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Normal form and global solutions for the Klein-Gordon-Zakharov equations

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1 Introduction and main results

We consider the Cauchy problem of the Klein-Gordon-Zakharov equations in three space dimensions:

$$\partial_t^2 u - \Delta u + u = -nu, \qquad t > 0,\ x \in \mathbf{R}^3, \tag{1.1}$$

$$\partial_t^2 n - \Delta n = \Delta |u|^2, \qquad t > 0,\ x \in \mathbf{R}^3, \tag{1.2}$$

$$u(0,x) = u_0(x),\ \partial_t u(0,x) = u_1(x), \tag{1.3}$$

$$n(0,x) = n_0(x),\ \partial_t n(0,x) = n_1(x),$$

where $\partial_t = \partial/\partial t$, and $u(t,x)$ and $n(t,x)$ are functions from $\mathbf{R}_+ \times \mathbf{R}^3$ to $\mathbf{C}^3$ and from $\mathbf{R}_+ \times \mathbf{R}^3$ to $\mathbf{R}$, respectively. The system (1.1)-(1.2) describes the propagation of strong turbulence of the Langmuir wave in a high frequency plasma (see [15]).

The usual Zakharov system

$$i\partial_t u + \Delta u = nu, \quad t > 0, \quad x \in \mathbf{R}^3, \tag{1.4}$$

$$\partial_t^2 n - \Delta n = \Delta |u|^2, \quad t > 0, \quad x \in \mathbf{R}^3 \tag{1.5}$$

is derived from (1.1)-(1.2) through the physical approximation procedure.

In the present paper we consider solving (1.1)-(1.3) around the *zero* solutions. There are many papers concerning the global existence of small solutions for the coupled systems of the Klein-Gordon and wave equations with quadratic nonlinearity (see, e.g., [1], [5]–[7], [9], [10], [12] and [13]). The methods to solve those systems can be classified into two groups (for a good review of this matter, see Strauss [14]). One is to use the Sobolev space with weight related to the generators of the Lorentz group. This was developed by Klainerman [9] and [10]. The combination of this method and the null condition technique has produced several nice applications to the hyperbolic systems of physical importance (see, e.g., Bachelot [1] and Georgiev [6]). However, this method does not seem to be directly applicable to (1.1)-(1.3). In fact, since the system (1.1)-(1.2) consists of the Klein-Gordon and wave equations with quadratic nonlinearity in three space dimensions, we need to use not only the Sobolev norms with weights related to the generators of the Lorentz group but also the null condition technique (see, e.g., Georgiev [5] and [6]), while the nonlinear terms in (1.1) and (1.2) do not seem to satisfy the null condition as they are. Another method is based on the theory of normal forms introduced by Shatah [12], which is an extension of Poincaré's theory of normal forms to the partial differential equations. In [16] the authors have applied the argument of normal form to (1.1)-(1.2) and proved the global existence of solutions to(1.1)-(1.3) for small initial data. In [16] the authors have also shown that these global solutions to (1.1)-(1.3) with small initial data approach the free solutions asymptotically as $t \to +\infty$. In this note we briefly describe the results obtained in [16].

Before we state the main results in this paper, we give several notations. For $1 \leq p \leq \infty$ and a nonnegative integer m, let L^p and $W^{m,p}$ denote the standard L^p and Sobolev spaces on $\mathbf{R}^3$, respectively. We put $H^m = W^{m,2}$. For $m \in \mathbf{R}$, we let $\dot{H}^m = (-\Delta)^{-m/2}L^2$. We put $\omega = (1-\Delta)^{1/2}$ and $\omega_0 = (-\Delta)^{1/2}$.

We have the following theorem concerning the global existence and asymptotic behavior of solutions to (1.1)-(1.3) for small initial data.

Theorem 1.1 *Let* $0 < \varepsilon \leq 10^{-4}$. *Assume that* $u_0 \in H^{25} \cap W^{15,6/(5+6\varepsilon)}$, $u_1 \in H^{24} \cap W^{14,6/(5+6\varepsilon)}$, $n_0 \in H^{24} \cap W^{14,28/27} \cap \dot{H}^{-1}$ *and* $n_1 \in H^{23} \cap W^{13,28/27} \cap \dot{H}^{-2}$. *Then, there exists a* $\delta > 0$ *such that if*

$$\begin{aligned}(1.6) \qquad & \|u_0\|_{H^{25}\cap W^{15,6/(5+6\varepsilon)}} + \|u_1\|_{H^{24}\cap W^{14,6/(5+6\varepsilon)}} \\ & + \|n_0\|_{H^{24}\cap W^{14,28/27}\cap \dot{H}^{-1}} + \|n_1\|_{H^{23}\cap W^{13,28/27}\cap\dot{H}^{-2}} \leq \delta,\end{aligned}$$

(1.1)-(1.3) have the unique global solutions (u,n) *satisfying*

$$(1.7) \qquad u \in \bigcap_{j=0}^{2} C^j([0,\infty); H^{25-j}),$$

$$(1.8) \qquad n \in \left[\bigcap_{j=0}^{2} C^j([0,\infty); H^{24-j})\right] \cap \left[\bigcap_{j=0}^{1} C^j([0,\infty); \dot{H}^{-1-j})\right],$$

$$(1.9) \qquad \sum_{j=0}^{1} \|\partial_t^j u(t)\|_{W^{13-j,6/(1-6\varepsilon)}} = O(t^{-(1+3\varepsilon)}) \quad (t\to\infty),$$

$$(1.10) \qquad \sum_{j=0}^{1} \|\partial_t^j n(t)\|_{W^{12-j,28}} = O(t^{-13/14}) \quad (t\to\infty),$$

where δ *depends only on* ε. *Furthermore, the above solutions* (u,n) *of (1.1)-(1.3) have the asymptotic sates* $u_{+0} \in H^{12}, u_{+1} \in H^{11}, n_{+0} \in H^{11}, n_{+1} \in H^{10}$ *such that*

$$\begin{aligned}(1.11) \qquad & \sum_{j=0}^{1} \|\partial_t^j (u(t) - u_+(t))\|_{H^{12-j}} \\ & + \sum_{j=0}^{1} \|\partial_t^j (n(t) - n_+(t))\|_{H^{11-j}} \to 0 \quad (t\to\infty),\end{aligned}$$

where

$$
\begin{aligned}
u_+(t) &= (\cos\omega t)u_{+0} + (\omega^{-1}\sin\omega t)u_{+1}, \\
n_+(t) &= (\cos\omega_0 t)n_{+0} + (\omega_0^{-1}\sin\omega_0 t)n_{+1}.
\end{aligned}
$$

Remark 1.1

(1) In three space dimensions, $\boldsymbol{S} \subseteq \dot{H}^{-1}$ but $\boldsymbol{S} \subseteq \dot{H}^{-2}$, where $\boldsymbol{S}$ is the Schwartz space on $\mathbf{R}^3$. For the details of the homogeneous Sobolev space $\dot{H}^m$, see [2, §6.3 in Chapter 6].

(2) $u_+(t)$ and $n_+(t)$ are the solutions of the free Klein-Gordon equation and the free wave equation with the initial conditions $(u_+(0), \partial_t u_+(0)) = (u_{+0}, u_{+1})$ and $(n_+(0), \partial_t n_+(0)) = (n_{+0}, n_{+1})$, respectively. The relation (1.11) implies that the solutions of (1.1)-(1.3) given by Theorem 1.1 behave like the free solutions as $t \to \infty$.

(3) In connection with the usual Zakharov system (1.4)-(1.5) for three space dimensions, it is conjectured that if the initial data are large, the solutions of (1.1)-(1.3) may not necessarily exist globally in time.

(4) In the case of one or two space dimensions, the global existence result for small initial data can be proved more easily than the case of three space dimensions. We do not need the time decay estimates to show the global existence of solutions in the one and two dimensional cases.

The following corollary is an immediate consequence of Theorem 1.1.

Corollary 1.2 *Let $0 < \varepsilon \le 10^{-4}$ and let m be a positive integer with $m \ge 25$. Assume that $u_0 \in H^m \cap W^{15,6/(5+6\varepsilon)}$, $u_1 \in H^{m-1} \cap W^{14,6/(5+6\varepsilon)}$, $n_0 \in H^{m-1} \cap W^{14,28/27} \cap \dot{H}^{-1}$, $n_1 \in H^{m-2} \cap W^{13,28/27} \cap \dot{H}^{-2}$ and (u_0, u_1, n_0, n_1) satisfy (1.6). Then, the solutions (u, n) given by Theorem 1.1 satisfy*

$$u \in \bigcap_{j=0}^{m} C^j([0,\infty); H^{m-j}), \tag{1.12}$$

$$n \in \bigcap_{j=0}^{m-1} C^j([0,\infty); H^{m-1-j}). \tag{1.13}$$

In addition, if $u_0, u_1, n_0, n_1 \in \cap_{j=1}^{\infty} H^j$, then we have

$$u(t,x),\ n(t,x) \in C^\infty([0,\infty) \times \mathbf{R}^3). \tag{1.14}$$

The unique existence and regularity of local solutions for (1.1)-(1.3) follows from the standard iteration argument. The crucial part of proofs of Theorem 1.1 and Corollary 1.2 is to establish the a priori estimates of the solutions for (1.1)-(1.3) in order to extend the local solutions globally in time. The global behavior of local solutions for (1.1)-(1.3) can not be controlled directly, since the quadratic nonlinear term in (1.1) does not provide a sufficient decay property for the three dimensional case. Here we use the argument of normal forms of Shatah [12] to transform the quadratic nonlinearity into the cubic one. However, in our problem the transformed cubic nonlinearity is represented in terms of the integral operator with singular kernel (see (2.4)-(2.7) in Section 2). The singularity of the integral kernel makes it difficult to solve (1.1)-(1.3). This is different form the case of the system containing only the Klein-Gordon equations, where the integral kernels of the resulting integral operators are regular (see [12]). Therefore, our main task in the proof of Theorem 1.1 is to evaluate the singularity of the integral kernel of the transformed cubic nonlinearity. This enables us to apply the usual $L^p - L^q$ estimate to the transformed system, which provides us with the sufficient decay properties of solutions to (1.1)-(1.3) for the proof of Theorem 1.1. In Section 2 we apply the method of normal forms to our problem in order to transform the quadratic nonlinearity into the cubic one. Detailed proof of Theorem 1.1 will be given somewhere.

2 Normal form

In this section we show that the transformation exists for the system (1.1)-(1.2). We write (1.1)-(1.2) as a first order system

$$\frac{d\boldsymbol{U}}{dt} = A\boldsymbol{U} + F(\boldsymbol{U}), \tag{2.1}$$

where

$$\boldsymbol{U} = \begin{pmatrix} u \\ n \\ \partial_t u \\ \partial_t n \end{pmatrix}, \quad A = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ \Delta - 1 & 0 & 0 & 0 \\ 0 & \Delta & 0 & 0 \end{pmatrix}, \quad F(\boldsymbol{U}) = \begin{pmatrix} 0 \\ 0 \\ -nu \\ \Delta|u|^2 \end{pmatrix}$$

We consider the transformation with the following form:

$$\begin{pmatrix} v \\ m \\ \partial_t v \\ \partial_t m \end{pmatrix} = \boldsymbol{V} = \boldsymbol{U} - \begin{pmatrix} [\boldsymbol{U}, B_1, \boldsymbol{U}] \\ [\boldsymbol{U}, B_2, \boldsymbol{U}] \\ [\boldsymbol{U}, B_3, \boldsymbol{U}] \\ [\boldsymbol{U}, B_4, \boldsymbol{U}] \end{pmatrix},$$

where B_j $(j = 1, \cdots, 4)$ are 4×4 matrices and

$$\begin{aligned} &[\boldsymbol{U}, B_j, \boldsymbol{U}] \\ &= \int_{\mathbf{R}^3 \times \mathbf{R}^3} (\bar{u}, n, \overline{\partial_t u}, \partial_t n)(y) B_j(x-y, x-z) \begin{pmatrix} u \\ n \\ \partial_t u \\ \partial_t n \end{pmatrix} (z) dy dz. \end{aligned} \tag{2.2}$$

Let us consider the case $j = 1$. We put

$$B_1 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ G_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & G_2 & 0 \end{pmatrix}$$

Here G_1 and G_2 are to be thought of as distributions and the integral representation (2.2) is interpreted in an appropriate way. Then, we have

$$\begin{aligned} v &= u - [n, G_1, u] - [\partial_t n, G_2, \partial_t u] \\ &= u - \sum_{j=1}^{3} \int_{\mathbf{R}^3 \times \mathbf{R}^3} n(y) G_1(x-y, x-z) u_j(z) dy dz \\ &\quad - \sum_{j=1}^{3} \int_{\mathbf{R}^3 \times \mathbf{R}^3} \partial_t n(y) G_2(x-y, x-z) \partial_t u_j(z) dy dz, \end{aligned}$$

where

$$u = \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix}.$$

We now compute $\partial_t^2 v = \partial_t^2 (u - [n, G_1, u] - [\partial_t n, G_2, \partial_t u])$:

$$\begin{aligned} \partial_t v &= \partial_t u - [\partial_t n, G_1, u] - [n, G_1, \partial_t u] \\ &\quad - [\partial_t^2 n, G_2, \partial_t u] - [\partial_t n, G_2, \partial_t^2 u] \\ &= \partial_t u - [\partial_t n, G_1, u] - [n, G_1, \partial_t u] \\ &\quad -[\Delta n + \Delta |u|^2, G_2, \partial_t u] - [\partial_t n, G_2, \Delta u - u + nu], \\ \partial_t^2 v &= \partial_t^2 u - [\Delta n, G_1, u] - 2[\partial_t n, G_1, \partial_t u] \\ &\quad -[n, G_1, \Delta u - u] - [\Delta \partial_t n, G_2, \partial_t u] - 2[\Delta n, G_2, \Delta u - u] \\ &\quad -[\partial_t n, G_2, \Delta \partial_t u - \partial_t u] + \text{(cubic terms)}. \end{aligned}$$

Moreover,

$$\begin{aligned} (-\Delta + 1) v &= -\Delta u + u + [\Delta n, G_1, u] + \sum_{j=1}^{3} 2[\partial_j n, G_1, \partial_j u] \\ &\quad +[n, G_1, \Delta u] - [n, G_1, u] + [\Delta \partial_t n, G_2, \partial_t u] \\ &\quad + \sum_{j=1}^{3} 2[\partial_j \partial_t n, G_2, \partial_j \partial_t u] + [\partial_t n, G_2, \Delta \partial_t u] \\ &\quad -[\partial_t n, G_2, \partial_t u]. \end{aligned}$$

Therefore

$$\begin{aligned} \text{(2.3)}\quad \partial_t^2 v - \Delta v + v &= -nu - 2[\Delta n, G_2, \Delta u] + 2[\Delta n, G_2, u] \\ &\quad + \sum_j 2[\partial_j n, G_1, \partial_j u] - 2[\partial_t n, G_1, \partial_t u] \\ &\quad + \sum_j 2[\partial_j \partial_t n, G_2, \partial_j \partial_t u] + \text{(cubic terms)}. \end{aligned}$$

We choose the distributions G_1 and G_2 so that all quadratic terms in (2.3) cancel out:

$$(2.4)\quad -nu - 2[\Delta n, G_2, \Delta u] + 2[\Delta n, G_2, u] + \sum_j 2[\partial_j n, G_1, \partial_j u]$$
$$-2[\partial_t n, G_1, \partial_t u] + \sum_j 2[\partial_j \partial_t n, G_2, \partial_j \partial_t u] = 0.$$

Here we define the Fourier transform of G_j by

$$\widehat{G_j}(p,q) = \int_{\mathbf{R}^3 \times \mathbf{R}^3} e^{-i(p\cdot y + q\cdot z)} G_j(y,z)\,dydz.$$

Then equation (2.4) becomes

$$\begin{aligned} -1 - 2|p|^2|q|^2\widehat{G_2}(p,q) - 2|p|^2\widehat{G_2}(p,q) - 2p\cdot q\widehat{G_1}(p,q) &= 0, \\ -\widehat{G_1}(p,q) - p\cdot q\widehat{G_2}(p,q) &= 0. \end{aligned}$$

Thus, we obtain

$$(2.5)\quad \widehat{G_1}(p,q) = \frac{p\cdot q}{2\{|p|^2|q|^2 - (p\cdot q)^2 + |p|^2\}},$$

$$(2.6)\quad \widehat{G_2}(p,q) = \frac{-1}{2\{|p|^2|q|^2 - (p\cdot q)^2 + |p|^2\}}.$$

We next consider the case $j = 2$. Similarly we put

$$m = n - [u, H_1, u] - [\partial_t u, H_2, \partial_t u].$$

As before, we obtain

$$(2.7)\quad \widehat{H_1}(p,q) = \frac{(p\cdot q - 1)|p+q|^2}{2\{|p|^2|q|^2 - (p\cdot q)^2 + |p+q|^2\}},$$

$$(2.8)\quad \widehat{H_2}(p,q) = \frac{-|p+q|^2}{2\{|p|^2|q|^2 - (p\cdot q)^2 + |p+q|^2\}}.$$

We have thus completed the construction of the normal form, as required.

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