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Dynamic Fuzzy Systems with Time Average Rewards

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Abstract

In this paper, using a fuzzy relation we define a dynamic fuzzy system with a bounded convex fuzzy reward on the positive orthant $\mathbf{R}_+^n$ of an n-dimensional Euclidean space. As a measure of the system's performance we introduce the time average fuzzy reward, which is characterized by the limiting fuzzy state under the contractive properties of the fuzzy relation. In one-dimensional case, the average fuzzy reward is expressed explicitly by the functional equations concerning the extreme points of its α-cuts. Also, a numerical example is given to illustrate the theoretical results.

Keywords : dynamic fuzzy system, time average fuzzy reward, contractive properties, fuzzy relational equation.

1. Introduction and notations

In the previous paper [3, 8, 9], we defined a dynamic fuzzy system using a fuzzy relation and gave limit theorems for the transition of fuzzy states of the system under the contractive and nonexpansive properties of the fuzzy relation. Here, the dynamic fuzzy system defined in [3, 8] will be extended to the one with a bounded fuzzy reward on the positive orthant $\mathbf{R}_+^n$ of an n-dimensional Euclidean space and the time average fuzzy reward is introduced as a measure of the system's performance and characterized by the limiting fuzzy state or by various fuzzy relational equations.

Let X and Y be convex compact subsets of some Banach space or the positive orthants $\mathbf{R}_+^n$. We denote by $\mathcal{C}(X)$ the collection of all closed convex compact subsets of X. Let ρ be the Hausdorff metric on $\mathcal{C}(X)$.

Throughout this paper we define a fuzzy set on X by its membership function $\tilde{s}: X \mapsto [0,1]$ and its α-cut by

$$\tilde{s}_\alpha := \{x \in X \mid \tilde{s}(x) \geq \alpha\} \ (\alpha > 0) \quad \text{and} \quad \tilde{s}_0 := cl\{x \in X \mid \tilde{s}(x) > 0\},$$

where cl means the closure of a set. For the details, we we refer to Novák [6] and Zadeh [11].

A fuzzy set $\tilde{s}$ on X is called convex if

$$\tilde{s}(\lambda x + (1-\lambda)y) \geq \tilde{s}(x) \wedge \tilde{s}(y) \quad \text{for any } x, y \in X \text{ and } \lambda \in [0,1],$$

where $a \wedge b = \min\{a, b\}$ for real numbers a and b. Also, a fuzzy relation $\tilde{p}$ defined on $X \times Y$ is called convex if

$$\tilde{p}(\lambda x_1 + (1-\lambda)x_2, \lambda y_1 + (1-\lambda)y_2) \geq \tilde{p}(x_1, y_1) \wedge \tilde{p}(x_2, y_2)$$

for any $x_1, x_2 \in X, y_1, y_2 \in Y$, and $\lambda \in [0,1]$.

Let $\mathcal{F}(X)$ be the set of all convex fuzzy sets $\tilde{s}$ on X, which are upper semi-continuous. Clearly $\tilde{s} \in \mathcal{F}(X)$ implies $\tilde{s}_\alpha \in \mathcal{C}(X)$ for all $\alpha \in [0,1]$.

The addition and the multiplicative operation of fuzzy sets are defined as follows (see [4]) : For any $\tilde{s}, \tilde{v} \in \mathcal{F}(X)$ and $\lambda \in \mathbf{R}_+ := [0, \infty)$,

$$(\tilde{s} + \tilde{v})(x) := \sup_{y,z \in X : y+z=x} \{\tilde{s}(y) \wedge \tilde{v}(z)\} \quad x \in X \tag{1.1}$$

and

$$(\lambda \tilde{s})(x) := \begin{cases} \tilde{s}(\frac{x}{\lambda}) & \text{if } \lambda > 0 \\ I_{\{0\}}(x) & \text{if } \lambda = 0, \end{cases} \quad x \in X, \tag{1.2}$$

where $I_A(\cdot)$ is the classical indicator function of a subset A of X.

It is easily seen that, for $\alpha \in [0,1]$,

$$(\tilde{s} + \tilde{v})_\alpha = \tilde{s}_\alpha + \tilde{v}_\alpha \quad \text{and} \quad (\lambda \tilde{s})_\alpha = \lambda \tilde{s}_\alpha,$$

where $A + B := \{x + y \mid x \in A, y \in B\}$ and $\lambda A := \{\lambda x \mid x \in A\}$ for any subsets A, B of X. The following results have appeared in Chen-wei Xu [1].

Lemma 1.1. ([1]).

(i) *For any* $\tilde{s}, \tilde{v} \in \mathcal{F}(X)$ *and* $\lambda \in [0, \infty)$,

$$\tilde{s} + \tilde{v} \in \mathcal{F}(X) \quad \textit{and} \quad \lambda \tilde{v} \in \mathcal{F}(X).$$

(ii) *Let* $\tilde{p}$ *be any lower semi-continuous convex fuzzy relation on* $X \times Y$. *Then*

$$\sup_{x \in X} \tilde{s}(x) \wedge \tilde{p}(x, \cdot) \in \mathcal{F}(Y) \quad \textit{for all } \tilde{s} \in \mathcal{F}(X).$$

Here, we give the notion of convergency for a sequence of fuzzy sets, which is used in Section 2.

Definition 1.1 ([3, 5]). Let $\{\tilde{v}_t\}_{t=0}^{\infty}$ be a sequence fuzzy sets in $\mathcal{F}(S)$. Then we write $\tilde{v}_t \to \tilde{v} \in \mathcal{F}(S)$ as $t \to \infty$ if

$$\lim_{t \to \infty} \sup_{\alpha \in [0,1]} \rho(\tilde{v}_{t,\alpha}, \tilde{v}_\alpha) = 0, \tag{1.3}$$

where $\tilde{v}_{t,\alpha}$ and $\tilde{v}_\alpha$ are α-cuts of $\tilde{v}_t$ and $\tilde{v}$ respectively.

Note that for a sequence of sets $\{A_t\}_{t=1}^{\infty} \subset \mathcal{C}(X)$ and $A \in \mathcal{C}(X)$, $\lim_{t \to \infty} A_t = A$ means that

$$\overline{\lim_{t \to \infty}} A_t = \underline{\lim_{t \to \infty}} A_t = A,$$

where

$$\overline{\lim_{t \to \infty}} A_t := \{z \in X \mid \underline{\lim_{t \to \infty}} d(z, A_t) = 0\},$$

$$\underline{\lim_{t \to \infty}} A_t := \{z \in X \mid \overline{\lim_{t \to \infty}} d(z, A_t) = 0\},$$

$$d(z, D) := \min_{z' \in D} d(z, z') \quad D \in \mathcal{C}(X)$$

and d is a metric on X. It is known ([2]) that $\lim_{t \to \infty} \rho(A_t, A) = 0$ iff $\lim_{t \to \infty} A_t = A$, so that $\tilde{v}_t$ converges to $\tilde{v}$ as $t \to \infty$ in the sense of (1.3) means that $\lim \tilde{v}_{t,\alpha} = \tilde{v}_\alpha$ uniformly for $\alpha \in [0,1]$.

Now, extending a discrete dynamic fuzzy system in [3], we consider the one with a fuzzy reward, which is characterized with the elements $(S, \tilde{q}, \tilde{r}, \tilde{s})$ as follows :

(i) The state space S is a convex compact subset of some Banach space. In general, the system is fuzzy, so that the state of the system is called a fuzzy state denoted as an element of $\mathcal{F}(S)$.

(ii) The law of the motion and the fuzzy reward for the system are denoted by the time invariant fuzzy relations $\tilde{q} : S \times S \mapsto [0,1]$ and $\tilde{r} : S \times \mathbf{R}_+^n \mapsto [0,1]$ respectively. We restrict the convex fuzzy number $\tilde{n} \in \mathcal{F}(\mathbf{R}_+^n)$ has the finite support contained in the interval $[0, M]^n \subset \mathbf{R}_+^n$, that is, $\mathcal{F}([0, M]^n) := \{\tilde{n} \in \mathcal{F}(\mathbf{R}_+^n) \mid \tilde{n}_0 \subset [0, M]^n\}$. We also assume that $\tilde{q} : S \times S \mapsto [0,1]$ and $\tilde{r} : S \times [0, M]^n \mapsto [0,1]$ are convex and continuous.

If the system is in a fuzzy state $\tilde{s} \in \mathcal{F}(S)$, a fuzzy reward $R(\tilde{s})$ is incurred and we move to a new fuzzy state $Q(\tilde{s})$, where $Q : \mathcal{F}(S) \mapsto \mathcal{F}(S)$ and $R : \mathcal{F}(S) \mapsto \mathcal{F}([0, M]^n)$ are defined by

$$R(\tilde{s})(z) := \sup_{x \in S} \tilde{s}(x) \wedge \tilde{r}(x, z) \quad z \in [0, M]^n \tag{1.4}$$

and

$$Q(\tilde{s})(y) := \sup_{x \in S} \tilde{s}(x) \wedge \tilde{q}(x, y) \quad y \in S. \tag{1.5}$$

Note that by Lemma 1.1(ii) the maps R and Q are well-defined.

(iii) The initial fuzzy state $\tilde{s} \in \mathcal{F}(S)$ is arbitrary.

For the dynamic fuzzy system $(S, \tilde{q}, \tilde{r}, \tilde{s})$, we can define a sequence of fuzzy rewards on $[0, M]^n$, $\{R(\tilde{s}_t)\}_{t=0}^{\infty}$, where

$$\tilde{s}_0 := \tilde{s} \quad \text{and} \quad \tilde{s}_{t+1} := Q(\tilde{s}_t) \quad (t \geq 0). \tag{1.6}$$

In Section 2, we define the time average fuzzy reward, which is characterized by the limiting fuzzy state under the contractive properties of of the fuzzy relation $\tilde{q}$.

In Section 3, the one-dimensional case is treated and by introducing relative value functions the average fuzzy reward is expressed by the functional equations concerning the extreme points of its α-cuts.

Also, a numerical example is given to illustrate the theoretical results in this paper.

2. The average fuzzy reward

In this paper we specify the time average reward as a measure of the system's performance and discuss its characterization under the contractive assumption given in [3].

We define the total T-time fuzzy reward $\tilde{R}_T(\tilde{s})$ by

$$\tilde{R}_T(\tilde{s}) := \sum_{t=0}^{T-1} R(\tilde{s}_t) \qquad T \geq 1, \tag{2.1}$$

where $\{\tilde{s}_t\}_{t=0}^{\infty}$ is given in (1.6).

Associated with the fuzzy relation $\tilde{q}$ and fuzzy reward $\tilde{r}$, are the corresponding maps $Q_\alpha : \mathcal{C}(S) \mapsto \mathcal{C}(S)$ $(\alpha \in [0,1])$ and $R_\alpha : \mathcal{C}(S) \mapsto \mathcal{C}([0, M]^n)$ $(\alpha \in [0,1])$ defined as follows : For $D \in \mathcal{C}(S)$,

$$Q_\alpha(D) := \begin{cases} \{y \in S \mid \tilde{q}(x, y) \geq \alpha \text{ for some } x \in D\} & \alpha > 0 \\ cl\{y \in S \mid \tilde{q}(x, y) > 0 \text{ for some } x \in D\} & \alpha = 0, \end{cases} \tag{2.2}$$

and

$$R_\alpha(D) := \begin{cases} \{z \in [0, M]^n \mid \tilde{r}(x, z) \geq \alpha \text{ for some } x \in D\} & \alpha > 0 \\ cl\{z \in [0, M]^n \mid \tilde{r}(x, z) > 0 \text{ for some } x \in D\} & \alpha = 0. \end{cases} \tag{2.3}$$

The iterates Q_α^t $(t \geq 0)$ are defined by setting $Q_\alpha^0 := I$ (identity) and iteratively,

$$Q_\alpha^t := Q_\alpha Q_\alpha^{t-1} \qquad t \geq 1.$$

We have the following lemma, which is easily verified by the ideas in the proof of [3, Lemma 1] and the property (1.1).

Lemma 2.1.

(i) $\tilde{R}_T(\tilde{s}) \in \mathcal{F}([0, TM]^n)$ *for* $T \geq 1$.

(ii) $\tilde{s}_{t,\alpha} = Q_\alpha^t(\tilde{s}_\alpha)$ *for* $t \geq 0$, *where* $\tilde{s}_{t,\alpha} = (\tilde{s}_t)_\alpha$.

(iii) $(\tilde{R}_T(\tilde{s}))_\alpha = \sum_{t=0}^{T-1} R_\alpha(\tilde{s}_{t,\alpha})$ *for* $T \geq 1$.

From Lemma 2.1(ii),(iii), $(\tilde{R}(\tilde{s}))_\alpha$ is dependent only on α, $\tilde{s}_\alpha$ and T, so that we put

$$\tilde{R}_{T,\alpha}(\tilde{s}_\alpha) := (\tilde{R}_T(\tilde{s}))_\alpha \quad \text{for } T \geq 0 \text{ and } \alpha \in [0,1].$$

From this $\tilde{R}_{T,\alpha}(\tilde{s}_\alpha)$ we try to estimate the increasing amount of fuzzy reward per unit time.

For $K > 0$ and $\alpha \in [0,1]$, we define

$$G_{K,\alpha} := \left\{ r \in \mathbf{R}^n_+ \;\middle|\; \begin{array}{l} \text{there exists } \{z_T\}_{T=1}^\infty \text{ such that} \\ z_T \in \tilde{R}_{T,\alpha}(\tilde{s}_\alpha) \text{ and } \|z_T - rT\| \leq K \text{ for all } T \geq 1 \end{array} \right\}, \tag{2.4}$$

where $\mathbf{R}^n_+$ is the positive orthant of a n-dimensional Euclidean space and $\|x\| = (\sum_{i=1}^n x_i^2)^{\frac{1}{2}}$ for $x = (x_1, x_2, \cdots, x_n) \in \mathbf{R}^n_+$. The following properties of $G_{K,\alpha}$ are formulated in a lemma.

Lemma 2.2. *Let $K > 0$. Then :*

(i) $\{G_{K,\alpha} | \alpha \in [0,1]\} \subset \mathcal{C}(\mathbf{R}^n_+)$.

(ii) $G_{K,\alpha} \subset G_{K,\alpha'}$ *for* $0 \leq \alpha' \leq \alpha \leq 1$.

(iii) $\lim_{\alpha' \uparrow \alpha} G_{K,\alpha'} = G_{K,\alpha}$ *for* $\alpha \in (0,1]$, *i.e.*, $\lim_{\alpha' \uparrow \alpha} \delta(G_{K,\alpha'}, G_{K,\alpha}) = 0$.

Proof. (ii) From Lemma 2.1(i), we have

$$\tilde{R}_{T,\alpha}(\tilde{s}_\alpha) \subset \tilde{R}_{T,\alpha'}(\tilde{s}_{\alpha'}) \quad \text{for } T \geq 0,\ 0 \leq \alpha' \leq \alpha \leq 1.$$

Therefore we obtain (ii) from the definition (2.4).

(i) Let $\alpha \in [0,1]$. We prove that $G_{K,\alpha}$ is closed and convex. Let $\{r_k\}_{k=1}^\infty$ be a sequence such that $r_k \in G_{K,\alpha}$ $(k \geq 1)$ and $r_k \to r$ $(k \to \infty)$. From (2.4), there exists a sequence $\{z_T^k\}$ such that

$$z_T^k \in \tilde{R}_{T,\alpha}(\tilde{s}_\alpha) \text{ and } \|z_T^k - r_k T\| \leq K \quad \text{for all } T \geq 1,\ k \geq 1. \tag{2.5}$$

Since $\tilde{R}_{T,\alpha}(\tilde{s}_\alpha)$ is compact, there exist a convergent subsequence of $\{z_T^k\}_{k=1}^\infty$ and its limit $z_T \in \tilde{R}_{T,\alpha}(\tilde{s}_\alpha)$. From (2.5), we obtain

$$\|z_T - rT\| \leq K \quad \text{for all } T \geq 1.$$

Therefore we get $r \in G_{K,\alpha}$ and $G_{K,\alpha}$ is closed.

Next let $r_1, r_2 \in G_{K,\alpha}$. From (2.4), there exists $z_T^1, z_T^2 \in \tilde{R}_{T,\alpha}(\tilde{s}_\alpha)$ such that

$$\|z_T^1 - r_1 T\| \leq K,\ \|z_T^2 - r_2 T\| \leq K \quad \text{for all } T \geq 1. \tag{2.6}$$

Let $T \geq 1$. Since $\tilde{R}_{T,\alpha}(\tilde{s}_\alpha)$ is convex, we have

$$\lambda z_T^1 + (1-\lambda) z_T^2 \in \tilde{R}_{T,\alpha}(\tilde{s}_\alpha) \quad \text{for all } \lambda \in [0,1]. \tag{2.7}$$

From (2.6), we have

$$\begin{aligned} &\|(\lambda z_T^1 + (1-\lambda) z_T^2) - (\lambda r_1 + (1-\lambda) r_2) T\| \\ \leq\ & \lambda \|z_T^1 - r_1 T\| + (1-\lambda)\|z_T^2 - r_2 T\| \\ \leq\ & \lambda K + (1-\lambda) K = K \quad \text{for all } \lambda \in [0,1]. \end{aligned}$$

From (2.7) and this inequality, we obtain $\lambda r_1 + (1-\lambda) r_2 \in G_{K,\alpha}$ for all $\lambda \in [0,1]$. Therefore $G_{K,\alpha}$ is convex. Thus we get (i).

(iii) Let $\alpha \in (0,1]$. It is sufficient to prove $\overline{\lim}_{\alpha' \to \alpha} G_{K,\alpha'} \subset G_{K,\alpha}$. Let $r \in \overline{\lim}_{\alpha' \to \alpha} G_{K,\alpha'}$. From (ii), there exist sequences $\{\alpha_k\}_{k=1}^\infty$ and $\{r_k\}_{k=1}^\infty$ such that

$$\alpha_k \uparrow \alpha\ (k \to \infty), \quad r_k \in G_{K,\alpha_k} (k = 1, 2, \cdots), \quad \text{and} \quad r_k \to r\ (k \to \infty).$$

From (2.4), there exists a sequence $\{z_T^k\}_{k=1}^\infty$ such that

$$z_T^k \in \tilde{R}_{T,\alpha_k}(\tilde{s}_{\alpha_k}) \text{ and } \|z_T^k - r_k T\| \leq K \quad \text{for all } T \geq 1,\ k = 1, 2, \cdots.$$

From the compactness, there exists a convergent subsequence of $\{z_T^k\}_{k=1}^{\infty}$ and its limit z_T. Then we obtain

$$\|z_T - rT\| \le K \quad \text{for all } T \ge 1. \tag{2.8}$$

Further, from Lemma 2.1(i), we have

$$\lim_{\alpha' \to \alpha} \tilde{R}_{T,\alpha'}(\tilde{s}_{\alpha'}) = \tilde{R}_{T,\alpha}(\tilde{s}_\alpha).$$

Therefore we also obtain $z_T \in \tilde{R}_{T,\alpha}(\tilde{s}_\alpha)$. Together with (2.8), we get $r \in G_{K,\alpha}$. We obtain $\overline{\lim}_{\alpha' \to \alpha} G_{K,\alpha'} \subset G_{K,\alpha}$. *q.e.d.*

From [3, Lemma 3], we can define a fuzzy number

$$\tilde{g}(\tilde{s})(r) := \sup_{\alpha \in [0,1]} \{\alpha \wedge I_{G_{K,\alpha}}(r)\} \quad r \in [0, M]^n \quad \text{for } \tilde{s} \in \mathcal{F}(S). \tag{2.9}$$

Then, $\tilde{g}(\tilde{s}) \in \mathcal{F}([0, M]^n)$ and $(\tilde{g}(\tilde{s}))_\alpha = G_{K,\alpha}$ for all $\alpha \in [0, 1]$.

We call $\tilde{g}(\tilde{s})$ *an average fuzzy reward* for the dynamic fuzzy systems, which depends on the initial fuzzy state $\tilde{s} \in \mathcal{F}(S)$ with suppression of K. In the remainder of this section, we will investigate the average fuzzy reward from the limiting behaviour of the fuzzy states. The following lemma is useful in the sequel.

Lemma 2.3. *Let $\{D_t\}_{t=1}^{\infty} \subset \mathcal{C}(S)$ and $D \in \mathcal{C}(S)$ such that $\lim_{t\to\infty} D_t = D$. Let $\alpha \in (0, 1]$. For any ϵ ($\alpha > \epsilon > 0$), there exists $T \ge 1$ such that*

$$R_{\alpha-\epsilon}(D) \supset R_\alpha(D_t) \quad \textit{for all } t \ge T.$$

Proof. Suppose that for some ϵ ($\alpha > \epsilon > 0$), there exist sequences $\{t_k\}_{k=1}^{\infty}$ and $\{z_k\}_{k=1}^{\infty}$ such that

$$t_k \to \infty \ (k \to \infty), \quad \text{and} \quad z_k \in R_\alpha(D_{t_k}) \setminus R_{\alpha-\epsilon}(D) \ (k = 1, 2, \cdots).$$

Then we have

$$\tilde{r}(x, z_k) < \alpha - \epsilon \quad \text{for all } x \in D, k = 1, 2, \cdots, \tag{2.10}$$

and there exists a sequence $\{x_k\}_{k=1}^{\infty}$ such that

$$x_k \in D_{t_k} \quad \text{and} \quad \tilde{r}(x_k, z_k) \ge \alpha \quad \text{for } k = 1, 2, \cdots. \tag{2.11}$$

From the compactness, we may assume that the sequences $\{x_k\}_{k=1}^{\infty}$ and $\{z_k\}_{k=1}^{\infty}$ are convergent. We put the limits $x^* = \lim_{k\to\infty} x_k$ and $z^* = \lim_{k\to\infty} z_k$. Then we have $x^* \in D$ since $\lim_{t\to\infty} D_t = D$. From (2.10),(2.11), we obtain

$$\tilde{r}(x^*, z^*) \ge \alpha \quad \text{and} \quad \tilde{r}(x, z^*) \le \alpha - \epsilon \quad \text{for all } x \in D.$$

It is a contradiction. Thus we get this lemma. *q.e.d.*

In order to characterizing the average fuzzy reward $\tilde{g}(\tilde{s})$, we need the following two assumptions, the first one is a contractive property concerning the fuzzy relation $\tilde{q}$ which guarantee the existence of the limiting fuzzy state and the second is a Lipschitz condition related with the fuzzy reward $\tilde{r}$.

Assumption *A* (Contraction and ergodic property).
There exists $t_0 \ge 1$ and β ($0 < \beta < 1$) satisfying that

$$\rho(Q_\alpha^{t_0}(D_1), Q_\alpha^{t_0}(D_2)) \le \beta\rho(D_1, D_2) \quad \text{for all } D_1, D_2 \in \mathcal{C}(S), \ \alpha \in [0, 1].$$

Assumption *B* (Lipschitz conditions).
There exists a constant $C > 0$ such that

$$\delta(R_\alpha(D_1), R_\alpha(D_2)) \le C\rho(D_1, D_2) \quad \text{for all } D_1, D_2 \in \mathcal{C}(S), \ \alpha \in [0, 1], \tag{2.12}$$

where δ is the Hausdorff metric on $\mathcal{C}([0, M]^n)$.

Lemma 2.4 [3, Theorem 1]

(i) *There exists a unique fuzzy state $\tilde{p} \in \mathcal{F}(S)$, which is independently of the initial fuzzy state $\tilde{s}$, satisfying*

$$\tilde{p}(y) = \max_{x \in S}\{\tilde{p}(x) \wedge \tilde{q}(x,y)\} \quad \text{for all } y \in S. \tag{2.13}$$

(ii) *For $\alpha \in [0,1]$, the α-cut $\tilde{p}_\alpha$ is a unique set of $\mathcal{C}(S)$ such that*

$$Q_\alpha(\tilde{p}_\alpha) = \tilde{p}_\alpha.$$

(iii) *Let $\alpha \in [0,1]$. It holds that*

$$\rho(Q_\alpha^t(D), \tilde{p}_\alpha) \leq \beta^{[t/t_0]} K_\alpha(D, \tilde{p}_\alpha) \quad \text{for all } D \in \mathcal{C}(S),\ t \geq 1,$$

where $K_\alpha(D, \tilde{p}_\alpha) := \sum_{l=0}^{t_0-1} \rho(Q_\alpha^l(D), \tilde{p}_\alpha)$ and, for a real number c, $[c]$ is the largest integer equal to or less than c.

Recently, Yoshida [10] has given the notion of α-recurrent set for the fuzzy relation and shown that the α-cut of the limiting fuzzy set $\tilde{p}$ in Lemma 2.4 is characterized as the maximum α-recurrent set.

Now, we can state one of main results, which shows that $\tilde{g}(\tilde{s})$ is represented using the limiting fuzzy state $\tilde{p}$.

Theorem 2.1. *Suppose that Assumptions A and B hold. For sufficient large all K, it holds that*

$$\tilde{g}(\tilde{s}) = R(\tilde{p}), \tag{2.14}$$

where $\tilde{p}$ is the limiting fuzzy state given in Leman 2.3. Further this is independent of the initial fuzzy state $\tilde{s}$.

Proof. Let $\alpha \in [0,1]$. First we show that

$$(\tilde{g}(\tilde{s}))_\alpha = G_{K,\alpha} \subset R_\alpha(\tilde{p}_\alpha) = (R(\tilde{p}))_\alpha. \tag{2.15}$$

By (2.9) and Lemma 2.1(iii), the two equality in (2.15) hold obviously. We prove $G_{K,\alpha} \subset R_\alpha(\tilde{p}_\alpha)$. Suppose that there exists $r \in G_{K,\alpha} \setminus R_\alpha(\tilde{p}_\alpha)$. Then

$$\tilde{r}(x,r) < \alpha \quad \text{for all } x \in \tilde{p}_\alpha.$$

From the continuity of $\tilde{r}$ and the compactness of $\tilde{p}_\alpha$, there exists $\epsilon > 0$ such that

$$\tilde{r}(x,r) \leq \epsilon < \alpha \quad \text{for all } x \in \tilde{p}_\alpha.$$

Therefore

$$r \notin R_{\frac{\alpha+\epsilon}{2}}(\tilde{p}_\alpha). \tag{2.16}$$

Since $R_{\frac{\alpha+\epsilon}{2}}(\tilde{p}_\alpha)$ is closed and convex, there exists a unique $z_0 \in R_{\frac{\alpha+\epsilon}{2}}(\tilde{p}_\alpha)$ such that

$$0 < \gamma := \|z_0 - r\| \leq \|z - r\| \quad \text{for all } z \in R_{\frac{\alpha+\epsilon}{2}}(\tilde{p}_\alpha). \tag{2.17}$$

From Lemma 2.1(ii) and Lemma 2.4(iii), we have

$$\tilde{s}_{t,\alpha} \to \tilde{p}_\alpha \quad (t \to \infty).$$

By Lemma 2.3, there exists $T^* > 0$ such that

$$R_{\frac{\alpha+\epsilon}{2}}(\tilde{p}_\alpha) \supset R_\alpha(\tilde{s}_{t,\alpha}) \quad \text{for all } t \geq T^*. \tag{2.18}$$

On the other hand, from $r \in G_{K,\alpha}$, there exists $\{r_T\}_{T=0}^\infty$ such that

$$r_T \in \tilde{R}_{T,\alpha}(\tilde{s}_\alpha) \text{ and } \|r_T - rT\| \leq K \text{ for all } T \geq 1. \tag{2.19}$$

From Lemma 2.1(iii), there exists a sequence $\{r_{T,t}\}$ such that

$$r_{T,t} \in R_\alpha(\tilde{s}_{t,\alpha}) \ (t = 0, 1, 2, \cdots, T-1) \text{ and } r_T = \sum_{t=0}^{T-1} r_{T,t} \quad \text{for all } T \geq 1. \tag{2.20}$$

Noting the supporting hyperplane of $R_{\frac{\alpha+\epsilon}{2}}(\tilde{p}_\alpha)$ at z_0, from (2.17),(2.18),(2.20) we have

$$\langle z_0 - r, r_{T,t} - r \rangle \geq \|z_0 - r\|^2 = \gamma^2 \quad \text{for all } t, T \ (T > t \geq T^*).$$

So

$$\left\langle z_0 - r, \sum_{t=T^*}^{T-1} (r_{T,t} - r) \right\rangle \geq (T - T^*)\gamma^2 \quad \text{for all } T > T^*,$$

where $\langle \cdot, \cdot \rangle$ is an inner product on $\mathbf{R}^n_+$: $\langle a, b \rangle = \sum_{i=1}^n a_i b_i$ for $a = (a_1, a_2, \cdots, a_n), b = (b_1, b_2, \cdots, b_n) \in \mathbf{R}^n_+$. By Cauchy-Schwartz inequality, we obtain

$$\left\| \sum_{t=T^*}^{T-1} (r_{T,t} - r) \right\| \geq (T - T^*)\gamma \quad \text{for all } T > T^*. \tag{2.21}$$

Since

$$\|r_T - rT\| \geq \left\| \sum_{t=0}^{T-1} (r_{T,t} - r) \right\| - \left\| \sum_{t=0}^{T^*-1} (r_{T,t} - r) \right\|$$

and

$$\left\| \sum_{t=0}^{T^*-1} (r_{T,t} - r) \right\| \leq 2nT^*M \quad \text{for all } T > T^*,$$

we have

$$\|r_T - rT\| = \left\| \sum_{t=0}^{T-1} (r_{T,t} - r) \right\| \to \infty \quad (T \to \infty).$$

This contradicts (2.19). Thus we obtain (2.15).

Next we prove

$$R_\alpha(\tilde{p}_\alpha) \subset G_{K,\alpha} \quad \text{for sufficient large all } K. \tag{2.22}$$

From Assumption B, we have

$$\delta(R_\alpha(\tilde{s}_{t,\alpha}), R_\alpha(\tilde{p}_\alpha)) \leq C\rho(\tilde{s}_{t,\alpha}, \tilde{p}_\alpha) \quad \text{for } t \geq 0. \tag{2.23}$$

From Lemmas 2.1(ii) and 2.4(iii), we also have

$$\rho(\tilde{s}_{t,\alpha}, \tilde{p}_\alpha) \leq \beta^{[t/t_0]} K_\alpha(\tilde{s}_\alpha, \tilde{p}_\alpha) \quad \text{for } t \geq 0. \tag{2.24}$$

From (2.23),(2.24) and the compactness of E, there exists a constant $C^* > 0$ such that

$$\delta(R_\alpha(\tilde{s}_{t,\alpha}), R_\alpha(\tilde{p}_\alpha)) \leq C^* \beta^t \quad \text{for } t \geq 0.$$

Therefore, for any $r \in R_\alpha(\tilde{p}_\alpha)$, there exists $\{r_t\}_{t=0}^\infty$ such that

$$r_t \in R_\alpha(\tilde{s}_{t,\alpha}) \text{ and } \|r_t - r\| \leq C^* \beta^t \quad \text{for } t \geq 0. \tag{2.25}$$

Then

$$\left\| \sum_{t=0}^{T-1} r_t - rT \right\| = \left\| \sum_{t=0}^{T-1} (r_t - r) \right\| \leq \sum_{t=0}^{T-1} \|r_t - r\| \leq \sum_{t=0}^{T-1} C^* \beta^t \leq \frac{C^*}{1-\beta} \quad \text{for all } T \geq 1.$$

Thus we get $r \in G_{K,\alpha}$ for all $K \geq C^*/(1-\beta)$. Therefore (2.22) holds for all $K \geq C^*/(1-\beta)$. Together with (2.15), we get (2.14) for sufficient large all K. It is trivial that (2.14) is independent of the initial fuzzy state $\tilde{s}$ from Lemma 2.4(i). *q.e.d.*

From now on we take $K \geq C^*/(1-\beta)$. The following corollary shows that $\tilde{g}(\tilde{s})$ is given as the limit of $\{R(\tilde{s}_t)\}_{t=0}^{\infty}$ by the method of Cesaro averaging.

Corollary 2.1. *Under the same condition as Theorem 2.1, it holds that*

$$\lim_{T\to\infty} \frac{1}{T}\tilde{R}_{T,\alpha}(\tilde{s}_\alpha) = (\tilde{g}(\tilde{s}))_\alpha \quad \text{for all } \alpha \in [0,1]. \tag{2.26}$$

Proof. Let $\alpha \in [0,1]$. Let $r \in (\tilde{g}(\tilde{s}))_\alpha = G_{K,\alpha}$. Then there exists $\{r_T\}_{T=0}^{\infty}$ such that

$$r_T \in \tilde{R}_{T,\alpha}(\tilde{s}_\alpha) \text{ and } \|r_T - rT\| \leq K \quad \text{for } T = 0,1,2,\cdots.$$

Then

$$\frac{r_T}{T} \in \frac{1}{T}\tilde{R}_{T,\alpha}(\tilde{s}_\alpha) \text{ and } \left\|\frac{r_T}{T} - r\right\| \leq \frac{K}{T} \quad \text{for } T = 1,2,\cdots.$$

Letting $T \to \infty$, we obtain

$$r \in \varliminf_{T\to\infty} \frac{1}{T}\tilde{R}_{T,\alpha}(\tilde{s}_\alpha).$$

Therefore

$$(\tilde{g}(\tilde{s}))_\alpha \subset \varliminf_{T\to\infty} \frac{1}{T}\tilde{R}_{T,\alpha}(\tilde{s}_\alpha).$$

Conversely we let $r \in \overline{\lim}_{T\to\infty} \frac{1}{T}\tilde{R}_{T,\alpha}(\tilde{s}_\alpha)$. Then there exists a subsequence $\{r_{T_j}\}_{j=1}^{\infty}$ such that

$$r_{T_j} \in \tilde{R}_{T_j,\alpha}(\tilde{s}_\alpha)\ (j = 1,2,\cdots) \text{ and } \lim_{j\to\infty} \frac{r_{T_j}}{T_j} = r. \tag{2.27}$$

Let $j = 1,2,\cdots$. From Lemma 2.1(iii), there exists $\{r_{T_j,t}\}_{t=1}^{\infty}$ such that

$$r_{T_j,t} \in R_\alpha(\tilde{s}_{t,\alpha})\ (t = 0,1,2,\cdots,T_j-1) \text{ and } r_{T_j} = \sum_{t=0}^{T_j-1} r_{T_j,t}.$$

In similar way to (2.25), from Lemma 2.4(iii) and Assumption B, there exist a constant $C^* > 0$ and $\{z_{T_j,t}\}_{j=1}^{\infty}$ such that

$$z_{T_j,t} \in R_\alpha(\tilde{p}_\alpha) \text{ and } \|z_{T_j,t} - r_{T_j,t}\| \leq C^*\beta^t \quad \text{for } t = 1,2,\cdots,T_j-1.$$

Since $R_\alpha(\tilde{p}_\alpha)$ is convex, putting $z_{T_j} := \sum_{t=0}^{T_j-1} z_{T_j,t}$, we have

$$\frac{z_{T_j}}{T_j} \in R_\alpha(\tilde{p}_\alpha) \text{ and } \left\|\frac{z_{T_j}}{T_j} - \frac{r_{T_j}}{T_j}\right\| \leq \frac{1}{T_j}\sum_{t=0}^{T_j-1} C^*\beta^t \leq \frac{C^*}{T_j(1-\beta)} \quad \text{for } j \geq 1. \tag{2.28}$$

From (2.27) and (2.28), we obtain

$$\left\|\frac{z_{T_j}}{T_j} - r\right\| \leq \left\|\frac{z_{T_j}}{T_j} - \frac{r_{T_j}}{T_j}\right\| + \left\|\frac{r_{T_j}}{T_j} - r\right\| \to 0\ (j \to \infty).$$

Since $R_\alpha(\tilde{p}_\alpha)$ is closed, we obtain $r \in R_\alpha(\tilde{p}_\alpha) = (\tilde{g}(\tilde{s}))_\alpha$ from Theorem 2.1. Thus we get (2.26). *q.e.d.*

3. One-Dimensional Case

In this section we consider the case of $n = 1$, i.e. $\tilde{r} \in \mathcal{F}(S \times [0,M])$, and characterize an average fuzzy reward $\tilde{g}(\tilde{s})$ by the functional equations concerning with the extremal points of its α-cuts. Throughout this section it is assumed that Assumptions A and B hold.

Since $\mathcal{C}([0,M])$ is the set of all closed intervals, we can write the map $R_\alpha : \mathcal{C}(S) \mapsto \mathcal{C}([0,M])$ by the following notation :

$$R_\alpha(D) := [\min R_\alpha(D), \max R_\alpha(D)] \quad \text{for all } D \in \mathcal{C}(S). \tag{3.1}$$

Let

$$\tilde{R}_{T,\alpha}(D) := \sum_{t=0}^{T-1} R_\alpha(Q_\alpha^t(D)) \quad \text{for } D \in \mathcal{C}(S).$$

Then, by Lemma 2.1(iii), it holds that

$$\min \tilde{R}_{T,\alpha}(D) = \sum_{t=0}^{T-1} \min R_\alpha(Q_\alpha^t(D)) \tag{3.2}$$

and

$$\max \tilde{R}_{T,\alpha}(D) = \sum_{t=0}^{T-1} \max R_\alpha(Q_\alpha^t(D)), \tag{3.3}$$

where

$$\tilde{R}_{T,\alpha}(D) = [\min \tilde{R}_{T,\alpha}(D), \max \tilde{R}_{T,\alpha}(D)].$$

From Lemma 2.4(iii) and Assumption B we observe that $R_\alpha(Q_\alpha^t(D))$ converges to $R_\alpha(\tilde{p}_\alpha)$ exponentially first as $t \to \infty$. Thus, by (3.2) and (3.2),

$$\underline{h}_\alpha(D) := \lim_{T\to\infty} (\min \tilde{R}_{T,\alpha}(D) - T \times \min R_\alpha(\tilde{p}_\alpha)) \tag{3.4}$$

and

$$\overline{h}_\alpha(D) := \lim_{T\to\infty} (\max \tilde{R}_{T,\alpha}(D) - T \times \max R_\alpha(\tilde{p}_\alpha)) \tag{3.5}$$

converge for all $D \in \mathcal{C}(S)$. The function $\underline{h}_\alpha$ ($\overline{h}_\alpha$ resp.) is called *a lower (upper) relative value function*, whose basic ideas are appearing in the theory of Markov decision processes (c.f. [7]). By Theorem 2.1, we have

$$\tilde{g}(\tilde{p})_\alpha = [\min R_\alpha(\tilde{p}_\alpha), \max R_\alpha(\tilde{p}_\alpha)], \tag{3.6}$$

where the extremal points are characterized in the following theorem.

Theorem 3.1. *Let $\alpha \in [0,1]$. Then the following (i) and (ii) hold.*

(i) *Let $\underline{h}_\alpha$ and $\overline{h}_\alpha$ be defined by (3.4) and (3.5). Then, the following equations hold:*

$$\underline{h}_\alpha(D) + \min R_\alpha(\tilde{p}_\alpha) = \min R_\alpha(D) + \underline{h}_\alpha(Q_\alpha(D)) \tag{3.7}$$

and

$$\overline{h}_\alpha(D) + \max R_\alpha(\tilde{p}_\alpha) = \max R_\alpha(D) + \overline{h}_\alpha(Q_\alpha(D)) \tag{3.8}$$

for all $D \in \mathcal{C}(S)$.

(ii) *Conversely, if there exist bounded functions $\underline{h}_\alpha$ and $\overline{h}_\alpha$ on $\mathcal{C}(S)$ and constants $\underline{K}_\alpha$ and $\overline{K}_\alpha$ satisfying that*

$$\underline{h}_\alpha(D) + \underline{K}_\alpha = \min R_\alpha(D) + \underline{h}_\alpha(Q_\alpha(D)) \tag{3.9}$$

and

$$\overline{h}_\alpha(D) + \overline{K}_\alpha = \max R_\alpha(D) + \overline{h}_\alpha(Q_\alpha(D)) \tag{3.10}$$

for all $D \in \mathcal{C}(S)$, then $\tilde{g}(\tilde{s})_\alpha = [\underline{K}_\alpha, \overline{K}_\alpha]$.

Proof. (i) (3.4) implies

$$\begin{aligned}\underline{h}_\alpha(D) &= \lim_{T\to\infty} \textstyle\sum_{t=0}^{T-1} (\min R_\alpha(Q_\alpha^t(D)) - \min R_\alpha(\tilde{p}_\alpha)) \\ &= \min R_\alpha(D) - \min R_\alpha(\tilde{p}_\alpha) \\ &\quad + \textstyle\sum_{t=1}^{\infty} (\min R_\alpha(Q_\alpha^{t-1}(Q_\alpha(D))) - \min R_\alpha(\tilde{p}_\alpha)) \\ &= \min R_\alpha(D) - \min R_\alpha(\tilde{p}_\alpha) + \underline{h}_\alpha(Q_\alpha(D)),\end{aligned}$$

which leads to (3.7). Also, (3.8) can be shown analogously to (3.7).

(ii) Let $\underline{h}_\alpha(D)$ and $\underline{K}_\alpha(D)$ be as in (3.9). Then, it holds that for each t ($t \geq 0$),

$$\underline{h}_\alpha(Q^t_\alpha(D)) + \underline{K}_\alpha = \min R_\alpha(Q^t_\alpha(D)) + \underline{h}_\alpha(Q^{t+1}_\alpha(D)). \qquad (3.11)$$

By summing (3.11) for $t = 0, 1, \cdots, T-1$, we get

$$\underline{h}_\alpha(D) + T \times \underline{K}_\alpha = \sum_{t=0}^{T-1} \min R_\alpha(Q^t_\alpha(D)) + \underline{h}_\alpha(Q^T_\alpha(D)).$$

So

$$\underline{K}_\alpha = \lim_{T\to\infty} \frac{1}{T} \sum_{t=0}^{T-1} \min R_\alpha(Q^t_\alpha(D)) \quad \text{for } D \in \mathcal{C}(S).$$

Thus, from Theorem 2.1 and Corollary 2.1,

$$\underline{K}_\alpha = \min R_\alpha(\tilde{p}_\alpha).$$

We also obtain $\overline{K}_\alpha = \max R_\alpha(\tilde{p}_\alpha)$ similarly. Therefore we get $\tilde{g}(\tilde{s})_\alpha = [\underline{K}_\alpha, \overline{K}_\alpha]$ by (3.6). *q.e.d.*

References

[1] Chen-wei Xu, On convexity of fuzzy sets and fuzzy relations, *Infor. Sci.*, **59** (1992) 92-102.

[2] K.Kuratowski, *Topology I* (Academic Press, New York, 1966).

[3] M.Kurano, M.Yasuda, J.Nakagami and Y.Yoshida, A limit theorem in some dynamic fuzzy systems, *Fuzzy Sets and Systems* **51** (1992) 83-88.

[4] L.Madan, L.Puri and Dan A.Palescu, The concept of normality for fuzzy random variables, *Ann. Prob.* **13** (1985) 1373-1379.

[5] S.Nanda, On sequences of fuzzy numbers, *Fuzzy Sets and Systems* **33** (1989) 123-126.

[6] V.Novák, *Fuzzy sets and their Applications* (Adom Hilger, Bristol-Boston, 1989).

[7] S.M.Ross, *Applied probability models with optimization applications* (Holden-Say, san Francisco, 1970).

[8] Y.Yoshida, M.Yasuda, J.Nakagami and M.Kurano, A potential of fuzzy relations with a linear structure: The contractive case. *Fuzzy Sets and Systems*,**60** (1993) 283-294.

[9] Y.Yoshida, M.Yasuda, J.Nakagami and M.Kurano, A potential of fuzzy relations with a linear structure: The unbounded case. *Sets and Systems*,**66** (1994) .

[10] Y.Yoshida, The recurrence of dynamic fuzzy systems, *proceeding of the 6th Bellman continuum conference*, to appear.

[11] L.A.Zadeh, Fuzzy sets, *Inform. Control* **8** (1965) 338-353.