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On the nonlinear degenerate elliptic PDEs with obstacles

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§1. Introduction

This article is a part of [10]. We are concerned with the nonlinear degenerate elliptic partial differential equations (PDEs) with obstacles.

Let $\Omega \subset \mathbb{R}^N$ be a bounded domain. We consider the following elliptic PDE:

$$(1.1) \qquad \begin{cases} \max\{u + F(x, Du, D^2u), u - \psi\} = 0 & \text{in } \Omega, \\ u = g & \text{on } \partial\Omega. \end{cases}$$

Here F is the Hamilton-Jacobi-Bellman operator:

$$F(x,p,X) = \sup_{\alpha\in\Lambda}\{-tr({}^t\sigma(x,\alpha)\sigma(x,\alpha)X) + \langle b(x,\alpha), p\rangle - f(x,\alpha)\},$$

where Λ is a compact metric space and trA and ${}^tA$ denote, respectively, the trace and the transposed matrix of A. The problem (1.1) is derived from the optimal stopping problems for diffusion processes. See [2] for more backgrounds.

It is easily seen by a simple example that, in general, the problem (1.1) has no classical solution.

In the case F is nondegenerate, we obtained the existence and uniqueness of solutions satisfying the boundary condition in the classical sense. By [11] and [17] there exists a unique solution of (1.1) in $W^{2,\infty}(\Omega) \cap C(\overline{\Omega})$. Applying the results in [4], we have a unique viscosity solution of (1.1).

However, in the case F is degenerate, especially on $\partial\Omega$, we cannot interpret the boundary condition in (1.1) in the classical sense. In [6] H. Ishii pointed out that in this case we should interpret the boundary condition in the "viscosity sense", which is naturally derived from the dynamic programming principle in the optimal control theory. Moreover he obtained the comparison principle and existence of viscosity solutions for Hamilton-Jacobi equations. Recently In [12], [13] M. A. Katsoulakis have proved the ones for second order degenerate elliptic PDEs without obstacles.

Our main aim here is to discuss the uniqueness and existence of viscostiy solutions of (1.1) and to apply them to the implicit boundary value problems. Since we consider the case F is degenerate, we interpret the boundary condition in the viscosity sense.

This article is organized in the following way. In Section 2 we state our assumptins and recall the notion of viscosity solutions of (1.1). In Section 3 we prove the comparison principle of viscosity solutions of (1.1). Section 4 is devoted to the existence of viscosity solutions of (1.1). In Section 5 we treat some implicit boundary value problems.

In what follows we surpress the term "viscosity" since we are mainly concerned with viscocity sub-, super- and solutions.

§2. Preliminaries

In this section we state our assumptions and give the definition of solutions of (1.1). We make the following assumptions.

(A.1) $\Omega \subset \mathbb{R}^N$ is a bounded domain with the smooth boundary $\partial\Omega$.

(A.2) $\sup_{\alpha\in\Lambda}\left\{\|\sigma(\cdot,\alpha)\|_{W^{1,\infty}(\overline{\Omega})}, \|b(\cdot,\alpha)\|_{W^{1,\infty}(\overline{\Omega})}, \|f(\cdot,\alpha)\|_{C(\overline{\Omega})}\right\} = K < +\infty.$

(A.3) $\psi,\ g \in C(\overline{\Omega})$ and $\psi \geqq g$ on $\partial\Omega$.

(A.4) For each $z \in \partial\Omega$, there exist $\alpha = \alpha(z) \in \Lambda$ satisfying

(i) $\alpha(\cdot) \in W^{1,\infty}(\overline{\Omega})$,

(ii) $tr({}^t\sigma(z,\alpha(z))\sigma(z,\alpha(z))D^2\rho(z)) - \langle b(z,\alpha(z)), D\rho(z)\rangle \geqq \eta$ for some $\eta > 0$,

(iii) $\langle {}^t\sigma(z,\alpha(z))\sigma(z,\alpha(z))D\rho(z), D\rho(z)\rangle = 0$,

(iv) There are unit vectors $\{\hat{e}_l\}_{1\leqq l\leqq N-1} \subset \mathbb{R}^N$ by which the tangent plane at z is spanned such that

$$\langle {}^t\sigma(z,\alpha(z))\sigma(z,\alpha(z))\hat{e}_l, \hat{e}_l\rangle = 0$$

except at most two vectors $\{\hat{e}_{l_1}, \hat{e}_{l_2}\}$.

(A.5) For each $z \in \partial\Omega$, there exist a constant $\eta > 0$ and $\beta = \beta(z) \in \Lambda$ satisfying

(i) $\beta(\cdot) \in W^{1,\infty}(\overline{\Omega})$,

(ii) $tr({}^t\sigma(z,\beta(z))\sigma(z,\beta(z))D^2\rho(z)) - \langle b(z,\beta(z)), D\rho(z)\rangle \leqq -\eta$

or

$\langle {}^t\sigma(z,\beta(z))\sigma(z,\beta(z))D\rho(z), D\rho(z)\rangle \geqq \eta.$

Remark 2.1. As to the assumption (A.4), see [12] and [13].

Next we give the definition of solutions of (1.1) and the equivalent proposition. For any function $u : \overline{\Omega} \to \mathbb{R}$, u^* and u_* denote, respectively, the upper semicontinuous (u.s.c.) envelope and the lower semicontinuous (l.s.c.) envelope of u:

$$u^*(x) = \lim_{r\to 0} \sup\{u(y) \mid |y-x| < r,\ y \in \overline{\Omega}\},$$
$$u_*(x) = \lim_{r\to 0} \inf\{u(y) \mid |y-x| < r,\ y \in \overline{\Omega}\}.$$

We define $J^{2,+}_{\overline{\Omega}}u(x)$, $J^{2,-}_{\overline{\Omega}}u(x)$ by

$$J^{2,+}_{\overline{\Omega}}u(x) = \left\{(p,X) \in \mathbb{R}^N \times \mathbb{S}^N \;\middle|\; u(x+h) \leqq u(x) + \langle p,h\rangle + \frac{1}{2}\langle Xh,h\rangle + o(|h|^2) \quad \text{as } x+h \in \overline{\Omega} \text{ and } h \to 0\right\},$$

$$J^{2,-}_{\overline{\Omega}}u(x) = \left\{(p,X) \in \mathbb{R}^N \times \mathbb{S}^N \;\middle|\; u(x+h) \geqq u(x) + \langle p,h\rangle + \frac{1}{2}\langle Xh,h\rangle + o(|h|^2) \quad \text{as } x+h \in \overline{\Omega} \text{ and } h \to 0\right\}.$$

It is observed that if $(p,X) \in J^{2,+}_{\overline{\Omega}}u(x)$ (resp., $\in J^{2,-}_{\overline{\Omega}}u(x)$), then there exists a function $\varphi \in C^2(\overline{\Omega})$ such that $u - \varphi$ takes a local maximum (resp., a local minimum) at x and $(D\varphi(x), D^2\varphi(x)) = (p, X)$. Conversly it is seen that, for any $\varphi \in C^2(\overline{\Omega})$, if $u - \varphi$ takes a local maximum (resp., local minimum) at $x \in \overline{\Omega}$, then $(D\varphi(x), D^2\varphi(x)) \in J^{2,+}_{\overline{\Omega}} u(x)$ (resp., $\in J^{2,-}_{\overline{\Omega}} u(x)$).

Definition 2.2. *Let $u : \overline{\Omega} \to \mathbb{R}$.*

(1) *We say u is a subsolution of (1.1) if $u^* < +\infty$ on $\overline{\Omega}$ and for all $x \in \overline{\Omega}$ and $(p, X) \in J^{2,+}_{\overline{\Omega}} u^*(x)$, u^* satisfies*

$$\max\{u^*(x) + F(x, p, X), u^*(x) - \psi(x)\} \leqq 0 \quad (x \in \Omega),$$

$$u^*(x) \leqq g(x) \text{ or } \max\{u^*(x) + F(x, p, X), u^*(x) - \psi(x)\} \leqq 0 \quad (x \in \partial\Omega).$$

(2) *We say u is a supersolution of (1.1) if $u_* > -\infty$ on $\overline{\Omega}$ and for all $x \in \overline{\Omega}$ and $(p, X) \in J^{2,-}_{\overline{\Omega}} u_*(x)$, u_* satisfies*

$$\max\{u_*(x) + F(x, p, X), u_*(x) - \psi(x)\} \geqq 0 \quad (x \in \Omega),$$

$$u_*(x) \geqq g(x) \text{ or } \max\{u_*(x) + F(x, p, X), u_*(x) - \psi(x)\} \geqq 0 \quad (x \in \partial\Omega).$$

(3) *We say u is a solution of (1.1) if u is both a sub- and a supersolution of (1.1).*

Next we mention the equivalent proposition to Definition 2.2. $\bar{J}^{2,+}_{\overline{\Omega}} u(x)$, $\bar{J}^{2,-}_{\overline{\Omega}} u(x)$ are the graph closure of $J^{2,+}_{\overline{\Omega}} u(x)$, $J^{2,-}_{\overline{\Omega}} u(x)$, respectiverly.

Proposition 2.3. *Assume (A.2) and (A.3). Let $u : \overline{\Omega} \to \mathbb{R}$.*

(1) *u is a subsolution of (2.1) if and only if $u^* < +\infty$ on $\overline{\Omega}$ and for all $x \in \overline{\Omega}$ and $(p, X) \in \bar{J}^{2,+}_{\overline{\Omega}} u^*(x)$, u^* satisfies*

$$\max\{u^*(x) + F(x, p, X), u^*(x) - \psi(x)\} \leqq 0 \quad (x \in \Omega),$$

$$u^*(x) \leqq g(x) \text{ or } \max\{u^*(x) + F(x, p, X), u^*(x) - \psi(x)\} \leqq 0 \quad (x \in \partial\Omega).$$

(2) *u is a supersolution of (2.1) if and only if $u_* > -\infty$ on $\overline{\Omega}$ and for all $x \in \overline{\Omega}$ and $(p, X) \in \bar{J}^{2,-}_{\overline{\Omega}} u_*(x)$, u_* satisfies*

$$\max\{u_*(x) + F(x, p, X), u_*(x) - \psi(x)\} \geqq 0 \quad (x \in \Omega),$$

$$u_*(x) \geqq g(x) \text{ or } \max\{u_*(x) + F(x, p, X), u_*(x) - \psi(x)\} \geqq 0 \quad (x \in \partial\Omega).$$

We omit the proofs of the above proposition. See [4; Section 7].

§3. Comparison principle of solutions

In this section we prove the comparison principle of solutions of (1.1). To do so, we use the similar techniques seen in [6], [4], [12] and [14] etc.

We note that, by (A.1) there exists a small constant $r_0 > 0$ such that, for any $z \in \partial\Omega$,

$$K_y = y + \cup_{0<s<r_0} B(sn(z), s) \subset \Omega, \text{ for all } y \in B(z, r_0) \cap \overline{\Omega}, \tag{3.1}$$

where $-n(z)$ is the outward unit normal to Ω at $x \in \partial\Omega$ and $B(x, r)$ denotes the open ball centered at x with radius r.

Theorem 3.1. *Assume* (A.1)-(A.5) *hold. Let u and v, be, respectively, a subsolution and a supersolution of* (1.1). *If any one of the followings holds, then $u^* \leqq v_*$ on $\overline{\Omega}$.*

(1) $\limsup_{K_z \ni x \to z} u^*(x) = u^*(z)$ *and* $\liminf_{K_z \ni x \to z} v_*(x) = v_*(z)$ *for each* $z \in \partial\Omega$.

(2) $\limsup_{K_z \ni x \to z} u^*(x) = u^*(z)$ *and* $u^*(z) \leqq g(z)$ *for each* $z \in \partial\Omega$.

(3) $\liminf_{K_z \ni x \to z} v_*(x) = v_*(z)$ *and* $v_*(z) \geqq g(z)$ *for each* $z \in \partial\Omega$.

Remark 3.2. We call the properties in Theorem 3.1 (1) nontangential upper- and lower semicomtinuity, respectively. See [12], [13].

Proof of Theorem 3.1. We may consider that u and v are, respectively, u.s.c. and l.s.c. on $\overline{\Omega}$. First let the condition (1) hold.

We suppose $\sup_{\overline{\Omega}}(u - v) = \theta > 0$ and get a contradiction. We may assume $u - v$ takes its strict maximum at $z \in \overline{\Omega}$, because, if otherwise, we can make it do so by using some perturbation techniques.

We divide our consideration into three cases.

Case 1. $z \in \partial\Omega$ and $v(z) < g(z)$.

Let $\{z_n\}_{n\in\mathbb{N}} \subset K_z$ be a sequence such that

$$z_n \to z, \quad u^*(z_n) \to u^*(z) \qquad (n \to +\infty).$$

We define the function $\Phi(x,y)$ on $\overline{\Omega} \times \overline{\Omega}$ by

$$\Phi(x,y) = u(x) - v(y) - \frac{\alpha_n}{2}|x - y - z_n + z|^2,$$

where $\alpha_n = s_0^2/|z_n - z|^2$ and $s_0 > 0$ satisfies $(3K^2 + K)s_0^2 < \theta$.

Let $(x_n, y_n) \in \overline{\Omega} \times \overline{\Omega}$ be a maximum point of Φ. Calculating as in [4; Section 3] we obtain the behaviors of x_n, y_n, $u(x_n)$, $v(y_n)$ as $n \to +\infty$:

$$\text{(3.4)} \qquad \begin{cases} x_n,\ y_n \to z,\ u(x_n) \to u(z),\ v(y_n) \to v(z), \\ \alpha_n|x_n - y_n - z_n + z|^2 \to 0,\ \sqrt{\alpha_n}|x_n - y_n| \to s_0. \end{cases}$$

We apply the maximum principle for semicontinuous functions to obtain X, $Y \in \mathbb{S}^N$ satisfying

$$(p_n, X) \in \bar{J}^{2,+}_{\overline{\Omega}} u(x_n),\ (p_n, Y) \in \bar{J}^{2,+}_{\overline{\Omega}} v(y_n),$$

and

$$\text{(3.5)} \qquad -3\alpha_n \begin{pmatrix} I & O \\ O & I \end{pmatrix} \leqq \begin{pmatrix} X & O \\ O & -Y \end{pmatrix} \leqq 3\alpha_n \begin{pmatrix} I & -I \\ -I & I \end{pmatrix},$$

where $p_n = \alpha_n(x_n - y_n - z_n + z)$.

We may consider $x_n \in \Omega$ for sufficiently large $n \in \mathbb{N}$ because (3.4) implies $|x_n - y_n - z_n + z| < r_0|z_n - z|$ for large $n \in \mathbb{N}$, where $r_0 > 0$ is the same constant as in (3.1). Moreover we have $v(y_n) < g(y_n)$ for large $n \in \mathbb{N}$ by (A.3), (3.4) and $v(z) < g(z)$. Hence using the fact that u and v are, respectively, a subsolution and a supersolution of (1.1), we obtain the following inequalities:

$$\max\{u(x_n) + F(x_n, p_n, X), u(x_n) - \psi(x_n)\} \leqq 0,$$

$$\max\{v(y_n) + F(y_n, p_n, Y), v(y_n) - \psi(y_n)\} \geqq 0.$$

By (A.2), (A.3) and (3.5) we calculate

$$\begin{aligned}\theta &\leqq u(x_n) - v(y_n)\\ &\leqq \max\{F(y_n, p_n, Y) - F(x_n, p_n, X), \psi(y_n) - \psi(x_n)\}\\ &\leqq \max\{(3K^2 + K)\alpha_n|x_n - y_n|^2 + \omega(|x_n - y_n|), \omega(|x_n - y_n|)\},\end{aligned}$$

where ω is a modulus of continuity of the functions f and ψ.

Recalling (3.4) and letting $n \to +\infty$, we obtain

$$\theta \leqq (3K^2 + K)s_0^2 < \theta,$$

which is a contradiction.

Case 2. $z \in \partial\Omega$, $u(z) > g(z)$.

As in *Case 1*, we define the function Φ by

$$\Phi(x, y) = u(x) - v(y) + \langle q, x\rangle - \frac{\alpha_n}{2}|x - y + z_n - z|^2 \qquad \text{on} \quad \overline{\Omega} \times \overline{\Omega}.$$

We can prove the remainder similarly to the above.

Case 3. $z \in \Omega$.

In this case the proof is standard. See [4; Section 3].

When the condition (2) (resp., (3)) holds, it is sufficiently to consider only *Case 2, 3* (resp., *Case 1, 3*) in the above arguments. Thus we obtain the result. ∎

§4. Existence of solutions

This section is devoted to the existence of solutions of (1.1). In doing so, the results in [12], [13] play an important role. For the case $\sigma(\cdot, \alpha) \equiv O$ ($\forall \alpha \in \Lambda$), see [6]. In the following we assume

(A.6) Λ is a compact metric space.

We prepare some notations.

W_t = standard N − dimensional Brownian motion.

$\mathcal{A} = \{\alpha_t : [0,+\infty) \to \Lambda : \text{progressively measurable}\}$.

$\mathcal{B} = \{\theta : \text{stopping time}\}$.

X_t : solution of

$$\begin{cases} dX_t = -b(X_t,\alpha_t)dt + \sqrt{2}\sigma(X_t,\alpha_t)dW_t, \ t > 0, \\ X_0 = x \in \overline{\Omega}. \end{cases}$$

$\tau = \inf\{t \geqq 0 \,|\, X_t \notin \overline{\Omega}\}$.

1_A = characteristic function for A.

Our existence result is stated as follows.

Theorem 4.1. *Assume* (A.1)-(A.6). *Then there exists a unique solution* $u \in C(\overline{\Omega})$ *of* (1.1) *and it is represented as the value function associated with the optimal stopping problem:*

$$u(x) = \inf_{\substack{\alpha\in\mathcal{A}\\ \theta\in\mathcal{B}}} E_x\left\{\int_0^{\tau\wedge\theta} f(X_t,\alpha_t)e^{-t}dt + 1_{\theta<\tau}\psi(X_\theta)e^{-\theta} + 1_{\theta\geqq\tau}g(X_\tau)e^{-\tau}\right\}.$$

To show this theorem, we consider the penalized problem for (1.1).

$$\text{(4.1)} \qquad \begin{cases} F(x,u_n,Du_n,D^2u_n) + n(u_n-\psi)^+ = 0 & \text{in } \Omega, \\ u_n = g & \text{on } \partial\Omega, \end{cases}$$

where $n \in \mathbb{N}$ and $r^+ = \max\{r, 0\}$.

Then applying the results in [13], for each $n \in \mathbb{N}$, there exists a unique solution $u_n \in C(\overline{\Omega})$ of (4.1) and it is characterized as follows:

(4.2)

$$u_n(x) = \inf_{\alpha\in\mathcal{A}} E_x\left\{\int_0^{\tau} \left(f(X_t,\alpha_t) - n(u_n(X_t)-\psi(X_t))^+\right)e^{-t}dt + g(X_\tau)e^{-\tau}\right\}.$$

Using (A.5) and the barrier argument, we have

$$\text{(4.3)} \qquad u_n \leqq g \quad \text{on} \quad \partial\Omega \quad \text{for all } n \in \mathbb{N}.$$

Since the operator nr^+ is monotone with respect to $n \in \mathbb{N}$ and $u_n \geqq -C$ for large $C > 0$, we obtain

$$(4.4) \qquad -C \leqq \cdots \leqq u_n \leqq \cdots \leqq u_2 \leqq u_1 \qquad \text{on} \quad \overline{\Omega}$$

by the comparison principle of solutions of (4.1). (cf. [4; Theorem 7.9].) Hence we can define the function u by

$$(4.5) \qquad u(x) = \lim_{n \to +\infty} u_n(x).$$

Then we get the following lemma.

Lemma 4.2. *The above function u is a u.s.c. subsolution of* (1.1).

Proof. It is easily seen that u is u.s.c. on $\overline{\Omega}$ by means of (4.4). Using (4.3) and letting $n \to +\infty$, we have $u \leqq g$ on $\partial\Omega$.

For any $\varphi \in C^2(\overline{\Omega})$, we assume that $u - \varphi$ attains a local maximum at $x_0 \in \overline{\Omega}$. We may consider $x_0 \in \Omega$ and that x_0 is a strict local maximum point of $u - \varphi$. Then there exists a $\delta > 0$ such that

$$u(x_0) - \varphi(x_0) > u(x) - \varphi(x) \qquad \text{for all } x \in \overline{B(x_0, \delta)}(\subset \Omega), \ x \neq x_0.$$

Let x_n be a maximum point of $u_n - \varphi$ on $\overline{B(x_0, \delta)}$. Then by the same argument as in G. Barles - B. Perthame [1; Lemma A.3].), we get

$$(4.6) \qquad x_n \to x_0, \quad u_n(x_n) \to u(x_0) \qquad (n \to +\infty).$$

Since u_n is a subsolution of (4.1), we obtain,

$$(4.7) \qquad F(x_n, u_n(x_n), D\varphi(x_n), D^2\varphi(x_n)) + n(u_n(x_n) - \psi(x_n))^+ \leqq 0.$$

It follows from (A.2) and (4.7) that there exists a constant $C > 0$ such that

$$n(u_n(x_n) - \psi(x_n))^+ \leqq C \qquad \text{for all } n \in \mathbb{N}.$$

Thus passing to the limit as $n \to +\infty$, we have, by (4.6)

$$u(x_0) - \psi(x_0) \leqq 0.$$

On the other hand, (4.7) implies $F(x_n, u_n(x_n), D\varphi(x_n), D^2\varphi(x_n)) \leqq 0$. Sending $n \to +\infty$, we observe

$$F(x_0, u(x_0), D\varphi(x_0), D^2\varphi(x_0)) \leqq 0.$$

Therefore we have completed the proof. ∎

According to [16; p.37], the formula (4.2) can be rewritten as the following:

$$u_n(x) = \inf_{\substack{\alpha\in\mathcal{A}\\ \theta\in\mathcal{B}}} E_x \left\{ \int_0^{\tau\wedge\theta} f(X_t, \alpha_t)e^{-t}dt + 1_{\theta<\tau}\psi_n(X_\theta)e^{-\theta} + 1_{\theta\geqq\tau} g(X_\tau)e^{-\tau} \right\},$$

where $a \wedge b = \min(a, b)$ and $\psi_n = \psi + (u_n - \psi)^+$.

Since $u \leqq \psi$ on $\overline{\Omega}$ by Lemma 4.2, we have the following lemma by (4.4) and Dini's Theorem.

Lemma 4.3. *$u_n \rightrightarrows u$ on $\overline{\Omega}$ as $n \to +\infty$ and the function u is represented as*

$$u(x) = \inf_{\substack{\alpha\in\mathcal{A}\\ \theta\in\mathcal{B}}} E_x \left\{ \int_0^{\tau\wedge\theta} f(X_t, \alpha_t)e^{-t}dt + 1_{\theta<\tau}\psi(X_\theta)e^{-\theta} + 1_{\theta\geqq\tau} g(X_\tau)e^{-\tau} \right\}.$$

We are now in a position to prove Theorem 4.1.

Proof of Theorem 4.1. We have only to show that u is a supersolution of (4.1).

For any $\varphi \in C^2(\overline{\Omega})$, we assume $u - \varphi$ takes a strict local minimum at $x_0 \in \overline{\Omega}$. We consider the case $x_0 \in \partial\Omega$. Then we may assume $u(x_0) < g(x_0)$, because, if otherwise, we have nothing to prove. Since $u \in C(\overline{\Omega})$ by Lemma 4.3, there exists a $\delta > 0$ satisfying

$$u(x) < g(x) \qquad x \in \overline{B(x_0, \delta)} \cap \partial\Omega,$$

$$u(x) < \psi(x) \qquad x \in \overline{B(x_0, \delta)} \cap \overline{\Omega}.$$

Moreover, Lemma 4.3 implies there exists a $n_0 \in \mathbb{N}$ satisfying, for all $n > n_0$,

$$u_n(x) < g(x) \qquad x \in \overline{B(x_0,\delta)} \cap \partial\Omega, \tag{4.8}$$

$$u_n(x) < \psi(x) \qquad x \in \overline{B(x_0,\delta)} \cap \overline{\Omega}. \tag{4.9}$$

Let $x_n \in \overline{B(x_0,\delta)} \cap \overline{\Omega}$ be a minimum point of $u_n - \varphi$ on $\overline{B(x_0,\delta)} \cap \overline{\Omega}$. By the same argument as in the proof of Lemma 4.1, we have

$$x_n \to x_0, \quad u_n(x_n) \to u(x_0) \qquad (n \to +\infty).$$

Therefore, using (4.8), (4.9) and the fact that u_n is a supersolution of (4.1), we obtain

$$F(x_n, u_n(x_n), D\varphi(x_n), D^2\varphi(x_n)) \geqq 0.$$

Sending $n \to +\infty$, we get

$$F(x_0, u(x_0), D\varphi(x_0), D^2\varphi(x_0)) \geqq 0.$$

Thus the proof is completed. ∎

§5. Implicit boundary value problems

In this section we apply Theorems 3.1 and 4.1 to the implicit boundary value problems.

I. *The impulse control problem.* We consider the following problem:

$$\begin{cases} \max\{u + F(x, Du, D^2u), u - Mu\} = 0 & \text{in } \Omega, \\ \max\{u - g, u - Mu\} = 0 & \text{on } \partial\Omega, \end{cases} \tag{5.1}$$

where the operator M is defind by

$$Mu(x) = \inf\{k(\xi) + u(x+\xi) \mid \xi \in (\mathbb{R}^+)^N, \ x + \xi \in \overline{\Omega}\}.$$

This problem arises in the impulse control problems for diffusion processes. For the impulse control and the related results, see [3], [19], [20] and [9] etc.

In addition to (A.1)-(A.6), we make the following assumptions.

(A.7) There exists a mapping $P : \overline{\Omega} \times (\mathbb{R}^+)^N \to (\mathbb{R}^+)^N$ satisfying

$$\begin{aligned} &x + P(x,\xi) \in \overline{\Omega} && \text{for all } (x,\xi) \in \overline{\Omega} \times (\mathbb{R}^+)^N, \\ &P(x,\xi) = \xi && \text{if } x + \xi \in \overline{\Omega}, \\ &P(\cdot,\xi) \in C(\overline{\Omega}) && \text{for each } \xi \geqq 0. \end{aligned}$$

(A.8) $k \in C((\mathbb{R}^+)^N)$ and there exists a constant $k_0 > 0$ such that $k(\xi) \geqq k_0$ for all $\xi \in (\mathbb{R}^+)^N$.

Remark 5.1. The assumption (A.7) needs to make sure that, whenever u is u.s.c. on $\overline{\Omega}$, so is Mu. See [9; Section 2].

We give the definition of solutions of (5.1).

Definition 5.2. *Let $u : \overline{\Omega} \to \mathbb{R}$.*

(1) *We say u is a subsolution of* (5.1) *if $u^* < +\infty$ on $\overline{\Omega}$ and for all $x \in \overline{\Omega}$ and $(p, X) \in J^{2,+}_{\overline{\Omega}} u^*(x)$, u^* satisfies*

$$\begin{aligned} &\max\{u^*(x) + F(x,p,X), u^*(x) - Mu^*(x)\} \leqq 0 \quad (x \in \Omega), \\ &\max\{u^*(x) - g(x), u^*(x) - Mu^*(x)\} \leqq 0 \\ &\quad \text{or } \max\{u^*(x) + F(x,p,X), u^*(x) - Mu^*(x)\} \leqq 0 \quad (x \in \partial\Omega). \end{aligned}$$

(2) *We say u is a supersolution of* (5.1) *if $u_* > -\infty$ on $\overline{\Omega}$ and for all $x \in \overline{\Omega}$ and $(p, X) \in J^{2,-}_{\overline{\Omega}} u_*(x)$, u_* satisfies*

$$\begin{aligned} &\max\{u_*(x) + F(x,p,X), u_*(x) - Mu_*(x)\} \geqq 0 \quad (x \in \Omega), \\ &\max\{u_*(x) - g(x), u_*(x) - Mu_*(x)\} \geqq 0 \\ &\quad \text{or } \max\{u_*(x) + F(x,p,X), u_*(x) - Mu_*(x)\} \geqq 0 \quad (x \in \partial\Omega). \end{aligned}$$

(3) *We say u is a solution of* (5.1) *if u is both a sub- and a supersolution of* (5.1).

We can prove the proposition equivalent to the above definition similar to that to Definition 2.2. We have the following theorem.

Theorem 5.3. *Assume* (A.1), (A.2), (A.4)-(A.8) *and* $g \in C(\overline{\Omega})$. *Then there exists a unique solution* $u \in C(\overline{\Omega})$ *of* (5.1).

Outline of proof. The comparson principle of solutions of (5.1) can be proved similarly to that of Theorem 3.1. Hence we show only the existence.

We may assume $f(\cdot, \alpha)$, $g \geqq 0$ on $\overline{\Omega}$. By [13] there exists a unique solution $u_0 \in C(\overline{\Omega})$ of

$$\begin{cases} F(x, u, Du, D^2u) = 0 & \text{in } \Omega, \\ u - g = 0 & \text{on } \partial\Omega. \end{cases}$$

Using Theorems 3.1 and 4.1 we can define the sequence $\{u_n\}_{n \in \mathbb{N}} \subset C(\overline{\Omega})$ inductively as follows:

$$u_n : \text{a unique solution of}$$
$$\begin{cases} \max\{F(x, u, Du, D^2u), u - Mu_{n-1}\} = 0 & \text{in } \Omega, \\ \max\{u - g, u - Mu_{n-1}\} = 0 & \text{on } \partial\Omega, \end{cases}$$

We see by Theorem 3.1 and the properties of the operator M the following estimates.

$$0 \leqq \cdots \leqq u_n \leqq \cdots \leqq u_2 \leqq u_1 \leqq u_0 \qquad \text{on } \overline{\Omega},$$

$$u_{n+1} - u_{n+2} \leqq (1 - \mu)^n \|u_0\|_{C(\overline{\Omega})} \qquad \text{on } \overline{\Omega},$$

for some $\mu \in (0, 1)$. Thus there exists a function $u \in C(\overline{\Omega})$ such that $u_n \rightrightarrows u$ on $\overline{\Omega}$. It follows from the stability of solutions and the comparison principle that u is a unique solution of (5.1). ∎

For the detail, see [10].

II. *The optimal switching problem.*

Next we treat the following system of elliptic PDEs:

$$
(5.2)\qquad \begin{cases} u=(u^1,\cdots,u^m),\ k\in\Gamma=\{1,\cdots,m\}, & \\ \max\{u^k+F^k(x,Du^k,D^2u^k),u^k-M^k[u]\}=0 & \text{in } \Omega, \\ \max\{u^k-g^k,u^k-M^k[u]\}=0 & \text{on } \partial\Omega, \end{cases}
$$

Here $m\ (>1)\in\mathbb{N}$ and F^k, M^k are defined by

$$
F^k(x,p,X)=\sup_{\alpha\in\Lambda}\{-tr({}^t\sigma^k(x,\alpha)\sigma^k(x,\alpha)X)+\langle b^k(x,\alpha),p\rangle-f^k(x,\alpha)\},
$$
$$
M^k[u](x)=\min\{u^l(x)+h^{kl}(x)\mid l\in\Gamma,\ l\neq k\}.
$$

This problem is associated with the optimal switching for diffusion processes. See [5], [18], [7], [14] and [15] for the related results.

As to the coefficients of F^k and h^{kl} (k, $l\in\Gamma$), we make the following assumptions.

(A.2)$'$ $\sup_{k\in\Gamma,\alpha\in\Lambda}\left\{\|\sigma^k(\cdot,\alpha)\|_{W^{1,\infty}(\overline{\Omega})},\|b^k(\cdot,\alpha)\|_{W^{1,\infty}(\overline{\Omega})},\|f^k(\cdot,\alpha)\|_{C(\overline{\Omega})}\right\}<+\infty$.

(A.9) $h^{kl}\in C(\overline{\Omega})$ and $h^{kl}>0$ on $\overline{\Omega}$ for k, $l=1,\cdots,m$.

Remark 5.4. The assumption (A.9) is needed to show the comparison principle of solutions of (5.2). It is called "no loop of zero cost condition". (cf. [18].)

We give the definition of solutions of (5.2). Let $u^*=(u^{1*},\cdots,u^{m*})$ and $u_*=(u_*^1,\cdots,u_*^m)$.

Definition 5.5. *Let $u:\overline{\Omega}\to\mathbb{R}^m$.*

(1) *We say u is a subsolution of (5.2) if $u^*<+\infty$ on $\overline{\Omega}$ and for all $k\in\Gamma$, $x\in\overline{\Omega}$ and $(p,X)\in J^{2,+}_{\overline{\Omega}}u^{k*}(x)$,*

$$
\max\{u^{k*}(x)+F^k(x,p,X),u^{k*}(x)-M^k[u^*](x)\}\leqq 0\quad (x\in\Omega),
$$
$$
\max\{u^{k*}(x)-g^k(x),u^{k*}(x)-M^k[u^*](x)\}\leqq 0
$$
$$
\text{or }\max\{u^{k*}(x)+F^k(x,p,X),u^{k*}(x)-M^k[u^*](x)\}\leqq 0\quad (x\in\partial\Omega).
$$

(2) *We say u is a supersolution of* (5.2) *if $u_* > -\infty$ on $\overline{\Omega}$ and for all $k \in \Gamma$, $x \in \overline{\Omega}$ and $(p, X) \in J^{2,-}_{\overline{\Omega}} u^k_*(x)$,*

$$\max\{u^k_*(x) + F^k(x,p,X), u^k_*(x) - M^k[u_*](x)\} \geqq 0 \quad (x \in \Omega),$$
$$\max\{u^k_*(x) - g^k(x), u^k_*(x) - M^k[u_*](x)\} \geqq 0$$
$$\text{or } \max\{u^k_*(x) + F^k(x,p,X), u^k_*(x) - M^k[u_*](x)\} \geqq 0 \quad (x \in \partial\Omega).$$

(3) *We say u is a solution of* (5.2) *if u is both a sub- and a supersolution of* (5.2).

The equivalent proposition to the above defintion can be shown similarly to that of Definition 2.2. Then we obtain the following theorem.

Theorem 5.6. *Assume* (A.1), (A.2)$'$, (A.4)-(A.6) *and* (A.9) *and $g^k \in C(\overline{\Omega})$ ($\forall k \in \Gamma$). Then there exists a unique solution $u \in C(\overline{\Omega})^m$ of* (5.2).

The strategy of the proof is similar to that of Theorem 5.3. Thus we leave it to the reader.

Remark 5.7. In [8], [14] and [15] we discussed the problem (5.2) from the viewpoint of monotone systems. As to the existence of solutions of (5.2), Theorem 5.4 provides another proof.

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