 0 < \delta \leq 1$, is the coefficient of Hölder's continuity. As a result we can conjecture that the dimension of the orbit becomes divergent (chaotic?)if the constant $\delta \downarrow 0$.

Definitions and notations. Let $S(t), \quad t \geq 0$, be a semigroup of continuous (generally nonlinear) operators on a Banach space $(X, |\cdot|)$.

A subset A is called invariant for the semigroup $S(t)$ if

$$S(t)A = A \qquad \text{for every } t \geq 0. \tag{1.1}$$

Furthermore, when the operators $S(t)$ are one-to-one on A and its inverse is continuous, $S(t)$ can be defined for all $t \in \mathcal{R}$ by $S(-t) := S(t)^{-1}$ and

$$S(t)A = A \qquad \text{for every } t \in \mathcal{R}, \tag{1.2}$$

then A is called strongly invariant.

The orbits through $x \in X$ are denoted by

$$\gamma^+(x) = \cup_{t \in \mathcal{R}^+} S(t)x, \quad \gamma(x) = \cup_{t \in \mathcal{R}} S(t)x$$

and the ω-limit set of x is defined by

$$\omega(x) = \{y \in X : y = \lim_{n \to \infty} S(t_n)x \quad \text{with } t_n \to \infty \quad \text{as } n \to \infty\}.$$

A subset A is called minimal under $S(t)$ if

$$\overline{\gamma^+(y)} = A \quad \text{for every } y \in A.$$

A subset A is callled equi-almost periodic under $S(t)$ if it is strongly invariant and, for each $\varepsilon > 0$ there exists an ε-almost period (abr. ε-a.p.) $l_\varepsilon > 0$ such that every interval of its length l_ε in $\mathcal{R}$ contains a point α with the property

$$|S(t+\alpha)y - S(t)y| \leq \varepsilon \quad \text{for all } \ t \in \mathcal{R}, \ y \in A. \tag{1.3}$$

Furthermore, under the same notations as above we say a function $f(t) : \mathcal{R} \to X$ is almost periodic if

$$|f(t+\alpha) - f(t)| \leq \varepsilon \quad \text{for all } \ t \in \mathcal{R}. \tag{1.4}$$

An attractor is a set $A \subset X$ that enjoys the following properties:

(i) A is invariant, which satisefies (1.1).

(ii) A possesses an open neighborhood U such that, for every $u_0 \in U$, $S(t)u_0$ converges to A as $t \to \infty$:

$$d(S(t)u_0, A) \to 0 \quad \text{as } t \to \infty$$

where $d(x, A) = \inf\{|x - y| : y \in A\}$.

In [2] Dafermos and Slemrod proved the following information on ω-limit sets:

Let $S(t)$ be a continuous semigroup of contractions on a closed subset C of X. (The definitions of orbits and ω-limit sets are considered in the closed set C.) If for some $x \in C$ $\omega(x)$ is nonempty, then it is minimal and strongly invariant. For each $t \in \mathcal{R}$, $S(t)$ is an isometry on $\omega(x)$. Furthermore, if $\omega(x)$ is compact, then it is equi-almost periodic.

Note that

$$\omega(A) = \cap_{s \geq 0} \overline{\cup_{t \geq s} S(t) A}.$$

If $A \subset X$ is an attractor, $\omega(A) = \omega(U) = \overline{A}$, since (i) implies $\omega(A) = \overline{A}$ and (ii) gives $\omega(U) \subset \overline{A}$. On the contrary, under some suitable compact conditions on $S(t)$, it is known (cf. [6]) that the ω limit set of some absorbing set becomes an attractor.

Our plan of this paper is as follows: In section 2 we estimate fractal dimensions of equi-almost periodic sets by the order of ε-almost period and the constant δ and in section 3 we calculate the ε-almost period of the 2-frequency quasiperiodic orbits and in section 4 we treat the n-frequency case ($n \geq 3$).

2. Hausdorff and fractal dimensions

The purpose of this section is to estimate fractal dimensions of some equi-almost periodic subsets in X.

Let A be a subset of X, then the d-dimensional Hausdorff measure $\mathcal{M}_d(A)$ is defined by

$$\mathcal{M}_d(A) = \lim_{\varepsilon \downarrow 0} \inf\{\sum_i r_i^d : A \subset \cup_i B_i, \quad r_i = radi(B_i) \leq \varepsilon\}$$

where the infimum is for all covering of A by a family $\{B_i\}$ of balls of X with radii $r_i \leq \varepsilon$.

The Hausdorff dimension of A, denoted by $\mathcal{D}_H(A)$, is defined by

$$\mathcal{D}_H(A) = \sup\{d > 0 : \mathcal{M}_d(A) = \infty\}.$$

Let $N_A(\varepsilon)$, $\varepsilon > 0$, denote the minimum number of balls of X radius ε which is necessary to cover a subset A of X. The fractal dimension of A, which is also called the capacity of A, is the number

$$\mathcal{D}_F(A) = \limsup_{\varepsilon \to 0} \frac{\log N_A(\varepsilon)}{\log 1/\varepsilon}.$$

The following alternative expression for $\mathcal{D}_F(A)$ can be given by

$$\mathcal{D}_F(A) = \inf\{d > 0 : \mu_d(A) = 0\}$$

where

$$\mu_d(A) = \limsup_{\varepsilon \to 0} \varepsilon^d N_A(\varepsilon).$$

The difference between the Hausdorff and the fractal dimensions lies in the fact that we consider, in one case, the covering of A by balls of radius $\leq \varepsilon$ and in the other case the covering of A by balls of radius ε. It is clear that $\mathcal{M}_d(A) \leq \mu_d(A)$ and so that $\mathcal{D}_H(A) \leq \mathcal{D}_F(A)$.

First we consider the dimension of the orbit of an almost periodic function.

Theorem 1. *Let $f(t) : \mathcal{R} \to X$ be an almost periodic function, which satisfies a Hölder condition: there exists a constant $\delta : 0 < \delta \leq 1$ such that*

$$\sup_{t,s\in\mathcal{R},t\neq s} \frac{|f(t)-f(s)|}{|t-s|^{\delta}} := c_0 < \infty. \tag{2.1}$$

If the ε-almost period of the function $f(t)$ satisfies the following estimate

$$l_\varepsilon \leq K\varepsilon^{-\vartheta} \tag{2.2}$$

for some $K > 0$ and $\vartheta > 0$, then the fractal dimension of its orbit $\Sigma := \cup_{t\in\mathcal{R}} f(t)$ satisfies

$$\mathcal{D}_F(\Sigma) \leq \vartheta + \frac{1}{\delta}. \tag{2.3}$$

proof. Denote

$$F := \cup\{f(t) : 0 \leq t \leq l_\varepsilon\},$$

then we have

$$\Sigma \subset F + B_\varepsilon(0) = \cup\{B_\varepsilon(f(t)) : 0 \leq t \leq l_\varepsilon\} \tag{2.4}$$

where $B_\varepsilon(z) = \{x \in X : |x - z| \leq \varepsilon\}$. In fact, for every $t \in \mathcal{R}$ there exists $\tau \in [t - l_\varepsilon, t]$ with the property

$$\sup_{\sigma\in\mathcal{R}} |f(\sigma+\tau) - f(\sigma)| \leq \varepsilon$$

and it follows that

$$|f(t) - f(t-\tau)| \leq |f((t-\tau)+\tau) - f(t-\tau)| \leq \varepsilon.$$

Since $f(t-\tau) \in F$, the above estimate implies (2.4).

Take a small number $\rho := (\varepsilon/c_0)^{1/\delta}$ and consider a partition of the interval $[0, l_\varepsilon]$:

$$0 = t_0 < t_1 < t_2 < \cdots < t_n < l_\varepsilon,$$
$$|t_i - t_{i-1}| = \rho, i = 1, ..., n, \quad l_\varepsilon - t_n < \rho$$

where $n = [l_\varepsilon/\rho]$. Since for each $t \in [0, l_\varepsilon]$ there exists a number $i : t_i \le t < t_{i+1}$, which satisfies

$$|f(t) - f(t_i)| \le c_0|t - t_i|^\delta \le c_0\rho^\delta = \varepsilon,$$

we have

$$F \subset \cup_{i=0}^{n} B_\varepsilon(f(t_i)). \tag{2.5}$$

From (2.4) and (2.5) we can estimate

$$\mu_d(\Sigma) \le (\frac{l_\varepsilon}{\rho} + 1)(2\varepsilon)^d \le K'(\varepsilon^{-\vartheta-\frac{1}{\delta}} + 1)\varepsilon^d.$$

Thus, taking the limit $\varepsilon \downarrow 0$, we have

$$\mu_d(\Sigma) < \infty$$

if $d \ge \vartheta + 1/\delta$. It follows that

$$\mathcal{D}_F(\Sigma) \le \vartheta + \frac{1}{\delta}. \qquad \square$$

Remark.1 In Theorem 1 it may be possible that uniform δ-Hölder condition (2.1) is substituted by the following local δ-Hölder condition; each $t_0 \in \mathcal{R}$ has a neighborhood $I_0 := (t_0 - \varepsilon_0, t_0 + \varepsilon_0)$ such that $f(t)$ is uniformly Hölder continuous on I_0, that is,

$$\sup_{t,s \in I_0; t \ne s} \frac{|f(t) - f(s)|}{|t - s|^\delta} := c_0(t_0) < \infty$$

where c_0 depends on $t_0 \in \mathcal{R}$. For instance, if we assume that c_0 is increasing as $|t_0| \uparrow ; c_0(|t_0|) \simeq |t_0|^\gamma$, $\gamma > 0$, then c_0 has the order of $\varepsilon^{-\gamma\vartheta}$ in the interval $[0, l_\varepsilon]$. Following the argument of the proof (putting $\rho := \varepsilon^{(1+\gamma\vartheta)/\delta}$), we obtain

$$\mathcal{D}_F(\Sigma) \le \vartheta(1 + \frac{\gamma}{\delta}) + \frac{1}{\delta}.$$

Here, we should note that almost periodic functions must be uniformly continuous (cf. [1]). While we have not yet found the proof for the assertion that the local Hölder continuity of an a.p. function yields the uniform Hölder continuity, the assumption of the uniform Hölder continuity may be considerable in the almost periodic case.

Next theorem is easily obtained by applying Theorem 1 to Dafermos-Slemrod results.

Theorem 2. *Let $S(t)$ be a semigroup of cotractions on a Banach space X and assume that for some $x \in X$ $\omega(x)$ is nonempty and compact. If for some $y \in \omega(x)$, $S(\cdot)y : \mathcal{R} \to X$ is uniformly δ-Hölder continuous and furthermore, if the ε-almost period of the equi-almost periodic set $\omega(x)$ satisfies*

$$l_\varepsilon \leq K\varepsilon^{-\vartheta} \tag{2.6}$$

for some $K > 0$ and $\vartheta > 0$, then we have

$$\mathcal{D}_F(\omega(x)) \leq \vartheta + \frac{1}{\delta}. \tag{2.7}$$

3. ε-a.p. of 2-frequency quasiperiodic functions

Let $g(t,s) : \mathcal{R} \times \mathcal{R} \to X$ be a piecewise periodic function such that

$$g(t+1,s) = g(t,s), \quad g(t,s+1) = g(t,s), \quad t,s \in \mathcal{R}$$

and consider a quasiperiodic function $f(t) = g(wt,t)$ where we take an irrational real number $w : w > 1$, which is a technical condition, and put $\tau = 1/w$. We assume Hölder conditions of the function $g(\cdot,\cdot)$ for constants $\delta_i : 0 < \delta_i \leq 1, i = 1,2$:

$$\begin{aligned} &|g(t,s) - g(t',s)| \leq |t-t'|^{\delta_1}, \\ &|g(t,s) - g(t,s')| \leq |s-s'|^{\delta_2}, \quad t,t',s,s' \in \mathcal{R} \end{aligned} \tag{3.1}$$

where, for simple terminology, we consider the case $c_0 = 1$ without losing its generality. In this section, applying Diophantine approximation (cf. [5]), we estimate the ε-almost period of the quasiperiodic function $f(t)$.

Consider the following continued fraction of the number τ:

$$\tau = \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \ddots}}} \qquad (a_i \in \mathbf{N}) \tag{3.2}$$

and take the rational approximation as follows: Let $m_0 = 1, n_0 = 0, m_{-1} = 0, n_{-1} = 1$ and define the pair of sequences of natural numbers

$$\begin{aligned} m_i &= a_i m_{i-1} + m_{i-2}, \\ n_i &= a_i n_{i-1} + n_{i-2}, \quad i \geq 1, \end{aligned}$$

then the elementary number theory gives the Diophantine approximation

$$|\tau - \frac{n_i}{m_i}| < \frac{1}{m_i m_{i+1}} < \frac{1}{m_i^2}. \tag{3.3}$$

On the other hand, τ is called badly approximable if there exists a positive constant c:

$$|\tau - \frac{p}{q}| > (\frac{c}{q})^2, \quad (0 < c < \frac{1}{\sqrt{5}}) \tag{3.4}$$

hold for infintely many rationals p/q. It is also known (cf. [5]) that τ is badly approximable if and only if the sequence $\{a_n\}$, which is obtained by the continued fraction of τ, is bounded.

Theorem 3. *Under the notations introduced above, assume that there exists a constant $K_0 > 1$ such that*

$$m_i > K_0 m_{i-1}, \qquad i = 1, 2, \cdots\ , \tag{3.5}$$

then the ε-almost period of $f(t)$ satisfies

$$l_\varepsilon \le K\varepsilon^{-\frac{1}{\sigma}}, \quad K > 0 \tag{3.6}$$

where $\sigma := \max\{\delta_1, \delta_2\}$ and for the fractal dimension of the orbit Σ of $f(t)$ we can estimate

$$\mathcal{D}_F(\Sigma) \le \frac{1}{\sigma} + \frac{1}{\sigma'} \tag{3.7}$$

where $\sigma' := \min\{\delta_1, \delta_2\}$

proof. We will prove the following two inequalities; $l_\varepsilon \le K\varepsilon^{-\frac{1}{\delta_i}}, i = 1, 2$. First, we consider the case $i = 2$, but, for simplicity of terminology, we use $\delta := \delta_2$

By taking a sufficiently large number k we define a small constant

$$\varepsilon_k = \frac{1}{1 - K_0^{-\delta}} \cdot (\frac{1}{m_{k+1}})^\delta,$$

then our main subject of the proof is to show that we can take $l_{\varepsilon_k} = m_{k+1}\tau$. Then (3.6) with $\varepsilon = \varepsilon_k$ holds and, by defining

$$l_\varepsilon = l_{\varepsilon_k} \quad \text{for} \quad \varepsilon_{k+1} < \varepsilon \le \varepsilon_k$$

we can obtain

$$l_\varepsilon = l_{\varepsilon_k} < K\varepsilon_k^{-\frac{1}{\delta}} \le K\varepsilon^{-\frac{1}{\delta}}.$$

Thus, for an interval $[a - m_{k+1}\tau, a]$ with an arbitrarily real number a, it is sufficient to find an element in the interval with the property (1.4).

First we consider the case where $a \ge m_{k+1}\tau$. Hereafter in the proof, for simplicity of terminology, we reset the numbering of suffix and the notation as follows:

$$m_{k+j} \to m_j, \quad a_{k+j} \to a_j, \cdots, \quad \text{and} \quad \varepsilon_k \to \varepsilon.$$

Fix the number $i : m_i\tau \le a < m_{i+1}\tau$, then, by considering the estimate

$$m_j < (a_j + 1)m_{j-1},$$

we can take a sequence of nonnegative integers $\{k_j\}$, $1 \le j \le i$, which satisfies

$$0 \le k_j \le a_j \quad \text{for} \quad 1 \le j \le i-1, \quad 1 \le k_i \le a_i$$

and

$$\begin{aligned} & k_i m_i \tau + k_{i-1} m_{i-1} \tau + \cdots + k_1 m_1 \tau \\ & \quad \le a < k_i m_i \tau + k_{i-1} m_{i-1} \tau + \cdots + k_2 m_2 \tau + (k_1+1) m_1 \tau. \end{aligned}$$

Define

$$\begin{aligned} m(k) &:= k_i m_i + k_{i-1} m_{i-1} + \cdots + k_1 m_1, \\ n(k) &:= k_i n_i + k_{i-1} n_{i-1} + \cdots + k_1 n_1 \end{aligned}$$

and note that $m(k)\tau \in [a - m_1\tau, a]$, then by Hölder continuity of g and Diophantine approximation we can obtain the following sequence of estimates:

$$\begin{aligned} & |f(t+m(k)\tau) - f(t)| \\ & \quad \le \; |g(w(t+m(k)\tau), t+m(k)\tau) - g(wt,t)| \\ & \quad = \; |g(wt+m(k), t+m(k)\tau) - g(wt,t)| \\ & \quad \le \; |g(wt+m(k), t+m(k)\tau) - g(wt+m(k), t+n(k))| \\ & \qquad\qquad + |g(wt+m(k), t+n(k)) - g(wt,t)| \\ & \quad = \; |g(wt+m(k), t+m(k)\tau) - g(wt+m(k), t+n(k))| \\ & \quad \le \; |m(k)\tau - n(k)|^\delta \\ & \quad \le \; (k_i m_i)^\delta |\tau - \frac{n_i}{m_i}|^\delta + \cdots + (k_1 m_1)^\delta |\tau - \frac{n_1}{m_1}|^\delta \\ & \quad \le \; m_{i+1}^\delta |\tau - \frac{n_i}{m_i}|^\delta + \cdots + m_2^\delta |\tau - \frac{n_1}{m_1}|^\delta \\ & \quad \le \; (\frac{1}{m_i})^\delta + \cdots + (\frac{1}{m_1})^\delta \end{aligned}$$

where we use an elementary inequality $(x+y)^\delta \le x^\delta + y^\delta, \quad x, y \ge 0$. Thus Hypothesis (3.5) yields

$$\begin{aligned} |f(t+m(k)\tau) - f(t)| &\le (\frac{1}{m_1})^\delta (1 + K_0^{-\delta} + K_0^{-2\delta} + \cdots) \\ &< (\frac{1}{m_1})^\delta \frac{1}{1 - K_0^{-\delta}} = \varepsilon \end{aligned}$$

for every $t \in \mathcal{R}$. Therefore, we can find the point $m(k)\tau$ with the required property in any interval $[a, a + l_\varepsilon]$ for $a \geq 0$.

For the interval $[a - l_\varepsilon, a]$, $a < 0$, we can take the element $-m(k)\tau$, since

$$|f(t + m(k)\tau) - f(t)| \leq \varepsilon \quad \text{for every} \quad t \in \mathcal{R}$$

yields

$$|f(t') - f(t' - m(k)\tau)| \leq \varepsilon \quad \text{for every} \quad t' = t + m(k)\tau \in \mathcal{R}$$

and $m(k)\tau \in [a', a' + l_\varepsilon]$, $a' > 0$, is equivalent to $-m(k)\tau \in [-a' - l_\varepsilon, -a']$. Finally, for the interval $[a, a + l_\varepsilon]$, $-l_\varepsilon < a < 0$, the null point plays its role for the property (1.4).

Next we treat the case $i = 1$, substituting the role of $m(k)\tau$ by that of $n(k)$. We use $\delta := \delta_1$ and the resetting notations for simplicity. Let

$$\varepsilon' := \frac{w^\delta}{1 - K_0^{-\delta}} \cdot (\frac{1}{m_1})^\delta$$

and consider the interval $[a - m_1\tau - \varepsilon_1, a + \varepsilon_1]$ where

$$\varepsilon_1 := \frac{1}{m_1(1 - K_0^{-1})},$$

then $n(k)$ is in this interval, since $|m(k)\tau - n(k)| < \varepsilon_1$ (cf. the estimation above) and $m(k)\tau \in [a - m_1\tau, a]$. We can show that the element $n(k)$ satisfies the property (1.4) by using the argument in the previous case $i = 2$ and applying the following sequence of inequalities.

$$\begin{aligned}
&|f(t + n(k)) - f(t)| \\
&\quad = |g(w(t + n(k)), t + n(k)) - g(wt, t)| \\
&\quad \leq |g(wt + wn(k), t + n(k)) - g(wt + wm(k)\tau, t + n(k))| \\
&\qquad\qquad + |g(wt + m(k), t + n(k)) - g(wt, t)| \\
&\quad = |g(wt + wn(k), t + n(k)) - g(wt + wm(k)\tau, t + n(k))| \\
&\quad \leq w^\delta |m(k)\tau - n(k)|^\delta \\
&\quad \leq w^\delta((\frac{1}{m_i})^\delta + \cdots + (\frac{1}{m_1})^\delta) \leq \varepsilon'.
\end{aligned}$$

Thus we can get the estimate

$$l_\varepsilon \leq K \min\{\varepsilon^{-\frac{1}{\delta_1}}, \varepsilon^{-\frac{1}{\delta_2}}\} = K\varepsilon^{-\frac{1}{\sigma}}, \quad \sigma = \max_i \delta_i.$$

To complete the proof by applying Theorem 1, it is sufficient to check the following Hölder condition.

$$\begin{aligned} |f(t) - f(s)| &= |g(wt, t) - g(ws, s)| \\ &\leq |g(wt, t) - g(wt, s)| + |g(wt, s) - g(ws, s)| \\ &\leq |t - s|^{\delta_2} + w^{\delta_1}|t - s|^{\delta_1} \\ &\leq (1 + w^{\delta_1})|t - s|^{\sigma'}, \quad \sigma' := \min_i \delta_i \end{aligned}$$

where we consider the case $|t - s| << 1$, which is sufficient to apply the proof of Theorem 1. $\square$

Remark 2. The assumption (3.5) is not so restrictive. For instance, it is satisfied if the irrational number τ is badly approximable. In fact, $\sup_j a_j \leq K$ implies that

$$\begin{aligned} m_j &= a_j m_{j-1} + m_{j-2} \\ &> (a_j + \frac{1}{a_{j-1} + 1}) m_{j-1} \\ &> (a_j + \frac{1}{K + 1}) m_{j-1} \geq (1 + \frac{1}{K + 1}) m_{j-1}. \end{aligned}$$

The following is the typical example of inappropriate cases.

$$a_j = \begin{cases} 1 & j = 2m \\ a_{2m+1} \to \infty & j = 2m + 1 \to \infty. \end{cases}$$

4. ε-a.p. of n-frequency quasiperiodic functions ($n \geq 3$)

Next we estimate the ε-almost period of n-frequency ($n \geq 3$) quasiperiodic functions by using the simultaneous approximation (cf. [5]):

Suppose that at least one of $\alpha_1, \alpha_2, \cdots, \alpha_n$ is irrational. Then there are infinitely many n-tuples of rational numbers

$$\frac{p_1}{q}, \cdots, \frac{p_n}{q}: \quad |\alpha_i - \frac{p_i}{q}| < \frac{1}{q^{1+1/n}} \quad (i = 1, \cdots, n). \tag{4.1}$$

Consider piecewisely Hölder continuous and 1-periodic function $g(\cdot, \cdots, \cdot) : \mathcal{R}^n \to X$ and, for simplicity, we assume that there exists a common constant $0 < \delta \leq 1$ such that

$$|g(t_1, t_2, \cdots, t_n) - g(t_1', t_2, \cdots, t_n)| \leq |t_1 - t_1'|^\delta, \quad t_1, t_1', t_i, i = 2, \cdots, n \in \mathcal{R}$$

and the same estimate holds for each of the other variables with the same constant δ. For periodicity we assume

$$\begin{aligned} g(t_1 + 1, t_2, \cdots, t_n) &= g(t_1, t_2 + 1, t_3, \cdots) \\ = \cdots &= g(t_1, t_2, \cdots, t_n + 1) = g(t_1, \cdots, t_n). \end{aligned}$$

For a given $\{n - 1\}$ tuples of irrational numbers $w_1, w_2, \cdots, w_{n-1}$, which are rationally independent and greater than 1, we define a quasiperiodic function by

$$f(t) = g(w_1 t, \cdots, w_{n-1} t, t) : \mathcal{R} \to X.$$

Our subject in this section is to estimate the ε-a.p. of $f(t)$.

It follows from the simultaneous approximation (4.1)(case $n := n - 1$) that there exist sequences $l_i, r_{k,i} \in \mathbf{N}, i = 1, 2, \cdots$, and $k = 1, \cdots, n - 1$, which satisfy

$$|w_k - \frac{r_{k,i}}{l_i}| < \frac{1}{l_i^{1+1/(n-1)}}, \quad k = 1, \cdots, n - 1.$$

Let $\tau_k = 1/w_k < 1$, then we have

$$|\tau_k r_{k,i} - l_i| < (\frac{1}{l_i})^{1/(n-1)}, \quad k = 1, \cdots, n - 1,$$

and it follows that

$$|\sum_{k=1}^{n-1} r_{k,i}\tau_k - (n-1)l_i| < (n-1)(\frac{1}{l_i})^{1/(n-1)}, \tag{4.2}$$

$$|\frac{1}{n-2}\sum_{k\neq j} r_{k,i}\tau_k - r_{j,i}\tau_j| < 2(\frac{1}{l_i})^{1/(n-1)}, \quad j = 1, \cdots, n-1. \tag{4.3}$$

Theorem 4. *Under the notations introduced above, assume that there exist constants $K_1, K_2 > 0$:*

$$K_1 l_{j-1} < l_j < K_2 l_{j-1} \quad \textit{for } j = 1, 2, \cdots \quad , \tag{4.4}$$

then the ε-almost period of $f(t)$ satisfies

$$l_\varepsilon \leq K' \varepsilon^{-\frac{n-1}{\delta}} \tag{4.5}$$

for some constant $K' > 0$ and consequently, for the orbit Σ of $f(t)$ we can estimate

$$\mathcal{D}_F(\Sigma) \leq \frac{n}{\delta}. \tag{4.6}$$

proof. For a sufficiently large i_0, define a small constant

$$\varepsilon := K_\delta l_{i_0+1}^{-\delta/(n-1)}$$

where

$$K_\delta = \frac{K_2^\delta}{1 - K_1^{-\delta/(n-1)}}\{2^\delta (n-2)^\delta (w_1^\delta + \cdots + w_{n-1}^\delta) + (n-1)^\delta\}, \tag{4.7}$$

then by proving $l_\varepsilon \simeq l_{i_0+1}$ we can obtain (4.5).

Hereafter in the proof we reset the numbering and define the notations as follows:

$$l_{i_0+j} \to l_j, \quad k_{i_0+j} \to k_j \quad \text{and so on,}$$
$$L_i := (n-1)l_i,$$
$$\varepsilon(l_1) := \frac{(n-1)K_2}{1 - K_1^{-1/(n-1)}}(\frac{1}{l_1})^{\frac{1}{n-1}},$$

then we take a real number $a > L_1+\varepsilon(l_1)$. Following the argument in the proof of Theorem 3 and considering an interval $[a-L_1-\varepsilon(l_1), a+\varepsilon(l_1)]$, we aim to search a point in this interval with the property (1.4).

We can find a number $i \geq 1$:

$$L_i \leq a < L_{i+1} \quad (\text{, which means } \; L_{i_0+i} \leq a < L_{i_0+i+1}).$$

By Hypothesis we can find a sequence $\{k_j\}, 1 \leq j \leq i$:

$$0 \leq k_j \leq K_2 \;\text{ for }\; 1 \leq j \leq i-1, \quad 1 \leq k_i \leq K_2$$

with the following property

$$k_1L_1 + \cdots + k_iL_i \leq a < (k_1+1)L_1 + k_2L_2 + \cdots + k_iL_i.$$

Put

$$\begin{aligned} s(i) &:= \sum_{j=1}^{i}\sum_{k=1}^{n} \tau_k r_{k,j} k_j, \\ n(i) &:= k_1L_1 + \cdots + k_iL_i, \end{aligned}$$

then from (4.2) we have

$$\begin{aligned} |s(i)-n(i)| &\leq \sum_{j=1}^{i} \Big|\sum_{k=1}^{n-1} \tau_k r_{k,j} - L_j\Big| k_j \\ &\leq \sum_{j=1}^{i} (n-1)\Big(\frac{1}{l_j}\Big)^{1/(n-1)} K_2 \\ &\leq \frac{(n-1)K_2}{1-K_1^{-1/(n-1)}}\Big(\frac{1}{l_1}\Big)^{\frac{1}{n-1}} = \varepsilon(l_1). \end{aligned}$$

It follows that

$$s(i) \in [a-L_1-\varepsilon(l_1), a+\varepsilon(l_1)],$$

since $n(i) \in [a-L_1, a]$. By using Hölder continuity and periodicity of g and applying approximation (4.3) we obtain the following sequence of estimates.

$$
\begin{aligned}
&|f(t+s(i))-f(t)| \\
\leq\ & |g(w_1(t+s(i)),\cdots,w_{n-1}(t+s(i)),t+s(i))-g(w_1t,\cdots,w_{n-1}t,t)| \\
=\ & |g(w_1t+w_1\sum_j\sum_{k=2}^{n-1}\tau_k r_{k,j}k_j,\, w_2t+w_2\sum_j\sum_{k\neq 2}\tau_k r_{k,j}k_j,\cdots,t+s(i)) \\
&\quad -g(w_1t,\cdots,w_{n-1}t,t)| \\
\leq\ & |g(w_1t+w_1\sum_j\sum_{k=2}^{n-1}\tau_k r_{k,j}k_j,\, w_2t+w_2\sum_j\sum_{k\neq 2}\tau_k r_{k,j}k_j,\cdots,t+s(i)) \\
&\quad -g(w_1t+(n-2)w_1\tau_1\sum_{j=1}^{i}r_{1,j}k_j,\, w_2t+w_2\sum_j\sum_{k\neq 2}\tau_k r_{k,j}k_j,\cdots,t+s(i))| \\
&\quad +|g(w_1t,w_2t+w_2\sum_j\sum_{k\neq 2}\tau_k r_{k,j}k_j,\cdots,t+s(i)) \\
&\quad -g(w_1t,w_2t+(n-2)w_2\tau_2\sum_{j=1}^{i}r_{2,j}k_j,\cdots,t+s(i))| \\
&\quad +|g(w_1t,w_2t,w_3t+w_3\sum_j\sum_{k\neq 3}\tau_k r_{k,j}k_j,\cdots,t+s(i)) \\
&\quad -g(w_1t,w_2t,w_3t+(n-2)w_3\tau_3\sum_{j=1}^{i}r_{3,j}k_j,\cdots,t+s(i))| \\
&\quad \vdots \\
&\quad +|g(w_1t,w_2t,\cdots,w_{n-1}t,t+s(i)))-g(w_1t,\cdots,w_{n-1}t,t)| \\
\leq\ & w_1^\delta\sum_{j=1}^{i}|\sum_{k\neq 1}\tau_k r_{k,j}-(n-2)\tau_1 r_{1,j}|^\delta k_j^\delta+w_2^\delta\sum_{j=1}^{i}|\sum_{k\neq 2}\tau_k r_{k,j}-(n-2)\tau_2 r_{2,j}|^\delta k_j^\delta \\
&\quad +\cdots+|s(i)-n(i)|^\delta \\
\leq\ & \{K_2w_1 2(n-2)\}^\delta\sum_j(\frac{1}{l_j})^{\delta/(n-1)}+\cdots \\
&\quad \cdots+\{K_2w_{n-1}2(n-2)\}^\delta\sum_j(\frac{1}{l_j})^{\delta/(n-1)}+\varepsilon(l_1)^\delta \\
\leq\ & (w_1^\delta+\cdots+w_{n-1}^\delta)\frac{\{2K_2(n-2)\}^\delta}{1-K_1^{-\delta/(n-1)}}(\frac{1}{l_1})^{\delta/(n-1)} \\
&\quad +\frac{\{K_2(n-1)\}^\delta}{(1-K_1^{-1/(n-1)})^\delta}(\frac{1}{l_1})^{\delta/(n-1)} \\
\leq\ & K_\delta l_1^{-\delta/(n-1)}=\varepsilon
\end{aligned}
$$

for every $t \in \mathcal{R}$ where K_δ is given by (4.7) and we use an elementary inequality

$$\left(\frac{1}{1-x}\right)^\alpha \leq \frac{1}{1-x^\alpha}, \quad 0 < x < 1, \quad 0 \leq \alpha \leq 1.$$

Thus we can admit the length of the interval $[a - L_1 - \varepsilon(l_1), a + \varepsilon(l_1)]$ as ε-almost period:

$$l_\varepsilon \simeq l_1 (= l_{i_0+1}) \leq K'\varepsilon^{-\frac{n-1}{\delta}}$$

for some constant $K' > 0$. Since we can apply the same argument as that in Theorem 3 for any interval with its length l_ε, we complete the proof. $\square$

Remark 3. If the function $g(\cdot, \cdots, \cdot)$ has various orders in Hölder conditions, say, $\delta_1, \delta_2, \cdots$, then the conclusion of Theorem 4 holds by putting

$$\delta := \min\{\delta_1, \cdots, \delta_n\}.$$

In fact, since the above argument gives

$$M_1 l_1^{-\frac{\delta_1}{n-1}} + M_2 l_1^{-\frac{\delta_2}{n-1}} + \cdots + M_n l_1^{-\frac{\delta_n}{n-1}} := \varepsilon,$$

we have $\varepsilon \leq K l_1^{-\delta/(n-1)}$.

Remark 4. Hypothesis (4.4) corresponds to the assertion in the 2-frequency case that the irrational number τ is badly approximable, but we have not yet found the corresponding property in the simultaneous approximations.

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