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Periodic behavior of solutions to a continuous casting problem

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1. Introduction

In this paper we consider a continuous casting problem

$$
(P)^{\nu}\qquad \begin{cases} \partial_t\eta+\nu\partial_z\eta-\Delta\theta=0 & \text{in } Q_\infty:=]0,\infty[\times\Omega,\\ \eta\in\beta(\theta) & \text{in } Q_\infty,\\ \dfrac{\partial\theta}{\partial n}+g(t,x,\theta)=0 & \text{on } \Sigma_\infty^N:=]0,\infty[\times\Gamma_N,\\ \theta=M & \text{on } \Sigma_\infty^0:=]0,\infty[\times\Gamma_0,\\ \theta=-m & \text{on } \Sigma_\infty^L:=]0,\infty[\times\Gamma_L, \end{cases}
$$

under periodic (in time) boundary condition

$$
g(t+T,x,\theta)=g(t,x,\theta)\quad \text{on } \Sigma_\infty^N\times\mathbf{R},
$$

for a given period $T>0$. Here $\Omega=]-l,l[\times]0,L[$, $\Gamma_N=\{l,-l\}\times]0,L[$, $\Gamma_0=]-l,l[\times\{0\}$, $\Gamma_L=]-l,l[\times\{L\}$, L, $l>0$, $x=(y,z)$; ν, m and M are given constants with $\nu\geq 0$ and m, $M>0$; β is a maximal monotone graph of the form

$$
\beta(r)=\begin{cases} \lambda+\displaystyle\int_0^r b(\tau)d\tau & \text{if } r>0,\\ [0,\lambda] & \text{if } r=0,\\ \displaystyle\int_0^r b(\tau)d\tau & \text{if } r<0, \end{cases}
$$

for a given constant $\lambda>0$ and a locally bounded measurable function b such that

$$
(1.1)\qquad b(r)\geq b_*>0\quad \text{for a.e. } r\in\mathbf{R}.
$$

Furthermore $g=g(t,x,\theta)$ is a given function on $\mathbf{R}_+\times\Gamma_N\times\mathbf{R}$ such that

(g1) $g(t,x,\cdot)$ is a nondecreasing function for a.e. $(t,x)\in\mathbf{R}_+\times\Gamma_N$;

(g2) $g(\cdot,\cdot,\theta) \in L^2_{loc}(\mathbf{R}_+; L^2(\Gamma_N))$ for all $\theta \in \mathbf{R}$;

(g3) For any $K > 0$ there is a constant $C_g(K) > 0$ such that

$$|g(t,x,\theta_1) - g(t,x,\theta_2)| \leq C_g(K)\,|\theta_1 - \theta_2|$$

for all θ_1, $\theta_2 \in [-K, K]$ and a.e. $(t,x) \in \mathbf{R}_+ \times \Gamma_N$;

(g4) There exist constants K_1, $K_2 > 0$ such that

$$g(t,x,-K_1) \leq 0, \quad g(t,x,K_2) \geq 0 \quad \text{for a.e. } (t,x) \in \mathbf{R}_+ \times \Gamma_N.$$

For details of continuous casting problems, see Rodrigues [5], Rodrigues-Yi [6], Yi [9] and the literatures in their references. We remark here that problem $(P)^0$ is a Stefan problem. For results to periodic solutions of Stefan problems we refer to Aiki *et al.* [1], Damlamian-Kenmochi [2] and Haraux-Kenmochi [3]. In the following chapters, we shall discuss problem $(P)^\nu$ due to Shinoda [7,8].

2. Main results

Throughout this paper we denote $Q_S =]0,S[\times\Omega$, $\Sigma_S^N =]0,S[\times\Gamma_N$, etc. for $S \in]0,+\infty]$. Now let us give a notion of a weak solution on an interval of the form $[0,S]$ or $[0,+\infty[$.

Definition 2.1. Let S be a positive number. Then a couple $(\theta,\eta) \in L^2(0,S;H^1(\Omega)) \times L^\infty(Q_S)$ is called a weak solution of $(P)^\nu$ on $[0,S]$ when the following four conditions are satisfied:

(w1) $\eta \in C_w([0,S];L^2(\Omega))$, that is, η is a weakly continuous function from $[0,S]$ to $L^2(\Omega)$;

(w2) $\theta = M$ a.e. on Σ_S^0 and $\theta = -m$ a.e. on Σ_S^L;

(w3) $\eta \in \beta(\theta)$ a.e. in Q_S;

(w4) for any $\varphi \in W_S := \{\varphi \in H^1(Q_S); \varphi(S,\cdot) = 0$ a.e. in Ω, $\varphi = 0$ a.e. on $\Sigma_S^D\}$,

$$-\int_{Q_S} \eta(\partial_t \varphi + \nu \partial_z \varphi) dx dt + \int_{Q_S} \nabla\theta \nabla\varphi dx dt + \int_{\Sigma_S^N} g(\cdot,\cdot,\theta)\varphi d\Gamma dt = \int_\Omega \eta(0,\cdot)\varphi(0,\cdot) dx,$$

where $\Sigma_S^D =]0,S[\times\Gamma_D$, $\Gamma_D = \Gamma_0 \cup \Gamma_L$. In the case when $S = +\infty$, (θ,η) is called a weak solution of $(P)^\nu$ on $\mathbf{R}_+$, if (θ,η) is a weak solution of $(P)^\nu$ on $[0,S]$ for any finite $S > 0$.

Definition 2.2. Let $0 < S \le +\infty$ and let (θ_0,η_0) be a pair of functions in $L^\infty(\Omega)$ satisfying $\eta_0 \in \beta(\theta_0)$ a.e. in Ω. Then we call a pair (θ,η) a weak solution for $CP(\theta_0,\eta_0)^\nu$ on $[0,S]$ ($\mathbf{R}_+$ if $S = +\infty$) if (θ,η) is a weak solution of $(P)^\nu$ on $[0,S]$ and the initial conditions $\theta(0,\cdot) = \theta_0$ and $\eta(0,\cdot) = \eta_0$ are satisfied, respectively.

Concerning the existence and the uniqueness results for $CP(\theta_0,\eta_0)^\nu$, we quote them from Rodrigues-Yi [6]. The first proposition assures the existence of a weak solution for $CP(\theta_0,\eta_0)^\nu$.

Proposition 2.1. (cf. [6;theorem 1]) *Let $(\theta_0,\eta_0) \in (L^\infty(\Omega))^2$ be any pair of functions such that $\eta_0 \in \beta(\theta_0)$ a.e. in Ω. Choose two positive constants $\widetilde{K}_1$ and $\widetilde{K}_2$ so that $\widetilde{K}_i \ge \max\{m, M, K_i\}$, $i = 1,2$, and that*

$$\beta(-\widetilde{K}_1) \le \eta_0(x) \le \beta(\widetilde{K}_2) \quad \text{for a.e. } x \in \Omega.$$

Then, there exists at least one weak solution (θ,η) for $CP(\theta_0,\eta_0)^\nu$ on $\mathbf{R}_+$ such that

$$\beta(-\widetilde{K}_1) \le \eta(t,x) \le \beta(\widetilde{K}_2) \quad \text{for a.e. } (t,x) \in Q_\infty,$$

hence

$$-\widetilde{K}_1 \le \theta(t,x) \le \widetilde{K}_2 \quad \text{for a.e. } (t,x) \in Q_\infty.$$

Remark 2.1. In view of the proof of [6;theorem 1] we may assume that the solution (θ, η) obtained in proposition 2.1 is constructed as a limit of an approximate solution $(\theta_\varepsilon, \beta_\varepsilon(\theta_\varepsilon))$ of

$$\begin{cases} \partial_t \beta_\varepsilon(\theta_\varepsilon) + \nu \partial_z \beta_\varepsilon(\theta_\varepsilon) - \Delta\theta_\varepsilon = 0 & \text{in } Q_\infty, \\ \dfrac{\partial \theta_\varepsilon}{\partial n} + g_\varepsilon(t, x, \theta_\varepsilon) = 0 & \text{on } \Sigma_\infty^N, \\ \theta_\varepsilon = M & \text{on } \Sigma_\infty^0, \\ \theta_\varepsilon = -m & \text{on } \Sigma_\infty^L, \\ \theta_\varepsilon(0, \cdot) = \theta_{0\varepsilon} & \text{in } \Omega, \end{cases}$$

in the sense that for some subsequence $\{\varepsilon_n\}$ of $\{\varepsilon\}$

$$\beta_{\varepsilon_n}(\theta_{\varepsilon_n}) \to \eta \quad \text{weakly* in } L^\infty_{loc}(\mathbf{R}_+; L^\infty(\Omega)); \tag{2.1}$$

$$\theta_{\varepsilon_n} \to \theta \quad \text{weakly in } L^2_{loc}(\mathbf{R}_+; H^1(\Omega)) \cap H^1_{loc}(Q_\infty); \tag{2.2}$$

$$g_{\varepsilon_n}(\cdot, \cdot, \theta_{\varepsilon_n}) \to g(\cdot, \cdot, \theta) \quad \text{in } L^2_{loc}(\mathbf{R}_+; L^2(\Gamma_N)). \tag{2.3}$$

Here $\{\beta_\varepsilon\}$, $\{g_\varepsilon\}$ and $\{\theta_{0\varepsilon}\}$ are smooth approximations to β, g and θ_0, respectively. Furthermore, $\{\beta_\varepsilon\}$ satisfies (1.1) with $b_\varepsilon = \beta'_\varepsilon$, $\beta_\varepsilon(0) = 0$, $\beta'_\varepsilon \le 1/\varepsilon$ and

$$\beta_\varepsilon(r) \to \beta(r) \quad \text{for any compact interval in } \mathbf{R}\backslash\{0\} \text{ as } \varepsilon \to 0;$$

$\{g_\varepsilon\}$ satisfies (g1)$\sim$(g4) and

$$g_\varepsilon(\cdot, \cdot, \theta) \to g(\cdot, \cdot, \theta) \quad \text{in } L^2_{loc}(\mathbf{R}_+; L^2(\Gamma_N))$$

uniformly with respect to θ on any compact set in $\mathbf{R}$ as $\varepsilon \to 0$;

$\{\theta_{0\varepsilon}\}$ satisfies the compatibility conditions

$$\theta_{0\varepsilon} = M \quad \text{on } \Gamma_0 \quad \text{and} \quad \theta_{0\varepsilon} = -m \quad \text{on } \Gamma_L \tag{2.4}$$

and

$$\beta_\varepsilon(\theta_{0\varepsilon}) \to \eta_0 \quad \text{in } L^2(\Omega) \text{ as } \varepsilon \to 0.$$

The second proposition is the continuous dependence of the weak solutions. This requires the following condition to a weak solution (θ, η) of $(P)^\nu$:

For some positive constants δ, $\rho > 0$,

$$\theta(t, y, z) \geq \rho > 0 \quad \text{a.e. in } Q_\infty^\delta := \{(t, y, z) \in Q_\infty; 0 < z < \delta\}. \tag{2.5}$$

Proposition 2.2. (cf. [6;theorem 2]) *Fix $\nu > 0$. Let (θ_1, η_1) and (θ_2, η_2) be two weak solutions for $CP(\theta_{10}, \eta_{10})^\nu$ and $CP(\theta_{20}, \eta_{20})^\nu$, respectively. If at least one of (θ_i, η_i) satisfies (2.5), then the following is valid:*

$$\int_{Q_\infty} |\eta_1 - \eta_2| \, dxdt \leq \frac{L}{\nu} \int_\Omega |\eta_{10} - \eta_{20}| \, dx. \tag{2.6}$$

As a direct corollary we have:

Corollary 2.1. *If at least one of the weak solution (θ, η) for $CP(\theta_0, \eta_0)^\nu$ on $\mathbf{R}_+$ satisfies (2.5), then (θ, η) is the only weak solution for $CP(\theta_0, \eta_0)^\nu$ on $\mathbf{R}_+$.*

Using well-known L^1-space technique, we have in the manner similar to that of [1]:

Proposition 2.3. *Let $\nu > 0$, and let (θ_1, η_1), (θ_2, η_2) be two weak solutions for $CP(\theta_{10}, \eta_{10})^\nu$ and $CP(\theta_{20}, \eta_{20})^\nu$ on $\mathbf{R}_+$ satisfying (2.5), respectively. Then we have*

$$\left|[\eta_1(t,\cdot) - \eta_2(t,\cdot)]^+\right|_{L^1(\Omega)} \leq \left|[\eta_1(s,\cdot) - \eta_2(s,\cdot)]^+\right|_{L^1(\Omega)} \quad \textit{for any } s,\ t \in \mathbf{R}_+ \textit{ with } s \leq t,$$

and

$$|\eta_1(t,\cdot) - \eta_2(t,\cdot)|_{L^1(\Omega)} \leq |\eta_1(s,\cdot) - \eta_2(s,\cdot)|_{L^1(\Omega)} \quad \textit{for any } s,\ t \in \mathbf{R}_+ \textit{ with } s \leq t.$$

In particular, if $\eta_{10} \leq \eta_{20}$ a.e. *in* Ω *then*

$$\eta_1 \leq \eta_2 \quad \textit{hence} \quad \theta_1 \leq \theta_2 \quad \text{a.e. } \textit{in } Q_\infty.$$

Remark 2.2. Propositions 2.1, 2.3 and corollary 2.1 are also valid for $\nu = 0$. We can prove them by using similar techniques to those in the proofs of [6;theorem 1,4;theorem 4.2,1;lemma 2.1], respectively.

Next we state a definition of a T-periodic weak solution of $(P)^\nu$ on $\mathbf{R}_+$.

Definition 2.3. Let T be a given positive number (period). Then (θ, η) is called a T-periodic weak solution of $(P)^\nu$ on $\mathbf{R}_+$ provided that (θ, η) is a weak solution of $(P)^\nu$ on $\mathbf{R}_+$ and satisfies the periodic conditions $\theta(t+T, \cdot) = \theta(t, \cdot)$ and $\eta(t+T, \cdot) = \eta(t, \cdot)$ for all $t \in \mathbf{R}_+$.

Finally we mention the main results for the T-periodic weak solution of $(P)^\nu$ on $\mathbf{R}_+$.

Theorem 2.1. *Let* $\nu > 0$. *Assume that the periodicity condition*

$$(2.7) \qquad g(t+T, x, \theta) = g(t, x, \theta) \quad \textit{for all } \theta \in \mathbf{R}_+ \textit{ and } \text{a.e. } (t, x) \in \mathbf{R}_+ \times \Gamma_N$$

holds. Then there exists one and only one T*-periodic weak solution* $(\theta_p^\nu, \eta_p^\nu)$ *of* $(P)^\nu$ *on* $\mathbf{R}_+$.

Theorem 2.2. *Assume that the same conditions as in theorem 2.1 hold. Then for any weak solution* (θ, η) *satisfying (2.5) for some positive constants* δ, $\rho > 0$, *we have*

$$\eta_p^\nu(t, \cdot) - \eta(t, \cdot) \to 0 \quad \textit{and} \quad \theta_p^\nu(t, \cdot) - \theta(t, \cdot) \to 0 \quad \textit{in } L^q(\Omega) \textit{ for all } q \geq 1 \textit{ as } t \to +\infty.$$

Remark 2.3. Yi [9] treated the periodic solutions under the Dirichlet boundary condition. He proved there the existence of periodic solutions using Schauder fixed point theorem.

Remark 2.4. There exists a T-periodic weak solution (θ_p^0, η_p^0) of $(P)^0$ on $\mathbf{R}_+$ under the periodicity condition (2.7). But for the uniqueness of T-periodic weak solutions of $(P)^0$ on $\mathbf{R}_+$, we can only prove that of $g(\cdot,\cdot,\theta_p^0)$ on Σ_∞^N and moreover that of θ_p^0 in Q_∞ (see [7,8] and also [2]).

3. Lemmas

In this chapter we prepare some lemmas to prove theorems 2.1 and 2.2.

Firstly we define a function $g_* = g_*(\theta)$ by $g_*(\theta) = C_g(K_3)[\theta + K_3]^+$ for $\theta \in \mathbf{R}$, where $K_3 = \max\{M, m, K_1, K_2\}$. Then the following is valid.

Lemma 3.1. *g_* defined as above is nondecreasing and satisfies*

$$g(t,x,\theta) \le g_*(\theta) \quad \textit{for all } \theta \le K_3 \textit{ and a.e. } (t,x) \in \mathbf{R}_+ \times \Gamma_N,$$

Next we construct a smooth function $\theta_* = \theta_*(x)$ satisfying for any $\varepsilon > 0$ the following system

$$\begin{cases} \nu\partial_z\beta_\varepsilon(\theta_*) - \Delta\theta_* \le 0 & \text{in } \Omega, \\ \dfrac{\partial\theta_*}{\partial n} + g_*(\theta_*) \le 0 & \text{on } \Gamma_N, \\ \theta_* \le M & \text{in } \overline{\Omega}, \\ \theta_* \le -K_3 & \text{on } \Gamma_L. \end{cases} \tag{3.1}$$

Choose a function $\chi = \chi(y) \in C^\infty([-l, l])$ such that

$$0 \le \chi \le \frac{M}{2} \quad \text{in }]-l, l[,$$

and

$$\pm\frac{\partial\chi}{\partial y}(\pm l) + g_*(M) \le 0.$$

For a positive parameter μ, let us define θ_* by

$$\theta_*(y,z) = -\mu z + \chi(y) + \frac{M}{2}.$$

Then we see that θ_* satisfies for some constants δ, $\rho > 0$

$$\theta_*(y,z) \geq \rho \quad \text{in } \Omega_\delta := \{(y,z) \in \Omega; 0 < z < \delta\}. \tag{3.2}$$

Moreover it is readily seen that (3.1) is fulfilled for sufficiently large μ dependent upon ν. Thus we have the following lemma.

Lemma 3.2. *There is a smooth function $\theta_* = \theta_*(x)$ on Ω which is independent of ε and satisfies (3.1) and (3.2) for some positive constants δ, ρ.*

Put $\eta_* = \beta(\theta_*)$. We remark that η_* is a.e. defined since the Lebesgue measure of the set $\{x \in \Omega; \theta_*(x) = 0\}$ is zero. Then we have:

Lemma 3.3. *The unique weak solution (θ, η) for $CP(\theta_*, \eta_*)^\nu$ on $\mathbf{R}_+$ satisfies (2.5) for some δ, $\rho > 0$.*

Proof. Let $\{\theta_{0\varepsilon}\} \subset C^\infty(\overline{\Omega})$ such that $\theta_* \leq \theta_{0\varepsilon}$ in Ω, $\beta_\varepsilon(\theta_{0\varepsilon}) \to \eta_*$ in $L^2(\Omega)$ as $\varepsilon \to 0$, and that (2.4) holds. Recalling proposition 2.1 and remark 2.1, we get a weak solution (θ, η) for $CP(\theta_*, \eta_*)^\nu$ on $\mathbf{R}_+$ as a limit of an approximate solution θ_{ε_n} corresponding to initial value $\theta_{0\varepsilon_n}$ in the sense of (2.1)~(2.3) for some subsequence $\{\varepsilon_n\}$ of $\{\varepsilon\}$. We note that for any $\varepsilon \in]0,1]$

$$\partial_t(\beta_\varepsilon(\theta_*) - \beta_\varepsilon(\theta_\varepsilon)) + \nu\partial_z(\beta_\varepsilon(\theta_*) - \beta_\varepsilon(\theta_\varepsilon)) - \Delta(\theta_* - \theta_\varepsilon) \leq 0 \quad \text{in } Q_\infty. \tag{3.3}$$

Now let us denote by $\{\sigma_m\}$ a sequence of smooth functions on $\mathbf{R}$ such that $\sigma_m(0) = 0$, and for any $r \in \mathbf{R}$, $\sigma_m'(r) \geq 0$, $-1 \leq \sigma_m(r) \leq 1$ and

$$\sigma_m(r) \to \sigma_0(r) := \begin{cases} 1 & \text{for } r > 0, \\ 0 & \text{for } r = 0, \\ -1 & \text{for } r < 0, \end{cases} \quad \text{as } m \to +\infty.$$

Multiply (3.3) by $\sigma_m([\theta_* - \theta_\varepsilon]^+)$ and integrate it over Q_t. By lemma 3.1 and 3.2,

$$
\begin{aligned}
&-\int_{Q_t} \Delta(\theta_* - \theta_\varepsilon)\sigma_m([\theta_* - \theta_\varepsilon]^+)dxd\tau \\
&\geq \int_{\Sigma_t^N} (g_*(\theta_*) - g_\varepsilon(\cdot,\cdot,\theta_\varepsilon))\sigma_m([\theta_* - \theta_\varepsilon]^+)d\Gamma d\tau \\
&\geq \int_{\Sigma_t^N} (g_*(\cdot,\cdot,\theta_\varepsilon) - g_\varepsilon(\cdot,\cdot,\theta_\varepsilon))\sigma_m([\theta_* - \theta_\varepsilon]^+)d\Gamma d\tau \\
&\to \int_{\Sigma_t^N} (g_*(\cdot,\cdot,\theta_\varepsilon) - g_\varepsilon(\cdot,\cdot,\theta_\varepsilon))\sigma_0([\theta_* - \theta_\varepsilon]^+)d\Gamma d\tau \quad \text{as } m \to +\infty.
\end{aligned}
$$

By the strict monotonicity of β_ε,

$$
\begin{aligned}
&\int_{Q_t} \partial_z(\beta_\varepsilon(\theta_*) - \beta_\varepsilon(\theta_\varepsilon))\sigma_m([\theta_* - \theta_\varepsilon]^+)dxd\tau \\
&\to \int_{Q_t} \partial_z(\beta_\varepsilon(\theta_*) - \beta_\varepsilon(\theta_\varepsilon))\sigma_0([\theta_* - \theta_\varepsilon]^+)dxd\tau \quad \text{as } m \to +\infty \\
&= \int_{Q_t} \partial_z(\beta_\varepsilon(\theta_*) - \beta_\varepsilon(\theta_\varepsilon))\sigma_0([\beta_\varepsilon(\theta_*) - \beta_\varepsilon(\theta_\varepsilon)]^+)dxd\tau \\
&= \int_0^t \int_{-l}^l [\beta_\varepsilon(\theta_*) - \beta_\varepsilon(\theta_\varepsilon)]^+ dx'd\tau \Big|_{z=0}^{L} = 0,
\end{aligned}
$$

and

$$
\begin{aligned}
&\int_{Q_t} \partial_t(\beta_\varepsilon(\theta_*) - \beta_\varepsilon(\theta_\varepsilon))\sigma_m([\theta_* - \theta_\varepsilon]^+)dxd\tau \\
&\to \int_{Q_t} \partial_t(\beta_\varepsilon(\theta_*) - \beta_\varepsilon(\theta_\varepsilon))\sigma_0([\theta_* - \theta_\varepsilon]^+)dxd\tau \quad \text{as } m \to +\infty \\
&= \int_{Q_t} \partial_t(\beta_\varepsilon(\theta_*) - \beta_\varepsilon(\theta_\varepsilon))\sigma_0([\beta_\varepsilon(\theta_*) - \beta_\varepsilon(\theta_\varepsilon)]^+)dxd\tau \\
&= \int_\Omega [\beta_\varepsilon(\theta_*) - \beta_\varepsilon(\theta_\varepsilon(t,\cdot))]^+ dx
\end{aligned}
$$

Therefore we have for all $t \in \mathbf{R}_+$

$$
\int_\Omega [\beta_\varepsilon(\theta_*) - \beta_\varepsilon(\theta_\varepsilon(t,\cdot))]^+ dx + \int_{\Sigma_t^N} (g_*(\cdot,\cdot,\theta_\varepsilon) - g_\varepsilon(\cdot,\cdot,\theta_\varepsilon))\sigma_0([\theta_* - \theta_\varepsilon]^+)d\Gamma d\tau \leq 0.
$$

Taking $\varepsilon = \varepsilon_n$ and letting $n \to +\infty$ we have by lemma 3.1

$$
\int_\Omega [\eta_* - \eta(t,\cdot)]^+ dx \leq 0 \quad \text{for all } t \in \mathbf{R}_+,
$$

which implies that

$$\eta_* \le \eta \quad \text{hence} \quad \theta_* \le \theta \quad \text{a.e. in } Q_\infty. \tag{3.4}$$

Because of lemma 3.2, we thus have

$$\theta(t,y,z) \ge \rho \quad \text{a.e. in } Q_\infty^\delta$$

for the same constants δ and ρ as in (3.2). By corollary 2.1 we see that (θ,η) is the unique weak solution for $CP(\theta_*,\eta_*)$ on $\mathbf{R}_+$. q.e.d.

4. Proof of main theorems

Let us prove theorems 2.1 and 2.2.

Proof of theorem 1.1. Firstly we construct a T-periodic weak solution of $(P)^\nu$ on $\mathbf{R}_+$. Let (θ,η) be as in lemma 3.3, that is, the unique weak solution for $CP(\theta_*,\eta_*)^\nu$ on $\mathbf{R}_+$. For each $m \in \mathbf{N}$ we denote by (θ_m,η_m) the weak solution for $CP(\theta(mT,\cdot),\eta(mT,\cdot))^\nu$ on $[0,T]$. By proposition 2.1 and (3.4), we have

$$\eta_* \le \eta \le \beta(K_3) \quad \text{a.e. in } Q_\infty.$$

In particular

$$\eta_* \le \eta(T,\cdot) \le \beta(K_3) \quad \text{a.e. in } \Omega.$$

Applying proposition 2.3 to η and η_1,

$$\eta_* \le \eta \le \eta_1 \le \beta(K_3) \quad \text{a.e. in } Q_T.$$

Recursive use of this procedure derives that

$$\eta_* \le \eta \le \eta_1 \le \eta_2 \le \cdots \le \eta_m \le \cdots \le \beta(K_3) \quad \text{a.e. in } Q_T,$$

hence

$$\theta_* \leq \theta \leq \theta_1 \leq \theta_2 \leq \cdots \leq \theta_m \leq \cdots \leq K_3 \quad \text{a.e. in } Q_T.$$

Then we can define $\eta_\infty(t,x) = \lim_{m\to+\infty} \eta_m(t,x)$ and $\theta_\infty(t,x) = \lim_{m\to+\infty} \theta_m(t,x)$ for a.e. $(t,x) \in Q_T$. It is easily verified that $\eta_\infty \in \beta(\theta_\infty)$ a.e. in Q_T, $\eta_\infty(0,\cdot) = \eta_\infty(T,\cdot)$ and $\theta_\infty(0,\cdot) = \theta_\infty(T,\cdot)$ a.e. in Ω. Further we have estimates

$$\eta_* \leq \eta_m \leq \beta(K_3) \quad \text{hence} \quad \theta_* \leq \theta_m \leq K_3 \quad \text{a.e. in } Q_T,$$

$$|\theta_m|_{L^2(0,T;H^1(\Omega))} \leq C_1,$$

and for any bounded subdomain A with $\overline{A} \subset Q_T$,

$$|\theta_m|_{H^1(A)} \leq C_2 := C_2(A),$$

where C_i, $i = 1, 2$ are positive constants independent of m. Then we easily see that $(\theta_\infty, \eta_\infty)$ is a weak solution of $(P)^\nu$ on $[0,T]$. Consequently, T-periodic extension $(\theta_p^\nu, \eta_p^\nu)$ of $(\theta_\infty, \eta_\infty)$ onto $\mathbf{R}_+$ is a T-periodic weak solution of $(P)^\nu$ on $\mathbf{R}_+$.

Next we prove the uniqueness of T-periodic weak solutions. To do this, we shall show that any T-periodic weak solution (θ, η) is equal to $(\theta_p^\nu, \eta_p^\nu)$ constructed as above. Since θ_p^ν satisfies (2.5), (2.6) holds for $\theta_1 = \theta_p^\nu$ and $\theta_2 = \theta$, from which it follows that

$$\int_{mT}^{(m+1)T} \int_\Omega \left|\eta_p^\nu - \eta\right| dx dt \to 0 \quad \text{as } m \to +\infty. \tag{4.1}$$

On the other hand, by T-periodicity of η_p^ν and η,

$$\int_0^T \int_\Omega \left|\eta_p^\nu - \eta\right| dx dt = \int_{mT}^{(m+1)T} \int_\Omega \left|\eta_p^\nu - \eta\right| dx dt.$$

So we must have $\int_0^T \int_\Omega \left|\eta_p^\nu - \eta\right| dx dt = 0$. Therefore $\eta_p^\nu = \eta$ a.e. in Q_T. Again, by T-periodicity of η_p^ν and η, $\eta_p^\nu = \eta$ a.e. in Q_∞. Hence $\theta_p^\nu = \theta$ a.e. in Q_∞. Thus the proof has been completed. q.e.d.

Proof of theorem 2. Let (θ, η) be an arbitrary weak solution of $(P)^\nu$ on $\mathbf{R}_+$ satisfying (2.5). From proposition 2.3 we find that

$$d := \lim_{t \to +\infty} \left|\eta_p^\nu(t, \cdot) - \eta(t, \cdot)\right|_{L^1(\Omega)}$$

exists. Further as $m \to +\infty$ we have

$$\int_{mT}^{(m+1)T} \int_\Omega \left|\eta_p^\nu - \eta\right| dx dt \geq T \left|\eta_p^\nu((m+1)T, \cdot) - \eta((m+1)T, \cdot)\right|_{L^1(\Omega)} \to dT.$$

Note that (4.1) also holds for η_p^ν and η, hence we deduce $d = 0$. That is $\eta_p^\nu(t, \cdot) - \eta(t, \cdot) \to 0$ in $L^1(\Omega)$. On account of the boundedness of η_p^ν and η in Q_∞, we obtain

$$\eta_p^\nu(t, \cdot) - \eta(t, \cdot) \to 0 \quad \text{in } L^q(\Omega) \text{ for all } q \geq 1 \text{ as } t \to +\infty.$$

From (1.1), it results that

$$b_* \left|\theta_p^\nu(t, x) - \theta(t, x)\right| \leq \left|\eta_p^\nu(t, x) - \eta(t, x)\right| \quad \text{for a.e. } (t, x) \in Q_\infty,$$

consequently

$$\theta_p^\nu(t, \cdot) - \theta(t, \cdot) \to 0 \quad \text{in } L^q(\Omega) \text{ for all } q \geq 1 \text{ as } t \to +\infty.$$

q.e.d.

In the rest of this chapter we study the convergence of the T-periodic weak solution of $(P)^\nu$ on $\mathbf{R}_+$ to that of the Stefan problem when $\nu \to 0$. The result is as follows.

Theorem 4.3. *Assume that (2.7) holds. When $\nu \to 0$, $(\theta_p^\nu, \eta_p^\nu)$ converges to some periodic solution (θ_p^0, η_p^0) of the Stefan problem $(P)^0$ in the following sense:*

$$\theta_p^\nu \to \theta_p^0 \quad \textit{weakly in } L^2(0, T; H^1(\Omega)) \textit{ and strongly in } L^q(Q_T) \textit{ for all } q \geq 1,$$

$$g(\cdot,\cdot,\theta_p^\nu) \to g(\cdot,\cdot,\theta_p^0) \quad \textit{in } L^2(\Sigma_T^N),$$

and there exists a subsequence $\{\nu_k\}$ of $\{\nu\}$ such that

$$\eta_p^{\nu_k} \to \eta_p^0 \quad \textit{weakly in } L^\infty(Q_T).$$

We claim that the following estimates hold for $\{(\theta_p^\nu, \eta_p^\nu)\}$:

$$\beta(-K_3) \le \eta_p^\nu(t,x) \le \beta(K_3) \quad \text{hence} \quad -K_3 \le \theta_p^\nu(t,x) \le K_3 \quad \text{a.e. in } Q_T,$$

$$|\theta_p^\nu|_{L^2(0,T;H^1(\Omega))} \le C_3,$$

and for any bounded subdomain A with $\overline{A} \subset Q_T$,

$$|\theta_p^\nu|_{H^1(A)} \le C_4,$$

where $C_i > 0$, $i = 3,4$, are constants independent of $\nu \in]0,1]$. Hence there exist a subsequence $\{\nu_k\}$ of $\{\nu\}$ and $(\theta,\eta) \in L^2(0,T;H^1(\Omega)) \times L^\infty(Q_T)$ such that

$$\eta_p^{\nu_k} \to \eta \quad \text{weakly* in } L^\infty(Q_T),$$

$$\theta_p^{\nu_k} \to \theta \quad \text{weakly in } L^2(0,T;H^1(\Omega)) \text{ and strongly in } L^q(Q_T) \text{ for all } q \ge 1, \tag{4.2}$$

$$g(\cdot,\cdot,\theta_p^{\nu_k}) \to g(\cdot,\cdot,\theta) \quad \text{in } L^2(\Sigma_T^N). \tag{4.3}$$

We easily see that (θ,η) is a weak solution of $(P)^0$ on $[0,T]$. Moreover, since $(\theta_p^\nu, \eta_p^\nu)$ is T-periodic, (θ,η) is also T-periodic. On account of remark 2.4 we can replace $\{\nu_k\}$ with $\{\nu\}$ in (4.2) and (4.3). Therefore T-periodic extension of (θ,η) onto $\mathbf{R}_+$ is a desired one.

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