

## Recurrent Sets for Dynamic Fuzzy Systems

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### 0. Introduction

This paper discusses a recurrent behavior of dynamic fuzzy systems defined by fuzzy relations on a Euclidean space. By introducing a recurrence for crisp sets, we give probability-theoretical properties for the fuzzy systems. When the fuzzy relations satisfy a contraction condition, the existence of the maximum recurrent set is shown. We also consider a monotonicity condition for the fuzzy relations as an extended case of a linear structure in one-dimensional fuzzy numbers. Then we present the existence of the arcwise connected maximal recurrent sets.

### 1. Notations

Let  $S$  be a metric space. We write a fuzzy set on  $S$  by its membership function  $\tilde{s} : S \mapsto [0, 1]$  and an ordinary set  $A (\subset S)$  by its indicator function  $1_A : S \mapsto \{0, 1\}$ . The  $\alpha$ -cut  $\tilde{s}_\alpha$  is defined by

$$\tilde{s}_\alpha := \{x \in S \mid \tilde{s}(x) \geq \alpha\} \quad (\alpha \in (0, 1]) \quad \text{and} \quad \tilde{s}_0 := \text{cl}\{x \in S \mid \tilde{s}(x) > 0\},$$

where  $\text{cl}$  denotes the closure of a set.  $\mathcal{F}(S)$  denotes the set of all fuzzy sets  $\tilde{s}$  on  $S$  satisfying the following conditions (i) and (ii) :

- (i)  $\tilde{s}_\alpha \in \mathcal{E}(S)$  for  $\alpha \in [0, 1]$ ;
- (ii)  $\bigcap_{\alpha' < \alpha} \tilde{s}_{\alpha'} = \tilde{s}_\alpha$  for  $\alpha \in (0, 1]$ ,

where  $\mathcal{E}(S) := \{A \mid A = \bigcup_{n=0}^\infty C_n, C_n \text{ are closed subsets of } S (n = 0, 1, 2, \dots)\}$ . We also define

$$\mathcal{G}(S) := \left\{ \text{fuzzy sets } \tilde{s} \text{ on } S \mid \text{there exists } \{\tilde{s}_n\}_{n \in \mathbf{N}} \subset \mathcal{F}(S) \text{ satisfying } \tilde{s} = \bigvee_{n \in \mathbf{N}} \tilde{s}_n \right\},$$

where  $\mathbf{N} := \{0, 1, 2, 3, \dots\}$  and for a sequence of fuzzy sets  $\{\tilde{s}_n\}_{n \in \mathbf{N}}$  on  $S$  we define

$$\bigwedge_{n \in \mathbf{N}} \tilde{s}_n(x) := \inf_{n \in \mathbf{N}} \tilde{s}_n(x) \quad \text{and} \quad \bigvee_{n \in \mathbf{N}} \tilde{s}_n(x) := \sup_{n \in \mathbf{N}} \tilde{s}_n(x) \quad x \in S.$$

Let a time space by  $\mathbf{N}$  and put  $\bar{\mathbf{N}} := \mathbf{N} \cup \{\infty\}$ . Let a state space  $E$  be a finite-dimensional Euclidean space. We put a path space by  $\Omega := \prod_{k=0}^\infty E$  and we write a sample path by  $\omega = (\omega(0), \omega(1), \omega(2), \dots) \in \Omega$ . We define a map  $X_n(\omega) := \omega(n)$  and a shift  $\theta_n(\omega) := (\omega(n), \omega(n+1), \omega(n+2), \dots)$  for  $n \in \mathbf{N}$  and  $\omega = (\omega(0), \omega(1), \omega(2), \dots) \in \Omega$ . We put

$\sigma$ -fields by  $\mathcal{M}_n := \sigma(X_0, X_1, \dots, X_n)$ <sup>1</sup> for  $n \in \mathbf{N}$  and  $\mathcal{M} := \sigma(\bigcup_{n \in \mathbf{N}} \mathcal{M}_n)$ <sup>2</sup>. Let  $\Delta$  be not a point of  $E$  and put  $E_\Delta := E \cup \{\Delta\}$ . We can extend the state space  $E$  to  $E_\Delta$ , setting  $\tilde{s}(\Delta) := 0$  for  $\tilde{s} \in \mathcal{G}(E_\Delta)$  and  $X_\infty(\omega) := \Delta$  for  $\omega \in \Omega$  ([10, Section 2]). Let  $\tilde{q}$  be an upper semi-continuous binary relation on  $E \times E$  satisfying the following normality condition :

$$\sup_{x \in E} \tilde{q}(x, y) = 1 \quad (y \in E) \quad \text{and} \quad \sup_{y \in E} \tilde{q}(x, y) = 1 \quad (x \in E).$$

We call  $\tilde{q}$  a fuzzy relation. We define a fuzzy expectation : For an initial state  $x \in E$  and an  $\mathcal{M}$ -measurable fuzzy set  $h \in \mathcal{F}(\Omega)$ ,

$$E_x(h) := \int_{\{\omega \in \Omega : \omega(0) = x\}} h(\omega) \, d\tilde{P}(\omega),$$

where  $\tilde{P}$  is the following possibility measure :

$$\tilde{P}(\Lambda) := \sup_{\omega \in \Lambda} \bigwedge_{n \in \mathbf{N}} \tilde{q}(X_n \omega, X_{n+1} \omega) \quad \Lambda \in \mathcal{M}$$

and  $\int d\tilde{P}$  denotes Sugeno integral (Sugeno [9]).

We need the first entry times (the first hitting times) of a set, which is adapted to the dynamic fuzzy system  $X := \{X_n\}_{n \in \mathbf{N}}$ , in order to define a recurrence of sets in Section 3. We define

$$\mathcal{E} := \{A \mid A \in \mathcal{E}(E) \text{ and } E \setminus A \in \mathcal{E}(E)\}$$

and we call a map  $\tau : \Omega \mapsto \overline{\mathbf{N}}$  an  $\mathcal{E}$ -stopping time if

$$\{\tau = n\} \in \mathcal{M}_n \cap \mathcal{E}(\Omega) \quad n \in \mathbf{N}.$$

For example, a constant stopping time i.e.  $\tau = n_0$  for some  $n_0 \in \mathbf{N}$ , is an  $\mathcal{E}$ -stopping time. For  $A \in \mathcal{E}$  we put

$$\tau_A(\omega) := \inf\{n \in \mathbf{N} \mid X_n(\omega) \in A\} \quad \omega \in \Omega;$$

$$\sigma_A(\omega) := \inf\{n \in \mathbf{N} \mid n \geq 1, X_n(\omega) \in A\} \quad \omega \in \Omega,$$

where the infimums of the empty set are understood to be  $+\infty$ . Then the first entry time  $\tau_A$  of  $A$  and the first hitting time  $\sigma_A$  of  $A$  are also  $\mathcal{E}$ -stopping times ([10, Lemma 1.5]).

Define a map  $P : \mathcal{G}(E) \mapsto \mathcal{G}(E)$  by

$$P\tilde{s}(x) := E_x(\tilde{s}(X_1)) = \sup_{y \in E} \{\tilde{q}(x, y) \wedge \tilde{s}(y)\} \quad x \in E \quad \text{for } \tilde{s} \in \mathcal{G}(E), \quad (1.1)$$

where we write binary operations  $a \wedge b := \min\{a, b\}$  and  $a \vee b := \max\{a, b\}$  for  $a, b \in [0, 1]$ . We call  $P$  a fuzzy transition defined by the fuzzy relation  $\tilde{q}$ . We also define  $n$ -steps fuzzy transitions  $P_n : \mathcal{G}(E) \mapsto \mathcal{G}(E)$ ,  $n \in \mathbf{N}$ , by

$$P_n \tilde{s} := E_n(\tilde{s}(X_n)) = \sup_{y \in E} \{\tilde{q}^n(\cdot, y) \wedge \tilde{s}(y)\} \quad \text{for } \tilde{s} \in \mathcal{G}(E),$$

<sup>1</sup>It denotes the smallest  $\sigma$ -field on  $\Omega$  relative to which  $X_0, X_1, \dots, X_n$  are measurable.

<sup>2</sup>It denotes the smallest  $\sigma$ -field generated by  $\bigcup_{n \in \mathbf{N}} \mathcal{M}_n$ .

where for  $n \in \mathbf{N}$

$$\tilde{q}^1(x, y) := \tilde{q}(x, y) \quad \text{and} \quad \tilde{q}^{n+1}(x, y) := \sup_{z \in E} \{ \tilde{q}^n(x, z) \wedge \tilde{q}(z, y) \} \quad x, y \in E.$$

Further for an  $\mathcal{E}$ -stopping time  $\tau$ , a fuzzy transition  $P_\tau : \mathcal{G}(E) \mapsto \mathcal{G}(E)$  is defined by

$$P_\tau \tilde{s} := E.(\tilde{s}(X_\tau)) \quad \text{for } \tilde{s} \in \mathcal{G}(E),$$

where  $X_\tau := X_n$  on  $\{\tau = n\}$ ,  $n \in \overline{\mathbf{N}}$ .

## 2. Transitive closures and $P$ -superharmonic fuzzy sets

We define a partial order  $\geq$  on  $\mathcal{G}(E)$  : For  $\tilde{s}, \tilde{r} \in \mathcal{G}(E)$

$$\tilde{s} \geq \tilde{r} \iff \tilde{s}(x) \geq \tilde{r}(x) \quad x \in E.$$

**Definition** ([10, Section 4]). A fuzzy set  $\tilde{s} (\in \mathcal{G}(E))$  is called  $P$ -harmonic ( $P$ -superharmonic) provided that

$$\tilde{s} = P\tilde{s} \quad (\tilde{s} \geq P\tilde{s} \text{ resp.}).$$

Clearly a constant fuzzy set,  $\tilde{s} = \beta$  for some  $\beta \in [0, 1]$ , is  $P$ -superharmonic. We represent the fuzzy set by  $\beta$  simply.

**Theorem 2.1.** *Let  $\tilde{s}$  be  $P$ -superharmonic and let a set  $A \in \mathcal{E}$ . Then  $P_{\tau_A} \tilde{s}$  is the smallest  $P$ -superharmonic fuzzy set which dominates  $\tilde{s} \wedge 1_A$ .*

We define an operator  $G := \bigvee_{n \in \mathbf{N}} P_n$  on  $\mathcal{G}(E)$ . Then we note that

$$PG1_{\{y\}}(x) = \bigvee_{n \geq 1} P_n 1_{\{y\}}(x) = \sup_{n \geq 1} \tilde{q}^n(x, y) \quad x, y \in E.$$

This is called a transitive closure ([3, Section 3.3]). In this paper we also call  $PG$  a transitive closure. Now we need to investigate the operator  $G$  in order to analyse the transitive closure  $PG := \bigvee_{n \geq 1} P_n$ . We have the following properties regarding  $G$ .

**Lemma 2.1** ([10, Lemma 4.1(ii)]). *Let  $\tilde{s} \in \mathcal{G}(E)$ . Then :*

(i) *It holds that*

$$G\tilde{s} = \tilde{s} \vee P(G\tilde{s});$$

(ii)  *$G\tilde{s}$  is the smallest  $P$ -superharmonic dominating  $\tilde{s}$ .*

For  $A \in \mathcal{E}(E)$  we introduce an operator  $I_A : \mathcal{G}(E) \mapsto \mathcal{G}(E)$  by

$$I_A \tilde{s} := \tilde{s} \wedge 1_A \quad \tilde{s} \in \mathcal{G}(E).$$

We define a sequence of hitting times  $\{\sigma_A^n\}_{n \in \mathbf{N}}$  of a set  $A (\in \mathcal{E})$  by

$$\sigma_A^n := \begin{cases} 0 & \text{if } n = 0 \\ \sigma_A^{n-1} + \sigma_A \circ \theta_{\sigma_A^{n-1}} & \text{if } n \geq 1. \end{cases}$$

Then  $\sigma_A^n$  means the first time to hit  $A$  after time  $\sigma_A^{n-1}$  (c.f. [8]).

**Proposition 2.1.** *Let  $A \in \mathcal{E}$ . Then :*

- (i)  $P_{\tau_A} \tilde{s} = GI_A \tilde{s}$  for  $P$ -superharmonic  $\tilde{s}$ ;
- (ii)  $P_{\sigma_A^n} \tilde{s} = (PGI_A)^n \tilde{s}$  for  $P$ -superharmonic  $\tilde{s}$  and  $n \in \mathbf{N}$ .

### 3. $\alpha$ -recurrent sets

**Definition.** Let  $\alpha \in (0, 1]$ . A set  $A \in \mathcal{E}(E)$  is called  $\alpha$ -recurrent provided :

- (a)  $A$  is non-empty;
- (b)  $P_{\sigma_B^n} 1 \geq \alpha$  on  $A$  for all  $n \in \mathbf{N}$  and all non-empty  $B \in \mathcal{E}$  satisfying  $B \subset A$ .

The  $\alpha$ -recurrence of a set  $A$  means that a possibility to transit infinite times from any point of  $A$  to any point of  $A$  is greater than  $\alpha$ .

We give simple necessary and sufficient criteria for  $\alpha$ -recurrence by the transitive closure  $PG$ .

**Proposition 3.1.** Let  $\alpha \in (0, 1]$  and let non-empty  $A \in \mathcal{E}(E)$ . Then the following statements are equivalent :

- (i)  $A$  is  $\alpha$ -recurrent;
- (ii)  $PG1_B \geq \alpha \wedge 1_A$  for non-empty  $B \in \mathcal{E}(E)$  satisfying  $B \subset A$ ;
- (iii)  $PG1_{\{y\}} \geq \alpha \wedge 1_A$  for  $y \in A$ .

We gives, by the fuzzy relation  $\tilde{q}$ , a representation of the union of all  $\alpha$ -recurrent sets.

**Theorem 3.1.** It holds that

$$\bigcup_{A \in \mathcal{E}(E) : \alpha\text{-recurrent sets}} A = \left\{ x \in E \mid \sup_{n \geq 1} \tilde{q}^n(x, x) \geq \alpha \right\} \quad \text{for } \alpha \in (0, 1].$$

### 4. The contractive case

We consider the contractive case in [5] and we give the maximum  $\alpha$ -recurrent set for the dynamic fuzzy system  $X$ .

Let  $E_c$  be a compact subset of  $E$ . We deal with a dynamic fuzzy system restricted on the compact space  $E_c$  according to [5]. Let  $\mathcal{C}(E_c)$  be the set of all closed subsets of  $E_c$  and let  $\rho$  be the Hausdorff metric on  $\mathcal{C}(E_c)$ . Let  $\mathcal{F}^0(E_c)$  be the set of all fuzzy sets  $\tilde{s}$  on  $E_c$  which are upper semi-continuous and satisfy  $\sup_{x \in E_c} \tilde{s}(x) = 1$ . Then we note  $\mathcal{F}^0(E_c) \subset \mathcal{F}(E_c)$ . Let  $\tilde{p}_0 \in \mathcal{F}^0(E_c)$  be a fuzzy set. Define a sequence of fuzzy sets  $\{\tilde{p}_n\}_{n=0}^{\infty}$  by

$$\tilde{p}_{n+1}(y) = \sup_{x \in E_c} \{\tilde{p}_n(x) \wedge \tilde{q}(x, y)\} \quad y \in E_c \quad \text{for } n \geq 0. \quad (4.1)$$

The fuzzy set  $\tilde{p}_0$ , in [5], is called an initial fuzzy state and the sequence  $\{\tilde{p}_n\}_{n=0}^\infty$  is called a sequence of fuzzy states. The fuzzy relation  $\tilde{q}$  is also restricted on  $E_c \times E_c$  and it is assumed to be continuous on  $E_c \times E_c$  and satisfy  $\tilde{q}(x, \cdot) \in \mathcal{F}^0(E)$ . Define a map  $\tilde{r}_\alpha : \mathcal{C}(E_c) \mapsto \mathcal{C}(E_c)$  ( $\alpha \in [0, 1]$ ) by

$$\tilde{r}_\alpha(D) := \begin{cases} \{y \mid \tilde{q}(x, y) \geq \alpha \text{ for some } x \in D\} & \text{for } \alpha > 0, D \in \mathcal{C}(E_c), D \neq \emptyset, \\ \text{cl}\{y \mid \tilde{q}(x, y) > 0 \text{ for some } x \in D\} & \text{for } \alpha = 0, D \in \mathcal{C}(E_c), D \neq \emptyset, \\ E_c & \text{for } 0 \leq \alpha \leq 1, D = \emptyset. \end{cases}$$

In the sequel we assume the following contraction property for the fuzzy relation  $\tilde{q}$  (see [5, Section 2]) : There exists a real number  $\beta \in (0, 1)$  satisfying

$$\rho(\tilde{r}_\alpha(A), \tilde{r}_\alpha(B)) \leq \beta \rho(A, B) \quad \text{for all } A, B \in \mathcal{C}(E_c) \text{ and all } \alpha \in [0, 1].$$

**Lemma 4.1** ([5, Theorem 1]).

(i) *There exists a unique fuzzy state  $\tilde{p} \in \mathcal{F}^0(E_c)$  satisfying*

$$\tilde{p}(y) = \max_{x \in E_c} \{\tilde{p}(x) \wedge \tilde{q}(x, y)\} \quad y \in E_c. \quad (4.2)$$

(ii) *The sequence  $\{\tilde{p}_n\}_{n=0}^\infty$  converges to a unique solution  $\tilde{p} \in \mathcal{F}^0(E_c)$  of (4.2) independently of the initial fuzzy state  $\tilde{p}_0$ . Namely,*

$$\lim_{n \rightarrow \infty} \tilde{p}_n = \tilde{p},$$

*where the convergence means  $\sup_{\alpha \in [0, 1]} \rho(\tilde{p}_{n, \alpha}, \tilde{p}_\alpha) \rightarrow 0$  ( $n \rightarrow \infty$ ) provided  $\tilde{p}_{n, \alpha}, \tilde{p}_\alpha$  are  $\alpha$ -cuts ( $\alpha \in [0, 1]$ ) for the fuzzy states  $\tilde{p}_n, \tilde{p}$  respectively.*

**Proposition 4.1.** *The  $\alpha$ -cut of the solution  $\tilde{p}$  of (4.2) is*

$$\tilde{p}_\alpha = \text{cl} \left\{ x \in E_c \mid \sup_{n \geq 1} \tilde{q}^n(x, x) \geq \alpha \right\} \quad \text{for } \alpha \in (0, 1].$$

Finally we prove that the closure of the union of all  $\alpha$ -recurrent sets equals to  $\alpha$ -cuts of the limit fuzzy state  $\tilde{p}$ . Now we compare (1.1) and (4.1). Using the inverse fuzzy relation  $\hat{q}$  ([3, Section 3.2]):

$$\hat{q}(x, y) := \tilde{q}(y, x) \quad x, y \in E_c,$$

we find that (4.1) follows

$$\tilde{p}_{n+1}(x) = \sup_{x \in E_c} \{\hat{q}(x, y) \wedge \tilde{p}_n(y)\} \quad x \in E_c \quad \text{for } n \geq 0.$$

Therefore we can apply the results in Sections 1 – 3 to a dynamic fuzzy system defined by the inverse fuzzy relation  $\hat{q}$ .

**Theorem 4.1.**

$$\tilde{p}_\alpha = \text{cl} \left\{ x \in E_c \mid \sup_{n \geq 1} \tilde{q}^n(x, x) \geq \alpha \right\} = \text{cl} \left( \bigcup_{A \in \mathcal{E}(E) : \alpha\text{-recurrent sets}} A \right) \quad \text{for } \alpha \in (0, 1].$$

*Further it is the maximum  $\alpha$ -recurrent set for  $X$ .*

## 5. The monotone case

In general, there does not always exist the maximum  $\alpha$ -recurrent set for the dynamic fuzzy system  $X$ , however we can consider the existence of the maximal  $\alpha$ -recurrent sets. In this section we deal with a case when the transition fuzzy relation  $\tilde{q}$  has a certain monotone property (see Section 6 for numerical examples). Then we prove the existence of at most countable arcwise connected maximal  $\alpha$ -recurrent sets.

In this section we use the notations in Sections 1 – 3. Further we introduce the following notations of  $\alpha$ -cuts ([5, Section 2]) :

$$\tilde{q}_\alpha(x) := \{y \in E \mid \tilde{q}(x, y) \geq \alpha\} \quad \text{for } x \in E \text{ and } \alpha \in (0, 1];$$

$$\tilde{q}_\alpha(A) := \bigcup_{x \in A} \tilde{q}_\alpha(x) \quad \text{for } A \in \mathcal{E}(E) \text{ and } \alpha \in (0, 1];$$

$$\tilde{q}_0(A) := \text{cl}\left(\bigcup_{\alpha > 0} \tilde{q}_\alpha(A)\right) \quad \text{for } A \in \mathcal{E}(E).$$

For  $\alpha \in (0, 1]$  and  $x \in E$  we define a sequence  $\{\tilde{q}_\alpha^m(x)\}_{m=1,2,\dots}$  :

$$\tilde{q}_\alpha^1(x) := \tilde{q}_\alpha(x); \quad \text{and} \quad \tilde{q}_\alpha^{m+1}(x) := \tilde{q}_\alpha(\tilde{q}_\alpha^m(x)) \quad \text{for } m = 1, 2, \dots.$$

We also need some elementary notations in the finite dimensional Euclidean space  $E$ :  $x + y$  denotes the sum of  $x, y \in E$  and  $\gamma x$  denotes the product of a real number  $\gamma$  and  $x \in E$ . We put  $A + B := \{x + y \mid x \in A, y \in B\}$  for  $A, B \in \mathcal{E}(E)$ . Then we define a half line on  $E$  by

$$l(x, y) := \{\gamma(y - x) \mid \text{real numbers } \gamma \geq 0\} \quad \text{for } x, y \in E.$$

**Definition.** We call a transition fuzzy relation  $\tilde{q}$  unimodal provided that  $\tilde{q}_\alpha(x)$  are bounded closed convex subsets of  $E$  for all  $\alpha \in (0, 1]$  and all  $x \in E$ .

**Definition.** We call a unimodal transition fuzzy relation  $\tilde{q}$  monotone provided that

$$\tilde{q}_\alpha(y) \subset \tilde{q}_\alpha(x) + l(x, y) \quad \text{for all } \alpha \in (0, 1] \text{ and all } x, y \in E.$$

From now on we deal with only unimodal fuzzy relations  $\tilde{q}$ , which is monotone and continuous on  $E \times E$ . The monotonicity is a natural extension of one-dimensional models with the linear structure in [12] and means that the fuzzy relations  $\tilde{q}$  keeps the partial order of fuzzy numbers (see (C.iii') in Section 6).

**Theorem 5.1.** *Assume that  $\tilde{q}$  is monotone. Let  $\alpha \in (0, 1]$ . Then*

$$\bigcup_{A \in \mathcal{E}(E) : \alpha\text{-recurrent sets}} A = \{x \in E \mid \tilde{q}(x, x) \geq \alpha\}.$$

We need the following assumption on  $\tilde{q}$ , which is technical but not so strong. It means that the function  $\tilde{q}$  does not have flat areas as a curved surface (Section 6).

**Assumption (A).** For  $\alpha \in (0, 1)$ ,

$$\text{int} \{(x, y) \in E \times E \mid \tilde{q}(x, y) \geq \alpha\} = \{(x, y) \in E \times E \mid \tilde{q}(x, y) > \alpha\},$$

where  $\text{int}$  denotes the interior of a set.

Since  $\tilde{q}$  is continuous,  $\{x \in E \mid \tilde{q}(x, x) \geq \alpha\}$  is represented by a disjoint sum of at most countable arcwise connected closed sets ([4]), we represent it by

$$\{x \in E \mid \tilde{q}(x, x) \geq \alpha\} = \bigcup_{n \in \mathbf{N}(\alpha)} F_{\alpha, n} \quad \text{for } \alpha \in (0, 1),$$

where  $F_{\alpha, n}$  are arcwise connected closed subsets of  $E$  and we put the index set  $\mathbf{N}(\alpha) (\subset \mathbf{N})$ .

**Theorem 5.2.** *We suppose Assumption (A). Let  $\alpha \in (0, 1)$ . Then maximal  $\alpha$ -recurrent sets for  $X$  are  $F_{\alpha, n}$  ( $n \in \mathbf{N}(\alpha)$ ).*

## 6. Numerical examples

Let a one-dimensional state space  $E = \mathbf{R}$ . We consider one-dimensional numerical examples. In Section 5 we have assumed the following conditions (C.i) — (C.iv):

(C.i)  $\tilde{q}$  is continuous on  $E \times E$ ;

(C.ii)  $\tilde{q}$  is unimodal;

(C.iii)  $\tilde{q}$  is monotone;

(C.iv)  $\tilde{q}$  satisfies Assumption (A).

When  $E = \mathbf{R}$ ,  $\mathcal{F}^0(\mathbf{R})$  means all fuzzy numbers on  $\mathbf{R}$ . From (C.ii),  $\tilde{q}_\alpha(x)$  are bounded closed intervals of  $\mathbf{R}$  ( $\alpha \in (0, 1], x \in \mathbf{R}$ ). So we write  $\tilde{q}_\alpha(x) = [\min \tilde{q}_\alpha(x), \max \tilde{q}_\alpha(x)]$ , where  $\min A$  ( $\max A$ ) denotes the minimum (maximum resp.) point of a interval  $A \subset \mathbf{R}$ . Then (C.iii) is equivalent to the following (C.iii') :

(C.iii')  $\min \tilde{q}_\alpha(\cdot)$  and  $\max \tilde{q}_\alpha(\cdot)$  are non-decreasing functions on  $\mathbf{R}$  for all  $\alpha \in (0, 1]$ .

Next we consider the following partial order  $\preceq$  on  $\mathcal{F}^0(\mathbf{R})$  (Nanda [6]) : For  $\tilde{s}, \tilde{r} \in \mathcal{F}^0(\mathbf{R})$ ,

$$\tilde{s} \preceq \tilde{r} \quad \text{means that} \quad \min \tilde{s}_\alpha \leq \min \tilde{r}_\alpha \quad \text{and} \quad \max \tilde{s}_\alpha \leq \max \tilde{r}_\alpha \quad \text{for all } \alpha \in (0, 1].$$

Then we can easily find that (C.iii) is equivalent to the following (C.iii'') :

(C.iii'') If  $\tilde{s}, \tilde{r} \in \mathcal{F}^0(\mathbf{R})$  satisfy  $\tilde{s} \preceq \tilde{r}$ , then  $Q(\tilde{s}) \preceq Q(\tilde{r})$ ,

where  $Q : \mathcal{F}^0(\mathbf{R}) \mapsto \mathcal{F}^0(\mathbf{R})$ , see (4.1), is defined by

$$Q\tilde{s}(y) = \max_{x \in \mathbf{R}} \{\tilde{s}(x) \wedge \tilde{q}(x, y)\}, \quad y \in \mathbf{R} \quad \text{for } \tilde{s} \in \mathcal{F}^0(\mathbf{R}).$$

(C.iii'') means that  $Q$  preserves the monotonicity on  $\mathcal{F}^0(\mathbf{R})$  with respect to the order  $\preceq$ . Finally (C.iv) means that the  $\alpha$ -slice  $\{x \in \mathbf{R} \mid \tilde{q}(x, x) = \alpha\}$  ( $\alpha \in (0, 1)$ ) are drawn by not areas but curved lines. We give an example of monotone fuzzy relations, which is not

contractive and does not have the linear structure in [12]. Then we calculate its maximal  $\alpha$ -recurrent sets.

**Example 6.1 (monotone case).** We give a fuzzy relation by

$$\tilde{q}(x, y) = (1 - |y - x^3|) \vee 0, \quad x, y \in \mathbf{R}.$$

Then  $\tilde{q}(x, y)$  satisfies the conditions (C.i) - (C.iv) (see Figure 6.1 for the fuzzy relation  $\tilde{q}(x, y)$  and Figure 6.2 for the  $\frac{3}{4}$ -slice).

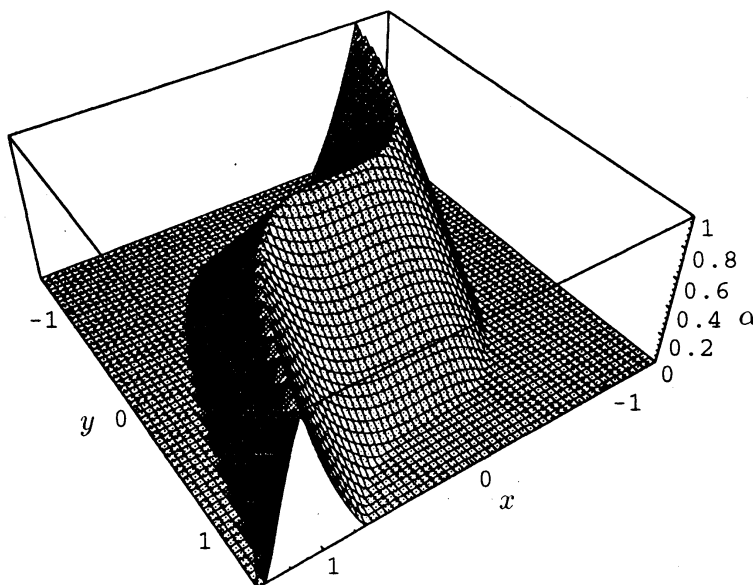


Fig. 6.1 : The monotone fuzzy relation  $\tilde{q}(x, y)$ .

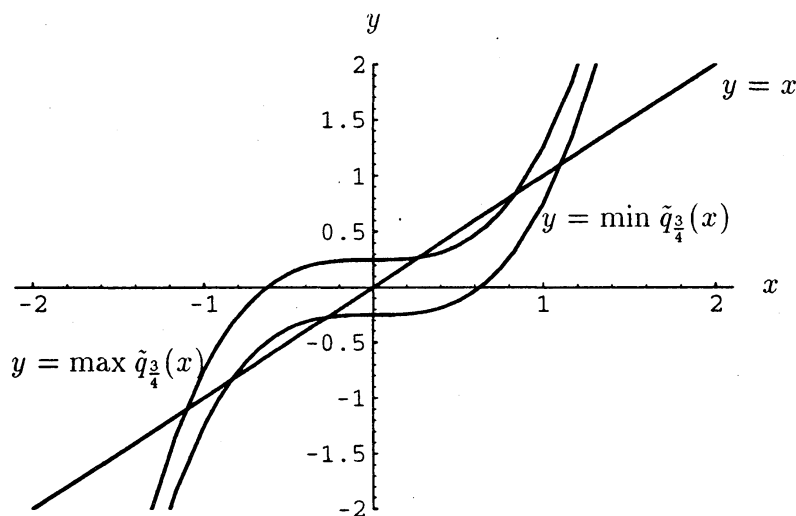


Fig. 6.2. The  $\frac{3}{4}$ -level sets  $\{(x, y) \mid \tilde{q}(x, y) = \frac{3}{4}\}$ .



Then we have

$$\tilde{q}(x, x) = (1 - |x - x^3|) \vee 0, \quad x \in \mathbf{R}.$$

Therefore  $\mathbf{N}(\frac{3}{4}) = \{0, 1, 2\}$  and

$$\begin{aligned} \left\{x \in \mathbf{R} \mid \tilde{q}(x, x) \geq \frac{3}{4}\right\} &= F_{\frac{3}{4},0} \cup F_{\frac{3}{4},1} \cup F_{\frac{3}{4},2} \\ &\approx [-1.10716, -0.837565] \cup [-0.269594, 0.269594] \cup [0.837565, 1.10716]. \end{aligned}$$

By Theorem 5.2, the maximal  $\frac{3}{4}$ -recurrent sets are given by three intervals

$$\begin{aligned} F_{\frac{3}{4},0} &\approx [-1.10716, -0.837565], \\ F_{\frac{3}{4},1} &\approx [-0.269594, 0.269594], \\ F_{\frac{3}{4},2} &\approx [0.837565, 1.10716]. \end{aligned}$$

## References

- [1] R.E.Bellman and L.A.Zadeh, Decision-making in a fuzzy environment, *Management Sci. Ser B.* **17** (1970) 141-164.
- [2] A.O.Esogbue and R.E.Bellman, Fuzzy dynamic programming and its extensions, *TIMS / Studies in Management Sci.* **20** (North-Holland, Amsterdam, 1984) 147-167.
- [3] G.J.Klir and T.A.Folger, *Fuzzy Sets, Uncertainty, and Information* (Prentice-Hall, London, 1988).
- [4] K.Kuratowski, *Topology I* (Academic Press, New York, 1966).
- [5] M.Kurano, M.Yasuda, J.Nakagami and Y.Yoshida, A limit theorem in some dynamic fuzzy systems, *Fuzzy Sets and Systems* **51** (1992) 83-88.
- [6] S.Nanda, On sequences of fuzzy numbers, *Fuzzy Sets and Systems* **33** (1989) 123-126.
- [7] J.Neveu, *Discrete-Parameter Martingales* (North-Holland, New York, 1975).
- [8] D.Revuz, *Markov Chains* (North-Holland, New York, 1975).
- [9] M.Sugeno, Fuzzy measures and fuzzy integral : a survey in M.M.Gupta, G.N.Saridis and B.R.Gaines, Eds., *Fuzzy Automata and Decision Processes* (North-Holland, Amsterdam, 1977) 89-102.
- [10] Y.Yoshida, Markov chains with a transition possibility measure and fuzzy dynamic programming, *Fuzzy Sets and Systems* **66** (1994) 39-57.
- [11] Y.Yoshida, The recurrence of dynamic fuzzy systems, *RIFIS Technical Report 80* (1994), Research Institute of Fundamental Information Science, Kyushu University, Japan.
- [12] Y.Yoshida, M.Yasuda, J.Nakagami and M.Kurano, A potential of fuzzy relations with a linear structure : The contractive case, *Fuzzy Sets and Systems* **60** (1993) 283-294.
- [13] Y.Yoshida, M.Yasuda, J.Nakagami and M.Kurano, A potential of fuzzy relations with a linear structure : The unbounded case, *Fuzzy Sets and Systems* **66** (1994) 83-95.