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2階Emden-Fowler型方程式系の振動問題

広島大総合科　宇佐美広介　(Hiroyuki USAMI)

1. Introduction and Results

This is a joint work with Professor Manabu Naito (Hiroshima University).

Let us consider the following binary elliptic system of the Emden-Fowler type in an exterior domain $\Omega \subset \mathbb{R}^N$:

$$\begin{aligned} \Delta u_1 + p_1(x)|u_2|^{\sigma_1 - 1}u_2 &= 0, \\ \Delta u_2 + p_2(x)|u_1|^{\sigma_2 - 1}u_1 &= 0, \end{aligned} \tag{S}$$

where we always assume the next conditions:

(A_1) σ_1, $\sigma_2 > 0$;

(A_2) $p_i \in C(\Omega;[0,\infty))$, and supp p_i is unbounded, $i = 1, 2$.

System (S) is called oscillatory if, for any $R > 0$, it has no solutions (u_1,u_2) satisfying $u_1u_2 > 0$ in $\Omega \cap \{x\ ;\ |x| > R\}$.

For the single elliptic equation

$$\Delta u + p(x)|u|^{\sigma-1}u = 0, \qquad \sigma > 0, \tag{0}$$

useful oscillation criteria have been obtained by many authors; see e. g. [3, 4, 10, 11]. Of course, (0) is called oscillatory if it has no positive solutions which are of constant sign near

∞. The next oscillation criteria are well-known:

Theorem 0. *Let $\sigma \neq 1$ and $\hat{p}$ be a continuous function satisfying $0 \le \hat{p}(r) \le \min_{|x|=r} p(x)$ for large $|x|$.*

(i) *Let $N = 2$. Then (0) is oscillatory if*

$$\int^{\infty} r(\log r)^{\sigma_*}\hat{p}(r)dr = \infty, \qquad \sigma_* = \min\{1,\sigma\}.$$

(ii) *Let $N \ge 3$. Then (0) is oscillatory if*

$$\int^{\infty} r^{N-1-\sigma^*(N-2)}\hat{p}(r)dr = \infty, \qquad \sigma^* = \max\{1,\sigma\}.$$

By considering the case where $p(x)$ has radial symmetry, we find that this theorem characterizes the oscillation situation of (0) in some sense. However, turning our attention to system (S), we realize that there exist few results which give effective criteria for oscillation of system (S). Motivated by this fact we make an attempt to give a contribution to this problem. Other related results of asymptotic theory for elliptic systems like (S) are found in [1, 2, 5, 6, 7, 12, 13].

First we introduce some notation. Let $\hat{p}_i$, $i = 1, 2$, be continuous functions such that

$$0 \le \hat{p}_i(r) \le \min_{|x|=r} p_i(x) \quad \text{for } |x| \text{ large.}$$

Define the functions $P_i(r)$, $i = 1, 2$, by

$$P_i(r) = \int_r^{\infty} s\hat{p}_i(s)ds \quad \text{if } N = 2; \text{ and}$$

$$P_i(r) = \int_r^{\infty} s^{-\sigma_i(N-2)-3}\hat{p}_i(s)ds \quad \text{if } N \ge 3.$$

It will be seen below that system (S) is oscillatory if $P_i \equiv \infty$

for some i. Hence assuming the existence of P_i, $i = 1, 2$, loses no generality.

Our oscillation criteria for (S) are as follows:

Theorem 1. *Let $\sigma_1, \sigma_2 > 1$. Suppose that there are constants $\lambda_i > 0$ $(i = 1, 2)$ and $\varepsilon > 0$ satisfying*

$$\lambda_1 + \lambda_2 = 1, \qquad \lambda_i\sigma_i - \lambda_j - \varepsilon > 0 \quad \text{for } i, j \in \{1,2\}, \ i \neq j,$$

and

$$\int^{\infty} r^{N-3}[P_1(r)]^{\lambda_1}[P_2(r)]^{\lambda_2} \cdot \left(\int^{r} s^{N-3}P_1(s)ds\right)^{\lambda_2\sigma_2-\lambda_1-\varepsilon}\left(\int^{r} s^{N-3}P_2(s)ds\right)^{\lambda_1\sigma_1-\lambda_2-\varepsilon} dr = \infty.$$

Then, (S) is oscillatory.

Theorem 2. (i) *Let $\sigma_1\sigma_2 < 1$ and $N = 2$. Suppose that for some $i, j \in \{1,2\}$, $i \neq j$,*

$$\int^{\infty} r(\log r)^{\sigma_i(\sigma_j+1)}\hat{p}_i(r)[P_j(r)]^{\sigma_i}dr = \infty.$$

Then, (S) is oscillatory.

(ii) *Let $\sigma_1\sigma_2 < 1$ and $N \geq 3$. Suppose that for some $i, j \in \{1,2\}$, $i \neq j$,*

$$\int^{\infty} r^{\sigma_1\sigma_2(N-2)-3}\,\hat{p}_i(r)[P_j(r)]^{\sigma_i}dr = \infty.$$

Then, (S) is oscillatory.

Theorem 3. *Let $\sigma_1\sigma_2 > 1$. Suppose that for some $i, j \in \{1,2\}$, $i \neq j$,*

$$\int^{\infty} r^{N-3}P_i(r)dr = \infty \quad \text{and} \quad \liminf_{r\to\infty} \frac{P_j(r)}{P_i(r)}\left[\int^{r} s^{N-3}P_i(s)ds\right]^{\sigma_j+1} > 0.$$

Then (S) is oscillatory.

Theorem 4. *Let* $\sigma_1\sigma_2 = 1$. *Suppose that*

$$\int^{\infty} r^{N-3} \min\{P_1(r),P_2(r)\}dr = \infty.$$

Suppose moreover that

$$\limsup_{r\to\infty} \left(\int^{r} s^{-1}\min\{P_1(r),P_2(r)\}dr - \sigma^{1/(\sigma+1)}\log\log r\right) > -\infty$$

if $N = 2$, *or*

$$\limsup_{r\to\infty} \left(\int^{r} s^{N-3}\min\{P_1(r),P_2(r)\}dr - (N-2)\sigma^{1/(\sigma+1)}\log r\right) > -\infty$$

if $N \geq 3$, *where* $\sigma = \max\{\sigma_1,\sigma_2\}$. *Then* (S) *is oscillatory.*

This report proceeds as follows. In § 2 we give a comparison principle to the effect that the existence of a positive solution of (S) guarantees the existence of a positive solution of an ordinary differential system ($\hat{S}$) in § 2 associated to (S). (Here and in the sequel, a vector function is defined to be positive if both components are positive.) Hence, the problem of finding oscillation criteria for (S) reduces to the problem of finding oscillation criteria for one-dimensional problem ($\hat{S}$). This problem is fully discussed in § 3. The author believes that the results in this section is of independent interest. The proofs of Theorems 1-4 are actually omitted in this report, because they can be easily carried out by combining Proposition 2 with Propositions 3-6 in § 3.

2. Reduction to One-Dimensional Problems

We begin with the following proposition.

Proposition 1. *Suppose that* (S) *has a positive solution* (u_1, u_2) *for* $|x| \geq R$, *with R sufficiently large. Then, there is a positive solution* (w_1, w_2) *of the ordinary differential system*

$$\begin{aligned} (r^{N-1}w_1')' + r^{N-1}\hat{p}_1(r)|w_2|^{\sigma_1 - 1}w_2 &= 0, \\ (r^{N-1}w_2')' + r^{N-1}\hat{p}_2(r)|w_1|^{\sigma_2 - 1}w_1 &= 0, \end{aligned} \qquad (\hat{S})$$

for $r \geq R$, *such that*

$$0 < w_i(r) \leq \min_{|x|=r} u_i(x), \quad r \geq R, \quad i = 1, 2.$$

Proposition 1 yields the following simple comparison principle on which our results are heavily based:

Proposition 2. *Elliptic system* (S) *is oscillatory if the one-dimensional system* $(\hat{S})$ *is oscillatory.*

The proof of Proposition 1 is similar to that of [8, Theorem 2.1]. We give only the sketch of the proof. The following lemma is needed to prove Proposition 1.

Lemma 1. *Let* $b > R$, *and* (u_1, u_2) *a positive solution of* (S) *defined on* $R \leq |x| \leq b$. *Then, there is a positive solution* (w_1, w_2) *of system* $(\hat{S})$ *on* $R \leq r \leq b$ *such that*

$$w_i(R) = \hat{u}_i(R), \quad w_i(b) = \hat{u}_i(b), \quad i = 1, 2; \text{ and,}$$

$$0 < w_i(r) \leq \hat{u}_i(r), \quad R \leq r \leq b, \ i = 1, 2,$$

where $\hat{u}_i(r) = \min_{|x|=r} u_i(x)$, $r \geq R$, $i = 1, 2$.

Proof of Proposition 1. Let $\{b_m\}$ be a sequence such that

$$R < b_1 < b_2 < \cdots < b_m < \cdots, \quad \text{and} \quad \lim_{m\to\infty} b_m = \infty.$$

By Lemma 1 we obtain a sequence $\{(w_{1m}, w_{2m})\}_{m=1}^{\infty}$ such that

$$\begin{aligned} &(r^{N-1}w_{1m}')' + r^{N-1}\hat{p}_1(r)w_{2m}^{\sigma_1} = 0, \\ & \qquad\qquad\qquad\qquad\qquad\qquad R \le r \le b_m; \\ &(r^{N-1}w_{2m}')' + r^{N-1}\hat{p}_2(r)w_{1m}^{\sigma_2} = 0, \end{aligned}$$

$$w_{im}(R) = \hat{u}_i(R), \quad w_{im}(b_m) = \hat{u}_i(b_m), \quad i = 1, 2; \ \textit{and},$$

$$0 < w_{im}(r) \le \hat{u}_i(r), \quad R \le r \le b_m, \quad i = 1, 2.$$

We can choose a subsequence $\{(w_{1\mu}, w_{2\mu})\}$ of $\{(w_{1m}, w_{2m})\}$ such that $\{(w_{1\mu}, w_{2\mu})\}$ converges to a positive function $\{(w_1, w_2)\}$ uniformly on each compact subinterval of $[R,\infty)$ as $\mu \to \infty$. This (w_1, w_2) gives a desired solution of $(\hat{S})$. For the detailed argument we refer the reader to [8].

3. Oscillation Theorems for One-Dimensional Problems

Instead of dealing with system $(\hat{S})$ directly, we shall transform it into the simple system of the form

$$\begin{aligned} &y_1'' + a_1(t)|y_2|^{\sigma_1 - 1}y_2 = 0, \\ &y_2'' + a_2(t)|y_1|^{\sigma_2 - 1}y_1 = 0. \end{aligned} \qquad (S_0)$$

When $N = 2$, consider the change of variables $t = \log r$, $z_i(t) = w_i(e^t)$, $i = 1, 2$. We then find that $(\hat{S})$ is equivalent to the system

$$\ddot{z}_1 + e^{2t}\hat{p}_1(e^t)|z_2|^{\sigma_1 - 1}z_2 = 0,$$

$$\ddot{z}_2 + e^{2t}\hat{p}_2(e^t)|z_1|^{\sigma_2-1}z_1 = 0,$$

where $\cdot = d/dt$. When $N \geq 3$, we make the change of valuables $t = r^{N-2}$, $z_i(t) = tw_i(t^{1/(N-2)})$, $i = 1, 2$. Then $(\hat{S})$ reduces to the system

$$\ddot{z}_1 + (N-2)^{-2}\ t^{-\sigma_1-N/(N-2)}\hat{p}_1(t^{1/(N-2)})|z_2|^{\sigma_1-1}z_2 = 0,$$

$$\ddot{z}_2 + (N-2)^{-2}\ t^{-\sigma_2-N/(N-2)}\hat{p}_2(t^{1/(N-2)})|z_1|^{\sigma_2-1}z_1 = 0,$$

where $\cdot = d/dt$. Note that the transformations used here keep the oscillatory property. For our purpose, it is sufficient to consider system (S_0).

Now, let us consider system (S_0) under the following basic assumptions:

(B_1) $\sigma_1, \sigma_2 > 0$;

(B_2) $a_i \in C([t_0,\infty);[0,\infty))$, and supp a_i is unbounded, $i = 1, 2$.

Define the functions $A_i(t)$, $i = 1, 2$, by

$$A_i(t) = \int_t^\infty a_i(s)ds, \qquad t \geq t_0.$$

As seen below, if $A_i \equiv \infty$ for some i, then (S_0) has no positive solutions.

Proposition 3. *Let* $\sigma_1, \sigma_2 > 1$. *Suppose that there are constants* $\lambda_i > 0$ $(i = 1, 2)$ *and* $\varepsilon > 0$ *satisfying*

$$\lambda_1 + \lambda_2 = 1, \qquad \lambda_i\sigma_i - \lambda_j - \varepsilon > 0 \quad \text{for } i,\ j \in \{1,2\},\ i \neq j,$$

and

$$\int^\infty [A_1(t)]^{\lambda_1}[A_2(t)]^{\lambda_2}$$

$$\cdot\left(\int^t A_1(s)ds\right)^{\lambda_2\sigma_2-\lambda_1-\varepsilon}\left(\int^t A_2(s)ds\right)^{\lambda_1\sigma_1-\lambda_2-\varepsilon} dt = \infty.$$

Then, (S_0) *is oscillatory.*

Proposition 4. *Let* $\sigma_1\sigma_2 < 1$. *Suppose that for some* $i, j \in \{1, 2\}$, $i \neq j$,

$$\int^\infty t^{\sigma_i(\sigma_j+1)} a_i(t)[A_j(t)]^{\sigma_i} dt = \infty.$$

Then, (S_0) *is oscillatory.*

Proposition 5. *Let* $\sigma_1\sigma_2 > 1$. *Suppose that for some* $i, j \in \{1, 2\}$, $i \neq j$,

$$\int^\infty A_i(t)dt = \infty \quad \text{and} \quad \liminf_{t\to\infty} \frac{A_j(t)}{A_i(t)} \left[\int^t A_i(s)ds\right]^{\sigma_j+1} > 0.$$

Then (S_0) *is oscillatory.*

Proposition 6. *Let* $\sigma_1\sigma_2 = 1$. *Suppose that*

$$\int^\infty \min\{A_1(t), A_2(t)\}dt = \infty \tag{1}$$

and

$$\limsup_{t\to\infty}\left(\int^t \min\{A_1(s), A_2(s)\}ds - \sigma^{1/(1+\sigma)}\log t\right) > -\infty, \tag{2}$$

where $\sigma = \max\{\sigma_1, \sigma_2\}$. *Then,* (S_0) *is oscillatory.*

Proof of Proposition 3. Suppose to the contrary that (S_0) has a solution (y_1, y_2) such that $y_1(t)y_2(t) > 0$ for $t \geq T$. We may assume that $y_1(t) > 0$ and $y_2(t) > 0$. Then by (S_0) we have $y_k'(t) \geq 0$, $t \geq T$, $k = 1, 2$, and

$$y_i'(\infty) - y_i'(t) + \int_t^\infty a_i(s)[y_j(s)]^{\sigma_i} ds = 0, \qquad t \geq T, \tag{3}$$

for $i, j \in \{1,2\}$, $i \neq j$. In view of $y_k(\infty) > 0$, this implies that the functions $A_k(t)$, $k = 1, 2$, are well defined, and

$$y_i'(t) \geq A_i(t)[y_j(t)]^{\sigma_i}, \quad t \geq T, \text{ for } i, j \in \{1,2\}, \; i \neq j. \tag{4}$$

Put $w(t) = y_1(t)y_2(t)$, $t \geq T$. Then $w(t) > 0$, $t \geq T$, and

$$w'(t) \geq A_1(t)[y_2(t)]^{\sigma_1+1} + A_2(t)[y_1(t)]^{\sigma_2+1}, \quad t \geq T. \tag{5}$$

Now, in the right hand side of (5), we apply the well-known inequality

$$X_1 + X_2 \geq C(\lambda_1,\lambda_2)X_1^{\lambda_1}X_2^{\lambda_2} \quad \text{for } X_1, X_2 \geq 0,$$

where $C(\lambda_1,\lambda_2) > 0$ is a constant. We then obtain

$$w'(t) \geq C_1[A_1(t)]^{\lambda_1}[A_2(t)]^{\lambda_2}$$

$$\cdot[y_2(t)]^{\lambda_1(\sigma_1+1)-1-\varepsilon}[y_1(t)]^{\lambda_2(\sigma_2+1)-1-\varepsilon}[w(t)]^{1+\varepsilon}, \quad t \geq T, \tag{6}$$

where $C_1 > 0$ is a constant. Since (4) shows that

$$y_i(t) \geq [y_j(T)]^{\sigma_i} \int_T^t A_i(s)ds, \quad t \geq T; \quad i, j \in \{1,2\}, \; i \neq j,$$

substituting these inequalities into (6) and integrating the resulting inequality, we have

$$\frac{[w(T)]^{-\varepsilon}}{\varepsilon} \geq C_2 \int_T^t [A_1(s)]^{\lambda_1}[A_2(s)]^{\lambda_2}$$

$$\cdot\left(\int_T^s A_1(r)dr\right)^{\lambda_2(\sigma_2+1)-1-\varepsilon}\left(\int_T^s A_2(r)dr\right)^{\lambda_1(\sigma_1+1)-1-\varepsilon} ds, \quad t \geq T,$$

where $C_2 = C_2(T) > 0$ is a constant. Letting $t \to \infty$, we have a contradiction. The proof is complete.

Proof of Proposition 4. Let (y_1,y_2) be a positive solution of (S_0) for $t \geq T$. We then obtain (3), and an integration implies that

$$y_i(t) \geq C_i t \int_t^\infty a_i(s)[y_j(s)]^{\sigma_i} ds, \quad t \geq T; \ i, \ j \in \{1,2\}, \ i \neq j, \quad (7)$$

where $C_i > 0$ is a constant. This yields $y_j(t) \geq C_j t A_j(t) \cdot [y_i(t)]^{\sigma_j}$, $t \geq T$. Substituting this inequality in (7), we have

$$\frac{y_i(t)}{t} \geq \hat{C} \int_t^\infty s^{\sigma_i(1+\sigma_j)} a_i(s)[A_j(s)]^{\sigma_i} \left[\frac{y_i(s)}{s}\right]^{\sigma_1\sigma_2} ds, \quad t \geq T,$$

where $\hat{C} > 0$ is a constant. Define $w(t)$ by the right hand side of the above. Then,

$$-w'(t) \geq \hat{C} t^{\sigma_i(1+\sigma_j)} a_i(t)[A_j(t)]^{\sigma_i}[w(t)]^{\sigma_1\sigma_2}, \qquad t \geq T,$$

from which, by an integration,

$$\frac{[w(T)]^{1-\sigma_1\sigma_2}}{1-\sigma_1\sigma_2} \geq \hat{C} \int_T^t s^{\sigma_i(1+\sigma_j)} a_i(s)[A_j(s)]^{\sigma_i} ds, \qquad t \geq T.$$

Letting $t \to \infty$, we get a contradiction. Hence the proof is complete.

In proving Proposition 3 we need the follwing lemma. The proof of this lemma is found in [9].

Lemma 2. *Let $\alpha < \beta$ be positive constants, and $q \in C[t_1,\infty)$ a positive function such that*

$$\liminf_{t\to\infty} t^{1+\beta} q(t) > 0.$$

Then, there exist no positive functions $w(t)$ which satisfy

$$w'(t) > 0, \ ([w'(t)]^\alpha)' \geq q(t)[w(t)]^\beta \quad \textit{for all large } t.$$

Proof of Proposition 5. Let (y_1,y_2), $t \geq T$, be a positive solution of (S_0). As seen before, we have (4). An integration gives

$$y_i(t) \geq y_i(T) + \int_T^t A_i(s)[y_j(s)]^{\sigma_i} ds, \qquad t \geq T.$$

We denote the right hand side of the above by $w_i(t)$:

$$w_i(t) \equiv y_i(T) + \int_T^t A_i(s)[y_j(s)]^{\sigma_i} ds, \qquad t \geq T.$$

Then, it is easy to see that

$$w_i' > 0, \quad \left(\left(\frac{w_i'}{A_i(t)}\right)^{1/\sigma_i}\right)' \geq A_j(t) w_i^{\sigma_j}, \qquad t \geq T.$$

The change of variable $\tau = \int_T^t A_i(s)ds$ transforms this inequality into

$$\frac{d}{d\tau}\left[\left(\frac{dw_i}{d\tau}\right)^{1/\sigma_i}\right] \geq \frac{A_j(t)}{A_i(t)} w_i^{\sigma_j}, \qquad \tau \geq 0.$$

Observe that $dw_i/d\tau > 0$, $\tau \geq 0$. Then, in view of Lemma 2, we reach a contradiction. The proof is complete.

Proof of Proposition 6. For simplicity we put $B(t) = \min\{A_1(t), A_2(t)\}$, $t \geq t_0$. We may suppose that $\sigma = \sigma_1$. Since (1) implies that $\int^\infty t a_1(t)dt = \infty$, we have $\int^\infty t^{\sigma_1} a_1(t)dt = \infty$.

To prove the theorem, suppose to the contrary that there is a positive solution (y_1, y_2), $t \geq T$, of (S_0). We first show that

$$\lim_{t\to\infty} y_2(t)/t = \lim_{t\to\infty} y_2'(t) = 0. \tag{8}$$

In fact, if this is not the case, the identity

$$y_1'(t) - y_1'(T) + \int_T^t a_1(s)[y_2(s)]^{\sigma_1} ds = 0, \qquad t \geq T,$$

shows that $\int^\infty a_1(t)[y_2(t)]^{\sigma_1} dt < \infty$, implying that $\int^\infty t^{\sigma_1} a_1(t)dt < \infty$. This contradiction proves (8). Exactly as in the proof of Proposition 5, we find that the function w defined by

$$w(t) \equiv y_2(T) + \int_T^t B(s)[y_1(s)]^{\sigma_2} ds \ (\le y_2(t)), \qquad t \ge T, \tag{9}$$

satisfies

$$\left[\left(\frac{w'}{B(t)}\right)^{\sigma}\right]' \ge B(t) w^{\sigma}, \qquad t \ge T. \tag{10}$$

Notice that (8) and (9) implies that

$$\lim_{t\to\infty} w(t)/t = 0. \tag{11}$$

The change of variable $\tau = \int_T^t B(s)ds$ transforms (10) into

$$\frac{d}{d\tau}\left[\left(\frac{dw}{d\tau}\right)^{\sigma}\right] \ge w^{\sigma}, \qquad \tau \ge 0.$$

Now, introduce the auxiliary function $v = v_C$ defined by

$$v(\tau) = C \exp\left(\sigma^{-1/(\sigma+1)}\tau\right), \qquad \tau \ge 0,$$

with $C > 0$ a constant. It is easily seen that, for any $C > 0$, v solves the half-linear equation

$$\frac{d}{d\tau}\left[\left(\frac{dv}{d\tau}\right)^{\sigma}\right] = v^{\sigma}, \qquad \tau \ge 0.$$

Since $v(0) = C$, and $v'(0) = C\sigma^{-1/(\sigma+1)}$, we can choose sufficiently small $C > 0$ so that $w(0) > v(0)$ and $w'(0) > v'(0)$. Then, by the well-known comparison principle, we have $w(\tau) \ge v(\tau)$, $\tau \ge 0$, namely,

$$w(t) \ge C \exp\left(\sigma^{-1/(\sigma+1)}\int_T^t B(s)ds\right), \qquad t \ge T.$$

On the other hand, condition (2) assures the existence of a constant $c_1 > 0$ and a sequence $\{t_n\} \subset [T,\infty)$ such that $t_n \uparrow \infty$ as $n \to \infty$, and

$$\frac{1}{t_n} \exp\left(\sigma^{-1/(\sigma+1)}\int_T^{t_n} B(s)ds\right) \ge c_1 \qquad \text{for } n \in \mathbb{N}.$$

From this, we have $\liminf_{n\to\infty} w(t_n)/t_n > 0$. This contradicts

(11), and hence the proof is complete.

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