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THE RATE OF CONVERGENCE OF A HOMOGENEOUS MARKOV CHAIN ARISING FROM TWO-QUEUE NETWORKS

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1. Introduction. This paper is concerned with the type of random walks in two dimensions considered by Malyshev [4], which describe a wide class of two-queue systems. This type of random walks have been studied since then in many articles (e.g., [1], [2], [3], [5], [8] and [12]). These papers are mainly concerned with necessary and sufficient conditions for ergodicity, null recurrence and transience of the random walks. In the present paper, we consider the rate of convergence of the random walks when they are ergodic.

In order to define this class of random walks for our convenience, we make use of Baccelli's formulation as follows (see [1]). The state space of the random walks is the lattice in the positive quarter plane $N \times N = \{x = (x^1, x^2) : x^i, i = 1, 2,$ are nonnegative integers$\}$ and has to be subdivided into the following non-overlapping regions:

D_0 is the origin $\{0\} \times \{0\}$,

D_1 $\left(D_2\right)$ denotes the open x-axis $N^* \times \{0\}$ $\left(\text{or y-axis } \{0\} \times N^*\right)$,

D_3 denotes the open positive quarter lattice: $N^* \times N^*$,

where N^* denotes the set of positive integers. There is a sequence of i.i.d. $Z \times Z$ -valued random variables $\{Y_n^i\}_{n\geq 1}$ associated with the region D_i: the random variable $Y_n^i = (\xi_n^i, \eta_n^i)$ is an increment to be added at time $n + 1$ if the current locatin is in D_i. We restrict the range of the random variables Y_n^i in order to keep the random walk in $N \times N$:

the random variables Y_n^0 belong to $N \times N$,

the random variables Y_n^1 $\left(\text{or } Y_n^2\right)$ to $\{-1, 0, 1, \ldots\} \times N$ $\left(\text{or } N \times \{-1, 0, 1, \ldots\}\right)$,

the random variables Y_n^3 to $\{-1, 0, 1, \ldots\} \times \{-1, 0, 1, \ldots\}$.

We assume that these random variables are mutually independent. We now define the random walk $\{Z_n = (z_n^1, z_n^2)\}$ in $N \times N$ by

$$(1.1)\quad \begin{cases} Z_0 = (z_0^1, z_0^2); \text{ where } Y = (z_0^1, z_0^2) \in N \times N \text{ is an independent initial random variable,} \\ Z_{n+1} = Z_n + W_{n+1}, \quad n = 0, 1, 2, \ldots \end{cases}$$

where W_{n+1} is given by

$$W_{n+1} = \sum_{i=0}^{3} I(Z_n \in D_i)\, Y_{n+1}^i .$$

Here, I(B) denotes the indicator function of a set B. We also assume that the random walk $\{Z_n\}$ is irreducible and aperiodic.

In the present paper, we shall obtain various rates of convegence results for the random walk (1.1) by modifying the method developed by Tweedie [11] and Thorisson [10] for one-dimensional random walk on R_+^1 (see also [9] for some extension).

2. Preliminaries.

First of all, we give necessary definitions on various rates of convergence for Markov chains. Let $\{X_n\}$ be a temporally homogeneous Markov chain on a countable state space S. We write

$$(2.1)\quad P^n(x, A) = P(X_n \in A | X_0 = x), \qquad x \in S, \quad A \subset S$$

for the n-step transition probabilities of $\{X_n\}$. Throughout the paper, we suppose that $\{X_n\}$ is irreducible and aperiodic.

We define $\{X_n\}$ to be ergodic if, for some probability measure π, and every $x \in S$,

$$\|P^n(x, \cdot) - \pi\| \longrightarrow 0, \qquad n \longrightarrow \infty, \tag{2.2}$$

where $\| \ \|$ denotes total variation of signed measures. Next, we define $\{X_n\}$ to be geometrically ergodic if there exists a $\rho < 1$ such that

$$\rho^{-n}\|P^n(x, \cdot) - \pi\| \longrightarrow 0, \qquad n \longrightarrow \infty, \tag{2.3}$$

for every $x \in S$. We define sub-geometric rates of convergence as follows, which are our main concern in this paper. Let Λ_0 denote the class of sequences $\psi: N \longrightarrow R_+$ which satisfy

(i) ψ is non-decreasing, with $\psi(j) \geq 2$ for all $j \geq 0$.

(ii) $\{\log \psi(j)\}/j$ is non-increasing and tends to 0 as $j \longrightarrow \infty$.

We denote by Λ the class of sequences $\psi: N \longrightarrow R_+$ for which there exists some $\psi_0 \in \Lambda_0$ such that

$$\liminf_{j\to\infty} \psi(j)/\psi_0(j) > 0, \qquad \limsup_{j\to\infty} \psi(j)/\psi_0(j) < \infty.$$

Examples of sequences $\psi \in \Lambda$ are

$$\psi(j) = j^{\alpha}(\log j)^{\beta}\exp(\gamma j^{\delta}) \vee 2 \tag{2.4}$$

for $0 < \delta < 1$ and either $\gamma > 0$ or $\gamma = 0$ and $\alpha > 0$. For $\psi \in \Lambda$, we call $\{X_n\}$ ergodic of order ψ if

$$\psi(n)\|P^n(x, \cdot) - \pi\| \longrightarrow 0, \qquad n \longrightarrow \infty, \tag{2.5}$$

for every $x \in S$.

We recall convenient criteria for sub-geometric rate of convergence in the following proposition, which is adapted from a more general case to the countable one.

Proposition 1 (Theorem 2.7 of [7] and Theorem 1 (iii) of [11]). If B is a finite subset of S and for some $\psi \in \Lambda$, $E_x[\psi^o(\tau_B)] < \infty$ for $x \in B$ where $\psi^o(n) = \Sigma_{j=1}^n \psi(j)$, and $E_y[\psi(\tau_B)] < \infty$ for all y, then $\{X_n\}$ is ergodic of order ψ. Here, we denote by τ_B the first hitting time on a set B for the Markov chain $\{X_n\}$.

From now on, for our convenience, we also denote by Λ_0 the class of functions $\psi:R_+ \longrightarrow R_+$ which satisfy both (i) and (ii) for any $t \geq 0$ instead of $j \in N$ in the above requirements for $\psi:N \longrightarrow R_+$ of Λ_0. We also make use of the similar convention for $\psi \in \Lambda$. We now introduce subclasses of functions of Λ_0 due to Thorisson [10] since we follow his approach in order to show sub-geometric rate of convergence results. The functions $\psi:[0, \infty] \longrightarrow [0, \infty]$ considered below are measurable, locally bounded and $\psi(\infty) = \infty$ (i.e., $\lim_{t\to\infty}\psi(t) = \infty$). Let $\tilde{\psi}$ be defined by $\tilde{\psi}(t) = \int_0^t \psi(s)ds$. Two functions ψ and θ are of the same order if

$$\limsup_{t\to\infty} \psi(t)/\theta(t) < \infty, \qquad \limsup_{t\to\infty} \theta(t)/\psi(t) < \infty.$$

This implies that $E[\psi(Y)] < \infty$ if and only if $E[\theta(Y)] < \infty$ for any nonnegative random variable Y.

Let Ψ_0 be the class of all concave non-decreasing ψ with $\psi(0) = 0$; Φ_0 the class of all convex ψ satisfying $\psi(2t) \leq c\psi(t)$ for some $c < \infty$ and $\psi = \tilde{\theta}$ where $\theta(0) = 0$ and $\theta(t) \uparrow \infty$ as $t \to \infty$. Throughout the paper, let φ be a function of the same order as $t \longrightarrow t^m\varphi_0(t)$ where m is a nonnegative integer and $\varphi_0 \in \Psi_0$. In this case, we write $\varphi(t) \simeq t^m\varphi_0(t)$. If we define φ_j recursively by $\varphi_j = \tilde{\varphi}_{j-1}$, $j \geq 1$, then φ is also of the same order as φ_m (see Lemma 1 (b) of [10]).

Let $Y_n^i = (\xi_n^i, \eta_n^i)$, $a^i = E(\xi_n^i)$ and $b^i = E(\eta_n^i)$ for $i = 1, 2, 3$. Throughout the paper, we make use of the following assumptions with no explicit mention of them. Malyshev [4] introduced Assumption 2, under which he proved ergodicity of the Markov chain (1.1) in the case of bounded jumps (see also [12]).

Assumption 1. $E(|Y_1^i|) < \infty$ for $i = 0, 1, 2, 3$.

Assumption 2. Suppose that $(a^3)^2 + (b^3)^2 > 0$ and one of the following conditions hold:

(2.6) $a^3 \geq 0$, $b^3 < 0$, $a^3b^1 - b^3a^1 < 0$
and $(a^2, b^2) \neq (0, b^2)$ with $b^2 \geq 0$,

(2.7) $a^3 < 0$, $b^3 \geq 0$, $b^3a^2 - a^3b^2 < 0$
and $(a^1, b^1) \neq (a^1, 0)$ with $a^1 \geq 0$,

(2.8) $a^3 < 0$, $b^3 < 0$, $a^3b^1 - b^3a^1 < 0$ and $b^3a^2 - a^3b^2 < 0$.

We write E_x and P_x for expectation and probability conditional on $X_0 = x$ respectively. For the inner product between the vectors x and y in $N \times N$, we make use of the notation $x \cdot y$. The next proposition obtained by Vaninskii and Lazareva [12] is the most important tool for our rate of convergence results.

Proposition 2 (Vaninskii and Lazareva [12]). Under Assumptions 1 and 2, there exist a finite subset K of $N \times N$, a vector $v = (v^1, v^2)$ with $v^1 > 0$ and $v^2 > 0$, and an N^*-valued function $n(x)$ of $x \in N \times N$ such that $n(x) = n_i$ for $x \in D_i$ and the inequality

(2.9) $\quad E_x\{v\cdot(Z_{n(x)} - x)\} \le -\varepsilon \qquad$ for any $x \in (N \times N)\setminus K$

holds for some $\varepsilon > 0$.

3. **The rate of convergence.** In this section, we obtain the rate of convergence results under Assumptions 1 and 2. For this purpose, we make use of an embedded chain $\{\hat{Z}_n\}$ which will be constructed as in §19.1.1 of [6]. Let $s(k)$ be defined by $s(0) = 0$ and

(3.1) $\quad s(k+1) = s(k) + n(Z_{s(k)}), \qquad k = 0, 1, 2, \ldots .$

Then, by virtue of the strong Markov property, we obtain that

(3.2) $\quad \hat{Z}_k := Z_{s(k)}, \qquad k = 0, 1, 2, \ldots$

is a Markov chain with transition law $\hat{P}$, where

(3.3) $\quad \hat{P}(x, A) = P^{n(x)}(x, A), \qquad x \in N \times N, \quad A \subset N \times N.$

We let τ_A (resp. $\hat{\tau}_A$) denote the first hitting time on a set A for the Markov chain $\{Z_n\}$ (resp. $\{\hat{Z}_n\}$).

When $Z_0 = x$, we denote by U_x the increment of $\{\hat{Z}_n\}$ to be added at time 1:

(3.4) $\quad U_x = \hat{Z}_1 - x = Z_{n(x)} - x = \sum_{j=1}^{n(x)} \sum_{i=0}^{3} I(Z_{j-1} \in D_i)\, Y_j^i .$

Then, the inequality (2.9) can be rewritten in the following form: For some $\varepsilon > 0$,

(3.5) $\quad E_x(v\cdot U_x) \le -\varepsilon \qquad$ for any $x \in (N \times N)\setminus K$

holds. We define the positive integer $\hat{n}$ by $\hat{n} = \max\{n_0, n_1, n_2, n_3\}$. Making use of (3.5), we can show ergodicity and geometric ergodicity as follows.

Theorem 1. (1)(Vaninskii and Lazareva [12]) $\{Z_n\}$ is ergodic.
(2) If, for some $\lambda > 0$, $E[\exp(\lambda|Y_1^i|)] < \infty$, $i = 0, 1, 2, 3$, then

$\{Z_n\}$ is also geometrically ergodic.

Remark 1. When Assumption 2 holds, the conclusion of (1) was proved by several authors: e.g., Malyshev [4], Malyshev and Menshikov [5], Rosenkrantz [8] and Fayolle [2] under various stronger assumptions than Assumption 1. The conclusion of (2) was proved by Malyshev and Menshikov [5] in the case of bounded jumps.

We now turn to discuss sub-geometric rates of convergence of $\{Z_n\}$ by making use of the approach developed by Tweedie [11] and Thorisson [10]. The next lemma readily follows from definiton of U_x and Assumption 1.

Lemma 1. If $\psi \in \Psi_0 \cup \Phi_0$ and $E[\psi(|Y_1^i|)] < \infty$, $i = 0, 1, 2, 3$, then $\sup_x E_x[\psi(|U_x|)] < \infty$.

The next lemma is adapted for the random walk (1.1) from Lemma 4 of [10] for the one-dimensional random walk on R_+^1 .

Lemma 2. Suppose that $\psi \in \Psi_0 \cup \Phi_0$, $E\{|Y_1^i|\ \psi(|Y_1^i|)\} < \infty$, $i = 0$, 1, 2, 3, and that there exist $c > 0$ and a finite set $\hat{K} \supset K$ such that

(3.6) $\qquad E_x\{\psi(\hat{\tau}_{\hat{K}})\} \leq c\ \psi(v \cdot x) \qquad$ for all large enough x.

Then the inequality (3.6) holds with ψ replaced by $\tilde{\psi}$ and $\hat{K}$ replaced by a finite set $\tilde{K} \supset \hat{K}$.

Proposition 3. If $E\{\varphi(|Y_1^i|)\} < \infty$, $i = 0, 1, 2, 3$, then there exists a finite set $L \supset K$ such that

$$E_x\{\varphi(\hat{\tau}_L)\} < c_1\ \varphi(v\cdot x), \qquad x \in (N \times N)\setminus L \tag{3.7}$$

holds for some $c_1 > 0$, and hence

$$E_x\{\varphi(\tau_L)\} < c_2\ \varphi(v\cdot x), \qquad x \in (N \times N)\setminus L \tag{3.8}$$

holds for some $c_2 > 0$.

By virtue of Proposition 3, we can easily obtain the following main result.

Theorem 2. If $E\{\tilde{\varphi}(|Y_1^i|)\} < \infty$, $i = 0, 1, 2, 3$, then $\{Z_n\}$ is ergodic of order φ.

REFERENCES

1. F.Baccelli, *Exponential martingales and Wald's formulas for two-queue networks*, J. Appl. Prob., **23** (1986), 812-819.
2. G.Fayolle, *On random walks arising in queueing systems: ergodicity and transience via quadratic forms as Lyapounov functions – part I*, QUESTA, **5** (1989), 167-184.
3. G.Fayolle, V.A.Malyshev and M.V.Menshikov, *Random walks in a quarter plane with zero drifts. I. Ergodicity and null recurrence*, Ann. Inst. H. Poinceré, **28** (1992), 179-194.
4. V.A.Malyshev, *Classification of two-dimensional positive random walks and almost linear semimartingales*, Soviet Math. Dokl., **13** (1972), 136-139.
5. V.A.Malyshev and M.V.Menshikov, *Ergodicity, continuity and analyticity of countable Markov chains*, Trans. Moscow Math. Soc.

39 (1979), 2-48; English Translation, 1981, Issue 1.

6. S.P.Meyn and R.L.Tweedie, *Markov Chains and Stochastic Stability*, Springer-Verlag, 1993.
7. E.Nummelin and P.Tuominen, *The rate of convergence on Orey's theorem for Harris recurrent Markov chains with applications to renewal theory*, Stoch. Proc. Appl. **15** (1983), 295-311.
8. W.Rosenkrantz, *Ergodicity conditions for two-dimensional Markov chains on the positive quadrant*, Prob. Theory Rel. Fields, **83** (1989), 309-319.
9. A.Tanikawa, *On the rate of convergence of a simple Markov chain arising in discrete-time storage systems* (in preparation).
10. H.Thorisson, *The queue GI/G/1: Finite moments of the cycle variables and uniform rates of convergence*, Stoch. Proc. Appl. **19** (1985), 85-99.
11. R.L.Tweedie, *Criteria for rates of convergence of Markov chains, with application to queueing and storage theory*, in *Probability, Statistics and Analysis*, London Math. Soc. Lecture Note Series **79** (ed. by J.F.C.Kingman and G.E.H.Reuter), Cambridge University Press, 1983, 260-276.
12. K.L.Vaninskii and B.V.Lazareva, *Ergodicity and nonrecurrence conditions of a homogeneous Markov chain in the positive quadrant*, Prob. Inf. Trans., **24** (1988), 82-86.