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Mathematical Models on the Competition for Some Territory

寺岡 義伸, University of Osaka Prefecture (Yoshinobu TERAOKA)
辻村 和彦, University of Osaka Prefecture (Kazuhiko TSUJIMURA)
山田 康吉, Mitsubishi Heavy Industry Co. (Yasuyoshi YAMADA)

Abstract

We consider a two-person game which is suggested from the logic of animal conflict, where the two players. Player I and II, compete to obtain a territory whose value is V(t) which depends on the passing time $t \in [0, \infty)$. The competition beings at their display, and both players continue their glares at each other, and it is the winner that can keep the glare longer than his opponent. In such a situation, it is natural to assume that each player spends the cost depending on how long he keeps his glare. Each has to select a time to give up the competition. This model yields us a certain class of non-zero sum infinite game on $[0, \infty) \times [0, \infty)$. There are two information patterns of information available to the players. We deal with the noisy version and the silent version here.

Key words: non-zero sum infinite game, nosh equilibrium strategy duopolistic problem, logic of animal conflict.

1 Introduction

We consider the two-person non-zero sum game which is suggested from the logic of animal conflict (see Smith [1]).

The two players, player I and II, compete to obtain a territory whose value is V(t) which depends on the passing time $t \in [0, \infty)$ The competition begins at their display, and both players continue their glares at each other, and it is the winner that can keep the glare longer than his opponent. Each has to select a time to give up, since we assume that each player spends the cost depending on how long he keeps his glare.

It is natural to consider the problem mentioned above, because this situation reflects many problems in usual duopolistic market, for example, competition of advertisement by two firms, race on getting the share of new product, competitive bidding, and so on. The model is described explicitly as follows :

Each of the two players (Player I, II) has to decide a stopping time to glare at his opponent in $[0, \infty)$. It is the winner that can keep the glare longer than his opponent, and he can obtain the territory which has the value V(t) depending on the passing time $t \in [0, \infty)$, While the loser is a player who escaped from this competition earlier, and he can not get anything. However, each of the both players have to spend the cost t when he keeps the glare until time t in $[0, \infty)$. When both of the two players stop the glare and give up to continue this competition at a same time t, player I and II share the value V(t) between them with the even ratio.

Now, We introduce two patterns of information available to the players. If a player is informed of his opponent's stopping tine as soon as his opponent gives up the competition, we say they are in a noisy version. If neither player learns when or whether his opponent has quitted the glare, we say both players are in silent version.

We shall discuss the three cases according to the information patterns mentioned above, as follows :

(i) Both players are in a noisy version

(ii) Both players are in a silent version

(iii) Player I is informed of II ' s stopping time whereas II does no learn when or whether I has quitted the glare, that is , I is the silent player and II is the noisy player.

Related to this paper, there are two works by Y. Teraoka [2,3] .

Throughout of this paper, we use notations on the expectations for real valued function $M_1(x,y)$ defined on $[0,\infty)\times[0,\infty)$ as follows :

$$M_i(x,y) = \int_0^\infty \int_0^\infty M_i(x,y)dF(x)dG(y)$$

and

$$M_i(x,y) = \int_0^\infty M_i(x,y)dG(y) \quad ; \quad M_i(x,y) = \int_0^\infty M_i(x,y)dF(x) .$$

Furthoremore, we also assume that the value of this territory is a non-negative, continuous, and non-increasing function with respect to the passing time t . It is natural to assume $V(0) > 0$.

2 Noisy Game

We establish two pure strategies for Player I and II as $x \in [0,\infty)$ and $y \in [0,\infty)$, respectively. Then the expected payoff kernels $M_1(x,y)$ for I and $M_2(x,y)$ for II are given by the following expressions,since both players are in a noisy version :

$$(1) \quad M_1(x,y) = \begin{cases} -x, & x < y \\ \frac{1}{2}V(x) - x, & x = y \\ V(y) - y, & x > y \end{cases} ;$$

$$(2) \quad M_2(x,y) = \begin{cases} -y, & y < x \\ \frac{1}{2}V(y) - y, & y = x \\ V(x) - x, & y > x \end{cases} .$$

There are no equilibrium strategies in the class of pure strategies for our non-zero sum infinite game (1) and (2) . Hence we shall find the equilibrium strategies in the class of mixed strategies.

Observing equations (1) and (2) , we can assume that both use the same mixed strategies (cdfs on $[0,\infty)$), so that each of the two players use $F(z)$ as his mixed strategy .

We suppose that $F(z)$ consists of a density part $f(z) > 0$ over $[0,\infty)$ and mass part $\alpha \geqq 0$ at $z=0$, then we have

$$(3) \quad M_2(F,y) = \begin{cases} \frac{1}{2}\alpha V(0) & y = 0 \\ \alpha' V(0) + \int_0^y \{V(x) - x\} f(x)dx + \int_y^\infty (-y)f(x)dx , & y > 0 \end{cases} .$$

setting

$$\frac{d}{dy} M_2(F,y) = 0 \qquad \text{for } y \in [0,\infty),$$

we obtain

$$\frac{f(y)}{1 - F(y)} = \frac{1}{V(y)} \qquad \text{for } y \in [0, r),$$

where

$$(4) \quad r = \sup \{ y \mid V(y)\} > 0 \} ,$$

since $V(t)$ is assumed to be a non-negative , continuous , and non-increasing function of t with $V(0)$

> 0.

Thus we get

$$(5)\qquad F^*(x) = \begin{cases} \alpha, & x = 0 \\ 1-(1-\alpha)\exp\{-\int_0^x \frac{dt}{V(t)}\}, & 0 < x < r \\ 1, & x \geqq r \end{cases}$$

by supplying possible jump $1 - F(1-0) \geqq 0$ at $x = r$.
inserting (5) into (3),
we have

$$(6)\qquad M_2(F^*, 0) = \begin{cases} \frac{1}{2}\alpha V(0), & y = 0 \\ \alpha V(0), & y > 0 \end{cases}$$

since $V(y) = 0$ for all $y \geqq r$.

Hence, setting $\alpha = 0$ leads to

$M_1(x, F^*) = 0$ for all $x \in [0,\infty)$

$M_2(F^*, y) = 0$ for all $y \in [0,\infty)$.

Thus we have Theorem 1.

The same arguments hold for $M_1(x,F)$, thus we get

$$M_1(x, F^*) = \begin{cases} \frac{1}{2}\alpha V(0), & x = 0 \\ \alpha V(0), & x > 0 \end{cases}.$$

Theorem 1. Let $F^*(z)$ be the following cdf over $[0,\infty)$:

$$F^*(z) = \begin{cases} 1 - \exp\{-\int_0^z \frac{dt}{V(t)}\}, & 0 \leqq x < r \\ 1, & x \geqq r, \end{cases}$$

where $r = \sup\{z \mid V(t) > 0\}$.

Then, (F^*, F^*) is an equilibrium mixed strategy for non-zero sum game (1) and (2) and the corresponding equilibrium values V_1^* for I and V_2^* for II are given by

$V_1^* = M_1(F^*, F^*) = 0$; $V_2^* = M_2(F^*, F^*) = 0$.

3 Silent Game

Here, we discuss the case where both players are in a silent version. We also establish pure strategies $x \in [0,\infty)$ for I and $y \in [0,\infty)$ for II. Then the expected payoff kernels $M_1(x,y)$ for I and $M_2(x,y)$ for II are given by the following:

$$(9)\qquad M_1(x,y) = \begin{cases} -x, & x < y \\ \frac{1}{2}V(x) - x, & x = y \\ V(x) - x, & x > y \end{cases}$$

$$(10)\qquad M_2(x,y) = \begin{cases} -y, & y < x \\ \frac{1}{2}V(y) - y, & y = x \\ V(y) - y, & y > x \end{cases}$$

We shall derive equilibrium strategies from the class of mixed strategies, too.

We suppose that each of player I and II uses a mixed strategy F(z) which consists of a density part $f(z) > 0$ over some interval (0,u) and a possible jump at $z=0$.

Then we have

$$(11) \quad M_2(F,y) = \begin{cases} \frac{1}{2}\alpha V(0) , & y = 0 \\ V(y)F(y) - y , & 0 < y < u \\ V(y) - y , & y > u . \end{cases}$$

observing (11), we can put

$$(12) \quad F^0(x) = \frac{x}{V(x)} , \qquad 0 < x < u^0 ,$$

where u^0 is the unique root of equation $t=V(t)$ in the interval (0,r).

Then we have

$$M_2 (F^0 , y) = \begin{cases} 0 , & 0 \leqq y \leqq u^0 \\ V(y) - y < 0 , & y > u^0. \end{cases}$$

In a similar fashion, we get

$$M_1 (x, F^0) = \begin{cases} 0 , & 0 \leqq x \leqq u^0 \\ V(x) - x < 0 , & x > u^0 . \end{cases}$$

Thus we get Theorem 2.

Theorem 2. Let u^0 be the unique root of equation $t=V(t)$ in the interval [0,r), and let

$$F^0(z) = \begin{cases} \frac{z}{V(z)} , & 0 \leqq z \leqq u^0 \\ 1 , & z \geqq u^0 . \end{cases}$$

Then $(F^0(x),G^0(y))$ is an equilibrium point of non-zero sum game (9) and (10) and the equilibrium values V_1^0 for I and V_2^0 for II are given by

$$V_1^0 = M_1(F^0, G^0) = 0 \quad ; \quad V_2^0 = M_2(F^0, G^0) = 0 .$$

4 Silent-Noisy Game

In this section, we shall discuss the case where Player I is informed of II' s stopping time where as II does not learn when or whether I has quitted the glare.

As well as in the previous sections, we suppose that $x \in [0,\infty)$ and $y \in [0,\infty)$ are the pure strategies for Player I and II, respectively, and cdfs F(x) and G(y) over $[0,\infty)$ are the mixed strategies for I and II, respectively.

Let $M_1(x,y)$ and $M_2(x,y)$ are the expected payoff kernels for I and II, respective, we have

$$(13) \quad M_1(x,y) = \begin{cases} -x , & x < y \\ \frac{1}{2}V(x) - x , & x = y \\ V(y) - y , & x > y \end{cases}$$

$$(14) \quad M_2(x,y) = \begin{cases} -y , & y < x \\ \frac{1}{2}V(y) - y , & y = x \\ V(y) - y , & y > x \end{cases}$$

There are no equilibrium pure strategies in the non-zero sum infinite game (13) and (14), too.

Therefore, we suppose that I and II use mixed strategies F(x) and G(y), respectively, which are in the class of the following cdfs by referring the previous sections :

F(x) consists of a density part $f(x)>0$ over an interval $[0,u) \subset [0,r)$ and mass parts $\alpha \geqq 0$ and $\beta \geqq 0$ at $x=0$ and $x=u$, respectively.

G(y) consists of a density part $g(y)>0$ over some interval $[0,u) \subset [0,r)$ and mass parts $\alpha' \geqq 0$ and $\beta' \geqq 0$ at $y=0$ and $y=u$, respectively.

Then we have

$$(15)\quad M_2(F, y) = \begin{cases} \frac{1}{2}\alpha V(0), & y=0 \\ V(y)F(y) - y, & 0<y<u \\ V(u)(1-\frac{1}{2}\beta) - u, & y=u \\ V(y) - y, & y>u \end{cases}$$

and

$$(16)\quad M_1(x, G) = \begin{cases} \frac{1}{2}\alpha' V(0), & x=0 \\ \alpha' V(0) + \int_0^x \{V(y)-y\}\, g(y)dy - x\, \{1-G(x), & 0<x<u \\ \alpha' V(0) + \int_0^u \{V(y)-y\}\, g(y)dy + \beta' \{\frac{1}{2}V(u) - u\}, & x=u \\ \alpha' V(0) + \int_0^u \{V(y)-y\}\, g(y)dy + \beta' \{V(u) - u\}, & x>u \end{cases}$$

From the same discussion on $M_2(F, y) = \text{const}$ for $y \in [0, u)$ we obtain

$$F^0(x) = \begin{cases} \frac{x}{V(x)}, & 0 \leqq u^0 \\ 1, & x \geqq u^0, \end{cases}$$

by putting $\alpha = \beta = 0$, where u^0 is the unique root of equation $t=V(t)$.

And we obtain

$$M_2(F^0, y) = \begin{cases} 0, & 0 \leqq y \leqq u^0 \\ V(y) - y < 0, & y > u^0. \end{cases}$$

Similar arguments on $M_1(x, G) = \text{const}$ for $x \in [0, u)$ give

$$g(y) = \frac{1}{V(y)} \exp\{\int_0^y \frac{dt}{V(t)}\}, \quad 0<y<u<r.$$

Considering $G(0) = \alpha'$ and $G(U) = 1$, we conclude that

$$(17)\quad G(y) = \begin{cases} \alpha', & y=0 \\ 1+\alpha' - \exp\{-\int_0^y \frac{dt}{V(t)}\}, & 0<y<u \\ 1, & y \geqq u \end{cases}$$

and

$$(18)\quad \beta' = \exp\{-\int_0^u \frac{dt}{V(t)}\} - \alpha'.$$

Inserting (17) and (18) into (16), and setting $u=u^0$ which is the unique root of the equation $t=V(t)$ in $[0, r)$, we get

$$
(19) \quad M_1(x,G) = \begin{cases} \frac{1}{2}\alpha' V(0) , & y = 0 \\ \alpha' V(0) + \alpha' u^0 , & u < y < u^0 \\ \alpha' V(0) + u^0 \left[\exp \left\{ - \int_0^u \frac{dt}{V(t)} \right\} - \frac{\beta'}{2} \right] , & y = u^0 \\ \alpha' V(0) + u^0 \left[\exp \left\{ - \int_0^u \frac{dt}{V(t)} \right\} - \beta' \right] , & y > u^0 . \end{cases}
$$

Hence, if we put $\beta'=0$ and then denote G^α instead G , the following equation holds:

$$
(20) \quad M_1(x,G) = \begin{cases} \frac{1}{2}\alpha' V(0) , & x = 0 \\ \alpha' \left[V(0) + V(u^0) \right] , & x > 0 \end{cases}
$$

Thus we have Theorem 3 .

Theorem 3. Let u^0 be the unique root of equation $t=V(t)$ in the interval $[0,r)$. and let

$$
\theta(y) = \exp \left\{ - \int_0^y \frac{dt}{V(t)} \right\} ,
$$

then consider the cdfs $F^0(x)$ and G^α given by

$$
F^0(x)= \begin{cases} \frac{x}{V(x)} , & 0 \leqq x < u^0 \\ 1 , & x \geqq u^0 \end{cases} ; \quad G^\alpha(y)= \begin{cases} 1- \{\theta(y) - \theta(u^0)\} , & 0 \leqq y < u^0 \\ 1 , & y \geqq u^0 \end{cases}
$$

The pair(F^0,G^α) is an equilibrium point of non-zero sum game (13) and (14), and the corresponding equilibrium values are given by

$$
M_1(F^0,G^\alpha)= \left[V(0) + V(u^0) \right] \theta(u^0) > 0 \quad ; \quad M_2(F^0,G^\alpha) = 0
$$

5. Simple Examples

We examine the simple examples, here. First, we deal with the case where $V(t)=1$ for all $t \geqq 0$. Since $r=\infty$, $u^0=1$, and

$$
\theta(z)= \exp \left\{ - \int_0^z \frac{dt}{V(t)} \right\} = e^{-z} , \qquad \text{for } z \geqq 0 .
$$

We have the following equilibrium strategies :

$$
F^*(z) = 1- e^{-z} , \qquad 0 \leqq z < \infty
$$

$$
F^0(z) = \begin{cases} z , & 0 \leqq z < 1 \\ 1 , & z \geqq 1 \end{cases}
$$

and

$$
G^\alpha(y)= \begin{cases} 1 + e^{-1} - e^{-y} , & 0 \leqq y < 1 \\ 1 , & y \geqq 1 \end{cases}
$$

The equilibrium values for silent-noisy game are given by

$$
M_1(F^0,G^\alpha)= \frac{e}{2} \fallingdotseq 0.736 \quad ; \quad M_2(F^0,G^\alpha)= 0
$$

Next, we shall discuss the case where

$$
V(t)= \begin{cases} 1 - t , & 0 \leqq t < 1 \\ 0 , & t \geqq 1 . \end{cases}
$$

We can get easily

$r = 1$, $u^0 = \frac{1}{2}$, and $\theta(z) = 1 - z$, for $0 \leqq z \leqq 1$,

and then we have

$$F^*(z) = \begin{cases} z , & 0 \leqq z < 1 \\ 1 , & z \geqq 1 , \end{cases}$$

$$F^0(z) = \begin{cases} \dfrac{z}{1-z} , & 0 \leqq z < \frac{1}{2} \\ 1 , & z \geqq \frac{1}{2} , \end{cases}$$

and

$$G^{\alpha}(y) = \begin{cases} y + \frac{1}{2} , & 0 \leqq y < \frac{1}{2} \\ 1 , & y \geqq \frac{1}{2} . \end{cases}$$

The equilibrium values for silent-noisy game are

$$M_1(F^0, G^{\alpha}) = \frac{3}{4} = 0.75 \quad ; \quad M_2(F^0, G^{\alpha}) = 0$$

Finally, we examine the case where $V(t) = e^{-t}$ for $t \geqq 0$.

We also get $r = \infty$, u^0 is the unique root of equation $t = e^{-t}$, i. e. , $u^0 \fallingdotseq 0.567$, and

$$\theta(z) = \exp(1 - e^{z}) \quad \text{for} \quad z \geqq 0 .$$

Hence, we have following equilibrium strategies :

$$F^*(z) = 1 - \exp(1 - e^{z}) , \quad 0 \leqq z < \infty$$

$$F^0(z) = \begin{cases} \dfrac{z}{e^{-z}} , & 0 \leqq z < u^0 \\ 1 , & z > u^0 , \end{cases}$$

and

$$G^{\alpha}(z) = \begin{cases} \exp(1 - e^{u^0}) , & z = 0 \\ 1 + \exp(1 - e^{u^0}) - \exp(1 - e^{z}) , & 0 < z < u^0 \\ 1 , & z \geqq u^0 \end{cases}$$

The equilibrium values for silent-noisy game are

$$M_1(F^0, G^{\alpha}) = (1 + e^{-u^0}) \exp(1 - e^{u^0}) \fallingdotseq 0.731 \quad ; \quad M_2(F^0, G^{\alpha}) = 0$$

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