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Periodic Solutions for Curve Evolution Equations

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1 Introduction.

This is a joint work with Prof. Giga of Hokkaido University.

We consider the quasilinear parabolic equation

$$u_t = u^2(u_{xx} + u - f) \quad \text{in } K, \tag{1}$$

where $K = (\mathbf{R}/2\pi\mathbf{Z}) \times (\mathbf{R}/T\mathbf{Z})$ with $T > 0$ and f is a positive function on K. The purpose of this paper is to prove the following result.

Theorem 1. If f is a positive continuous function on K with $f_t \in C(K)$ such that

$$\int_0^{2\pi} f(x,t)e^{ix}dx = 0 \quad \text{for all } t, \tag{2}$$

then there exists a positive solution $u \in \bigcap_{p>1} W_p^{2,1}(K)$ of the equation (1) satisfying the condition

$$\int_0^{2\pi} \frac{e^{ix}}{u(x,t)}dx = 0 \quad \text{for all } t \in \mathbf{R}. \tag{3}$$

We remark that the assumption (2) is necessarily satisfied provided that there is a positive solution of (1) satisfying (3). In fact, multiplying $u^{-2}e^{ix}$ with (1) and integrating over $(0, 2\pi)$ yields

$$-\frac{d}{dt}\int_0^{2\pi} \frac{e^{ix}}{u}dx = -\int_0^{2\pi} fe^{ix}dx.$$

If u satisfies the constraint (3), f must satisfy (2).

Our main result yields the existence of a periodic-in-time solution (up to translation)for an evolution equation of curves whose normal speed equals the curvature minus a given time periodic function depending on curves through its normals. Let $\{\Gamma_t\}$ be a smooth one parameter family of closed, embedded curves in a plane bounding a bounded domain. Let $\mathbf{n}$ denote the inward unit normal vector field on Γ_t. Let V denote the normal velocity of Γ_t in the direction of $\mathbf{n}$. We consider an equation for Γ_t of the form

$$V = k - q(\mathbf{n}, t), \tag{4}$$

where k is the inward curvature and q is a given function. The equation (4) is an example of curvature flow equation with anisotoropy ([13]). If Γ_t is convex, one can parameterize Γ_t by a Gauss map by introducing $\theta, 0 \leq \theta \leq 2\pi$ such that $\mathbf{n} = (\cos\theta, \sin\theta)$. The evolution of curvature k is expressed as

$$k_t = k^2(V_{\theta\theta} + V)$$

if we use $\theta-$ cordinates ([13]). Applying this identity to (4) yields an evolution equation of curvature

$$k_t = k^2(k_{\theta\theta} + k - (Q_{\theta\theta} + Q)) \quad \text{with} \quad Q(\theta, t) = q(\cos\theta, \sin\theta, t), \tag{5}$$

where k and Q are $2\pi-$ periodic in θ. We next recover (4) form (5). For k a curve parametrized by the Gauss map is given by

$$Z(\theta, t) = (\int_0^\theta \frac{\sin\sigma}{k(\sigma, t)} d\sigma, -\int_0^\theta \frac{\cos\sigma}{k(\sigma, t)} d\sigma).$$

If k solves (5), then integarating by parts yields

$$\frac{\partial Z}{\partial t} = ((k-Q)\cos\theta - (k_\theta - Q_\theta)\sin\theta - (k-Q)|_{\theta=0}, (k-Q)\sin\theta + (k_\theta - Q_\theta)\cos\theta - (k_\theta - Q_\theta)|_{\theta=}$$

Translate Z by

$$X_0(t) = (\int_0^t (k - Q)(0, \tau) d\tau, \int_0^t (k_\theta - Q_\theta)(0, \tau) d\tau),$$

so that new curve $X(\theta, t) = Z(\theta, t) + X_0(t)$ fulfills

$$V = \mathbf{n} \cdot \frac{\partial X}{\partial t} = (\cos\theta, \sin\theta) \cdot \frac{\partial X}{\partial t} = k - q.$$

We thus obtained the curve

$$\Gamma_t = \{X(\theta, t) : 0 \leq \theta \leq 2\pi\}$$

satisfying (4). The equation (4) and (5) are equivalent through X. However to be Γ_t is closed we need $X(0,t) = X(2\pi, t)$ which is equivalent to the constraint

$$\int_0^{2\pi} \frac{e^{ix}}{k(\theta, t)} d\theta = 0.$$

If we set $u = k, x = \theta$, this is nothing but the constraint (3). Since the condition (2) is automatically satisfied for $f = Q_{\theta\theta} + Q$, Theorem 1 yields a periodic-in-time solution Γ_t (up to translation in space) of (4).

We also note that f is a positive function if and only if the Frank diagram of q is strictly convex (see [12]).

The initial value problem for (5) with $q = 0$ was derived in [9] and extensively studied by Gage and Hamilton [11] for the curve shortening problem. Since a circle shrinks to a point in a finite time for the curve shortening equation (4) with $q = 0$, the curvature may blow up in a finite time. Blow up profiles for convex immersed curves were classified by Angenent [2] based on a result of [1] under the self-similar growth assumption for curvatures. There may happen that curvature growth is faster than self-similar rate. Its asysmptotic profile is studied in [2] via (4) with $q = 0$. Recently, more precise profile is obtained by Angenent and Velazquez [3] by studying (4) itself. The iunitial boundary value problem for higher dimensional version of (1) with $f = 0$

$$u_t = u^2(\Delta u + u)$$

in a bounded domain with zero boundary data was studied in [8] and [10] for positive initial data. The existence of blow up phenomena depends on the first eigenvalue of the Laplace operator with zero boundary condition. These authors studied whether a solution blows up and they estimated the size of blow up sets. However it seems that there are no results concerning the periodic problem for the equation (1).

We make use of the Leray-Schauder degree theory to show this theorem. The existence of periodic solutions for semilinear parabolic equations was obtained by the degree theory in Esteban [6], [7], Hirano and the second author [14] and so on. But constracting homotopies to solve the equation(1) is more difficult than that in the above papers because the equation (1) is degenerate and our desired solution should satisfy the constraint (3).

We shall select desired solution by introducing a kind of penalty method since not all solutions satisfy the constraint (3). Explaining heuristically, for small $\varepsilon > 0$, we consider the penalized equation

$$u_t = u^2(u_{xx} + u + \frac{\varepsilon}{u} - f) \quad \text{in } K. \tag{6}$$

For a solution u of this equation, we observe that the condition (2) implies

$$-\frac{d}{dt}\int_0^{2\pi} \frac{e^{ix}}{u} dx = \varepsilon \int_0^{2\pi} \frac{e^{ix}}{u} dx$$

by multiplying (6) with $u^{-2}e^{ix}$ and integrating over $(0, 2\pi)$. Since u is periodic in time, this implies that u satisfies the constraint (3). We modify the term $\frac{\varepsilon}{u}$ so that the solutions has a uniform bound in the next section. A penalty method is adapted in various evolution equations to introduce constraints of solutions. For example, it was used to constract a solution u satisfying a constraint $|u| = 1$ for the harmonic gradient flow equations in Chen [4], Chen and Struwe [5] and Keller, Rubinstein and Sternberg [15].

2 Upper bound for solutions of approximate equations.

The Leray-Schauder degree theory is adapted to show Theorem 1. To do that, we introduce the following approximate equation

$$u_t = (u + \varepsilon^2)^2(u_{xx} + \frac{u^2}{(u+\varepsilon^2)^2}(u + \frac{\varepsilon}{\xi_\varepsilon(u)} - f)) \quad \text{in } K, \tag{7}$$

where $\frac{1}{m} < \min_K f$ and ξ_ε is a smooth increasing function on $\mathbf{R}$ such that

$$\xi_\varepsilon(s) = s + \varepsilon^2 \quad \text{for all } s \geq m\varepsilon$$

and

$$\max(s+\varepsilon^2, m\varepsilon) \leq \xi_\varepsilon(s) \leq C\max(s+\varepsilon^2, m\varepsilon) \quad \text{for all } s > 0.$$

We first observe that any positive solution of

$$u_t = u^2(u_{xx} + u + \frac{\varepsilon}{u} - f) \quad \text{in } K$$

satisfies the constraint (3), so we modify this equation so that it is a uniformly parabolic and the Leray-Schauder degree in a large and a small ball can be computed.

For $\tau \in [0,1]$, we consider the equation

$$u_t = (u+\varepsilon^2)^2[u_{xx} + \frac{u^2}{(u+\varepsilon^2)^2}\{u + \tau(\frac{\varepsilon}{\xi_\varepsilon(u)} - f)\} + (1-\tau)\beta] \quad \text{in } K, \tag{8}$$

where $\beta > 0$.

We assume that f is a smooth function. Then it follows that each positive solution of (8) is smooth. Our purpose in the present section is to show the following result.

Theorem 2. There exists $M = M(|f|_\infty, |f_t|_\infty) > 0$ such that $\max_K u \leq M$ for each $\varepsilon > 0, \tau \in [0,1]$ and each positive solution u of (8).

We first get an estimate of Harnack type in space direction to prove this theorem. The Harnack inequality was used in [10] for the equation $u_t = u^2(\Delta u + u)$.

Lemma 1. Suppose that there is $M_0 > 0$ such that $\max_K u \geq M_0$ for any $\varepsilon > 0, \tau \in [0,1]$ and any positive solution u of (8). Then there exists $C_0 = C_0(|f|_\infty, |f_t|_\infty) > 0$ such that for each $\varepsilon > 0, \tau \in [0,1]$ and each positive solution u of (8),

$$(u(x,t_0) + \varepsilon^2)^2 \geq (M+\varepsilon^2)^2 - C_0(M+\varepsilon^2)^2(x-x_0)^2 \quad \text{for all } x,$$

where $M = \max_K u = u(x_0, t_0)$.

Proof. Put $v = u + \varepsilon^2$ and

$$g(v,x,t) = \frac{(v-\varepsilon^2)^2}{v^2}\{v - \varepsilon^2 + \tau(\frac{\varepsilon}{\xi_\varepsilon(v-\varepsilon^2)} - f)\} + (1-\tau)\beta].$$

Letting $z = \frac{v_t}{v}$, it follows that

$$z_x = \frac{v_{tx}}{v} - \frac{v_t v_x}{v^2}$$

and

$$z_{xx} = \frac{v_{txx}}{v} - \frac{2v_x z_x}{v} - \frac{v_t v_{xx}}{v^2}$$

from (8). Differenciating $z = v(v_{xx} + g)$,

$$z_t = v^2 z_{xx} + 2vv_x z_x + 2z^2 + v(g_v v - g)z + g_t v.$$

Let $(\hat{x}, \hat{t})$ be a minimizer of z in K. Then we have

$$2vz^2 + v(g_v v - g)z + g_t v \leq 0$$

at $(\hat{x}, \hat{t})$ and hence

$$z \geq -\frac{v\{(g_v v - g) + |g_v v - g|\}}{4} - (\frac{v|g_t|}{2})^{1/2}$$

at $(\hat{x}, \hat{t})$. Therefore there are $c_0 = c_0(|f|_\infty, |f_t|_\infty) > 0, c_1 = c_1(|f|_\infty, |f_t|_\infty) > 0$ such that

$$\min z \geq -c_0(M + \varepsilon^2) - c_1(M + \varepsilon^2)^{1/2}.$$

By the assumption, there is $c_2 = c_2(|f|_\infty, |f_t|_\infty) > 0$ such that

$$\min z \geq -c_2(M + \varepsilon^2). \tag{9}$$

From $vv_{xx} = z - vg$, it follows that

$$vv_{xx} \geq -c_2(M + \varepsilon^2) - (M + \varepsilon^2) \max_{v \leq M + \varepsilon} g.$$

Consequently, there is $C_0 = C_0(|f|_\infty, |f_t|_\infty) > 0$ such that $vv_{xx} \geq C_0(M + \varepsilon^2)^2$. Then we see

$$\frac{1}{2}(v^2)_{xx} = v_x^2 + vv_{xx} \geq C_0(M + \varepsilon^2)^2.$$

This implies the assertion of this lemma.

We next obtain integral bounds for solutions of (8).

Lemma 2. There are $C_1 = C_1(|f|_\infty, |f_t|_\infty) > 0$ and $C_2 = C_2(|f|_\infty, |f_t|_\infty) > 0$ such that

$$\int_0^T \int_0^{2\pi} (u + \varepsilon^2) dx dt \le C_1$$

and

$$\int_0^T \int_0^{2\pi} \frac{u_t^2}{(u+\varepsilon^2)^2} dx dt \le C_2$$

for each $\varepsilon > 0, \tau \in [0, 1]$ and each positive solution u of (8).

Proof. Multiplying (8) with $\dfrac{1}{(u+\varepsilon^2)^2}$ and $\dfrac{u_t}{(u+\varepsilon^2)^2}$ and integrating over K respectively, we obtain these integral bounds.

From Lemma 1 and 2, Theorem 2 can be shown.

Proof of Theorem 2. Assume there are no upper bounds for solutions of (8). From Lemma 1, it follows that

$$\int_0^{2\pi} (u(x, t_0) + \varepsilon^2)^2 dx \ge \frac{1}{2}(M + \varepsilon^2)^2. \tag{10}$$

Take $t_1 \in [0, T]$ with $\displaystyle\int_0^{2\pi} (u(x, t_1) + \varepsilon^2)^2 dx \le \frac{MC_1}{T}$. By Lemma 2, we get

$$\begin{aligned} \int_0^{2\pi} (u(x, t_0) + \varepsilon^2)^2 dx &\le \int_0^{2\pi} (u(x, t_1) + \varepsilon^2)^2 dx + \int_0^T \int_0^{2\pi} 2(u + \varepsilon^2) u_t dx dt \\ &\le \frac{MC_1}{T} + 2M^{3/2} C_1^{1/2} C_2^{1/2}. \end{aligned}$$

This contradicts (10). Therefore the assertion of this thoerem holds.

3 Lower bound for solutions of approximate equations.

We begin this section with another inequality of Harnack type in time direction.

Lemma 3. There is $C_3 = C_3(|f|_\infty, |f_t|_\infty) > 0$ such that for any $\varepsilon > 0$ and any positve solution u of (7),

$$u(x,t) + \varepsilon^2 \leq e^{-C_3(M+\varepsilon^2)(t-s)}(u(x,s) + \varepsilon^2) \tag{11}$$

for all s, t with $s - T \leq t \leq s$ and $x \in [0, 2\pi]$, where M is an upper bound obtained in Theorem 2.

Proof. From (9), it followe that $\dfrac{u_t}{u + \varepsilon^2} \geq -c_2(M + \varepsilon^2)$ in K. Integrating this inequality over (t, s), we obtain (11).

The following result about the distance of zeros of a solution for an ordinary differential inequality is crucial in our proof of Theorem 3.

Lemma 4. Let $U \in C^1([0, \beta])$ be nonnegative and not identically zero, $U(0) = U(\beta) = 0$ and $U_x(0) = 0$ or $U_x(\beta) = 0$. If $U_{xx} + U \geq 0$ in $(0, \beta)$, then $\beta > \pi$.

Proof. Suppose that $\beta \leq \pi$. Then we have

$$\int_0^\beta \sin(\frac{\pi x}{\beta})(U_{xx} + U) \leq \int_0^\beta \sin(\frac{\pi x}{\beta})\{U - (\frac{\pi}{\beta})^2 U\} dx \leq 0.$$

From $U(0) = U(\beta) = 0$, it follows that $U(x) = c\sin(\frac{\pi x}{\beta})$ in $[0, \beta]$ for some $c > 0$. This contradicts that $U_x(0) = 0$ or $U_x(\beta) = 0$. Therefore $\beta > \pi$.

The following result is concerned with the constraint (3).

Lemma 5. Ther exists $C_4 = C_4(|f|_\infty) > 0$ such that

$$|\int_0^T \int_0^{2\pi} \{\frac{u^2}{(u+\varepsilon^2)^2} \cdot \frac{\varepsilon}{\xi_\varepsilon(u)} + (1 - \frac{u^2}{(u+\varepsilon^2)^2})f\} \sin(x - \alpha) dx dt| \leq C_4 \varepsilon$$

for each $\alpha \in [0, 2\pi], \varepsilon > 0$ and each positive solution u of (7).

Proof. Integrating (7) over K, this follows from $\int_0^{2\pi} f \sin(x-\alpha)dx = 0$.

Using Lemma 3, 4 and 5, we can obtain a positive lower bound for solutions of (7).

Theorem 3. There exists $\delta = \delta(|f|_\infty, |f_t|_\infty) > 0$ such that $\min_K u \geq \delta$ for any $\varepsilon > 0$ and any positive solution u of (7).

Proof. On the contrary, assume that there are sequences $\varepsilon_n \to 0$ and $\{u_n\}$ for which u_n is a solution of (7) with $\varepsilon = \varepsilon_n$ such that $\min_K u_n \to 0$ as $n \to \infty$. We easily see $\max_K u_n \geq \min_K f - \frac{1}{m}$ for all n. Put $U_n(x) = \int_0^{2\pi} u_n(x,t)dt$ for $x \in [0, 2\pi]$. Integrating (11) over $(s-T, t)$ and $(t, t+T)$ respectively, we have $C_5 = C_5(|f|_\infty, |f_t|_\infty) > 0$ and $C_6 = C_6(|f|_\infty, |f_t|_\infty) > 0$ such that

$$C_5(U_n(x) + T\varepsilon_n^2) \leq u_n(x,t) + \varepsilon_n^2 \leq C_6(U_n(x) + T\varepsilon_n^2) \tag{12}$$

for all $(x,t) \in K$ and n . Therefore it holds that

$$\max_K U_n \geq \frac{1}{C_6}(\min_K f - \frac{1}{m} + \varepsilon_n^2) - T\varepsilon_n^2$$

for all n. Multiplying (7) with $\frac{1}{(u_n + \varepsilon_n^2)^2}$ and integrating over $(0, T)$, there is $C_7 = C_7(|f|_\infty, |f_t|_\infty) > 0$ such that

$$0 \leq U_{nxx} + U_n \leq C_7$$

for all $x \in (0, 2\pi)$ and n. By $|U_{nxx}|_\infty \leq C_7 + MT$ for each n, we may assume that U_n converges strongly to some U in $C^1([0, 2\pi])$. Then we get $U \geq 0$, $U \not\equiv 0$ and $U_{xx} + U \geq 0$. Letting $U_n(x_n) = \min_{x \in [0,2\pi]} U_n(x)$, it follows that $U_n(x_n) \to 0$ from (12). Since we may suppose that x_n converges to some x_0, we see $U(x_0) = 0$ and $U_x(x_0) = 0$. Take $\beta > 0$ such that $U(x_0 + \beta) = U_x(x_0 + \beta) = 0$ and $U > 0$ in $(x_0, x_0 + \beta)$. According to Lemma 4, we have $\beta > \pi$. Since $\{u_n\}$ is bounded in $H^1(K)$, we may assume that u_n converges to some u a.e. in K. It is immediate that $U(x) = \int_0^T u(x,t)dt$ and $u(x,t) > 0$ a.e. in $(x_0, x_0+\beta) \times (0,T)$ from (12). Taking $0 < \sigma < \beta - \pi$, there is $\rho > 0$ such that $U(x) \geq 2\rho$ for all $x \in [x_0+\sigma, x_0+\sigma+\pi]$. Therefore $U_n(x) \geq \rho$ for all $x \in [x_0+\sigma, x_0+\sigma+\pi]$ and sufficiently large n. Since $u_n(x,t) \geq C_5\rho - \varepsilon_n^2$ in $[x_0+\sigma, x_0+\sigma+\pi] \times [0,T]$ by (12), there is $C_8 = C_8(|f|_\infty, |f_t|_\infty) > 0$ such that

$$|\int_0^T \int_{x_0+\sigma}^{x_0+\sigma+\pi} \{\frac{u_n^2}{(u_n+\varepsilon_n^2)^2} \cdot \frac{\varepsilon_n}{\xi_{\varepsilon_n}(u_n)} + (1 - \frac{u_n^2}{(u_n+\varepsilon_n^2)^2})f\} \sin(x - (x_0+\sigma))dxdt| \leq C_8\varepsilon_n \tag{13}$$

for sufficiently large n. On the other hand, it holds that

$$U_n(x) \leq U_n(x_n) + C_9(x - x_n)^2$$

for all x, where $C_9 = (C_7 + M)T$. Letting $n \to \infty$, we get

$$U(x) \leq C_9(x - x_0)^2$$

and hence

$$u(x,t) \leq C_6C_9(x - x_0)^2$$

for all $(x,t) \in K$. Consequently, it holds that

$$\begin{aligned}
&\limsup_{n\to\infty} \frac{1}{\varepsilon_n} \int_{x_0+\sigma-\pi}^{x_0+\sigma} \int_0^T \{\frac{u_n^2}{(u_n+\varepsilon_n^2)^2} \cdot \frac{\varepsilon_n}{\xi_{\varepsilon_n}(u_n)} + (1 - \frac{u_n^2}{(u_n+\varepsilon_n^2)^2})f\} \\
&\qquad\qquad \cdot \sin(x - (x_0+\sigma))dxdt \\
\leq\ & -\sin(\frac{\sigma}{2}) \liminf_{n\to\infty} \frac{1}{\varepsilon_n} \int_{x_0}^{x_0+\sigma/2} \int_0^T \{\frac{1}{\max(u_n, m\varepsilon_n)} \cdot \frac{u_n^2}{(u_n+\varepsilon_n^2)^2}\}dxdt \\
\leq\ & -\sin(\frac{\sigma}{2}) \int_{x_0}^{x_0+\sigma/2} \int_0^T \frac{1}{u} dxdt \\
=\ & -\infty.
\end{aligned}$$

This inequality and (13) contradict Lemma 5. This completes the proof.

4 Proof of the Main theorem.

We take $b_\varepsilon > 0$ satisfying

$$b_\varepsilon s + \frac{s^2}{(s+\varepsilon^2)^2}(s + \frac{\varepsilon}{\xi_\varepsilon(s)} - f) \geq 0 \quad \text{for all } s > 0.$$

The following result is obtained in (see [12]).

Lemma 6. For any $v \in C(K)$, there is the unique solution $u \in \bigcap_{p>1} W_p^{2,1}(K)$ of

$$u_t = (u+\varepsilon^2)^2(u_{xx} - b_\varepsilon u + v) \quad \text{in } K. \tag{14}$$

Furthermore the operator S associating the solution u of (14) with v is compact from $C(K)$ into itself.

We define two functions ϕ and $\tilde{\phi}$ by

$$\phi(s) = \begin{cases} b_\varepsilon s + \dfrac{s^2}{(s+\varepsilon^2)^2}(s + \dfrac{\varepsilon}{\xi_\varepsilon} - f) & \text{for } s \geq 0 \\ 0 & \text{for } s < 0 \end{cases}$$

and

$$\tilde{\phi}(s) = \begin{cases} b_\varepsilon s + \dfrac{s^2}{(s+\varepsilon^2)^2} s + \beta) & \text{for } s \geq 0 \\ \beta & \text{for } s < 0. \end{cases}$$

We calculate degrees of $I - S \circ \phi$ in a small and a large ball in $C(K)$ and then show that the degree in the large ball exsept for the small ball is not zero. This argument was used for a semilinear parabolic equation with superlinear nonlinearity in [6] and [7].

Lemma 7. There is $r > 0$ such that $\deg\,(I - S \circ \phi, B_r(0), 0) = 1$, where $B_r(0)$ denotes the open ball with radius r centered at 0 in $C(K)$.

Proof. We first see that there is $r > 0$ such that $\max_K u \geq 2r$ for each $\varepsilon > 0, \tau \in [0,1]$ and each fixed point u of $S \circ (\tau\phi)$. In fact, any fixed point u of $S \circ (\tau\phi)$ satisfies

$$u_t = (u+\varepsilon^2)^2(u_{xx} + (\tau - 1)b_\varepsilon u + \tau \frac{u^2}{(u+\varepsilon^2)^2}(u + \frac{\varepsilon}{\xi_\varepsilon(u)} - f)) \quad \text{in } K \tag{15}$$

by the maximum principle. Suppose that $\max_K u_n \to 0$ for some $\varepsilon_n \to 0$, $\tau_n \in [0,1]$ and fixed points u_n of $S \circ (\tau_n \phi)$ with $\varepsilon = \varepsilon_n$. Multiplying (15) with $\dfrac{1}{(u_n + \varepsilon_n^2)^2}$ and integrating over K, we have a contradiction. Therefore there exists $r > 0$ such that $\max_K u \geq 2r$ for all $\varepsilon > 0, \tau \in [0,1]$ and any fixed point u of $S \circ (\tau\phi)$. According to the homotopy invariance of the Leray-Schauder degree, we obtain $\deg\,(I - S \circ \phi, B_r(0), 0) = 1$.

Lemma 8. There is $R > r$ such that $\deg\,(I - S \circ \phi, B_R(0), 0) = 0$.

Proof. Choose $R > M$, where M is an upper bound obtained in Theorem 2. By Lemma 2, there are no fixed points of $S \circ (\tau\phi + (1-\tau)\tilde{\phi})$ on the boundary of $B_R(0)$ for all $\varepsilon > 0$ and $\tau \in [0,1]$. We also observe that $\deg\,(I - S \circ \tilde{\phi}, B_R(0), 0) = 0$ since $I - S \circ \tilde{\phi}$ has no fixed points in $C(K)$. From the homotopy invariance of the Leray-Schauder degree, the assertion of this lemma follows.

By Lemma 7 and 8, it holds that

$$\deg\,(I - S \circ \phi, B_R(0) \backslash B_r(0), 0) = -1.$$

Therefore the approximate equation (7) has a positive solution u_ε for each $\varepsilon > 0$.

Now we can prove our main theorem under the above preparation.

Proof of Theorem 1. Since $\{u_\varepsilon\}$ has an upper and a positive lower bound by Theorem 2 and 3, we may assume that $\{u_\varepsilon\}$ weakly converges to some u in $W_p^{2,1}(K)$ with $p > 3$. Then u is a positive solution of (1). It remains to show that u satisfies the constraint (3). Since $\{u_\varepsilon\}$ is bounded away from zero, the equation (7) is written as

$$u_{\varepsilon t} = (u_\varepsilon + \varepsilon^2)^2 \Big(u_{\varepsilon xx} + \frac{u_\varepsilon^2}{(u_\varepsilon + \varepsilon^2)^2}\Big(u_\varepsilon + \frac{\varepsilon}{u_\varepsilon + \varepsilon^2} - f\Big)\Big) \quad \text{in } K.$$

Multiplying this equation with $\dfrac{\sin x}{(u_\varepsilon + \varepsilon^2)^2}$ and integrating over $(0, 2\pi)$, we have

$$-\frac{d}{dt} \int_0^{2\pi} \frac{\sin x}{u_\varepsilon + \varepsilon^2} dx = \varepsilon \int_0^{2\pi} \frac{\sin x}{u_\varepsilon + \varepsilon^2} dx + v_\varepsilon(t),$$

where

$$v_\varepsilon(t) = \int_0^{2\pi} \{(\frac{u_\varepsilon^2}{(u_\varepsilon+\varepsilon^2)^2} - 1)u_\varepsilon + (1 - \frac{u_\varepsilon^2}{(u_\varepsilon+\varepsilon^2)^2})f\} \sin x dx.$$

Then there is $C = C(|f|_\infty) > 0$ such that $|v_\varepsilon(t)| \leq C\varepsilon^2$ for all t. Therefore we obtain $|\int_0^{2\pi} \frac{\sin x}{u_\varepsilon + \varepsilon^2} dx| \leq C\varepsilon$ for all t. Letting $\varepsilon \to 0$, we see u satisfies the condition (3).

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