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PERIODIC SOLUTIONS OF SOME DIFFUSIVE FUNCTIONAL DIFFERENTIAL EQUATIONS

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§1. Introduction.

In this paper, we shall consider the diffusive functional differential equation

$$\frac{\partial u}{\partial t}(t,x) = D(x)\Delta u(t,x) + f(t,x,u_t(\cdot,x)) \quad \text{in } (0,\infty)\times\Omega, \tag{1.1}$$

together with the boundary condition

$$\frac{\partial u}{\partial n}(t,x) = \kappa(x)(K-u(t,x)) \quad \text{on } (0,\infty)\times\partial\Omega, \tag{1.2}$$

and discuss the existence of periodic solutions of (1.1) and (1.2). Here $u=(u_1,\cdots,u_N),\Omega$ is a bounded domain in R^ℓ with smooth boundary $\partial\Omega$ (e.g., $\partial\Omega\in C^{2+\alpha}$ for some $\alpha\in(0,1)$), and Δ and $\partial/\partial n$ respectively denote the Laplacian operator in R^ℓ and the exterior normal derivative at $\partial\Omega$. Moreover, K is a (positive) constant vector in $R^N, D(x) = diag(d_1(x),\cdots,d_N(x))$ with $d_i\in C^\alpha(\bar\Omega)$ and $d_i(x)>0$ on $\bar\Omega, \kappa(x) = diag(\kappa_1(x),\cdots,\kappa_N(x))$ with $\kappa_i\in C^{1+\alpha}(\partial\Omega)$ and $\kappa_i(x)\ge 0$ on $\partial\Omega$ and $u_t(\cdot,x)$ is a function mapping $R_- := (-\infty,0]$ into R^N defined by $u_t(\theta,x)=u(t+\theta,x)$ for $\theta\in R_-$. The subject is intimately related to the work of Zhang [14], as well as the one of Burton and Zhang [2]. In [14], Zhang has treated the equation (1.1) together with the Dirichlet boundary condition, and by using an a priori $H^1(\Omega)$-bound for periodic solutions he has deduced the existence of periodic solutions which satisfy (1.1) in the sense of $L^2(\Omega)$. The purpose of this paper is to discuss the existence of periodic solutions which satisfy (1.1) and (1.2) in the classical sense and whose values are in a bounded region in R^N. Especially, we feel interest in periodic classical solutions whose range is contained in the positive cone in R^N, which are called positive periodic solutions of (1.1) and (1.2). In a clear reason, the existence of positive periodic solutions would be an important subject in connection with biology, ecology or other fields. In the analysis of the subject, we need to have a $C(\bar{\Omega})$-bound rather than an $H^1(\Omega)$-bound. Roughly speaking, in this paper we shall employ the following strategy to deduce the existence of positive periodic solutions of (1.1) and (1.2). First we consider the Banach space $X = C(\bar{\Omega})$ equipped with the supremum norm and the (unbounded) linear operator A which is the closure in X of the operator $D\Delta$ with domain $\mathcal{D}(D\Delta) = \{\xi \in C^2(\bar{\Omega}; R^N) : \partial\xi/\partial n + \kappa\xi = 0$ on $\partial\Omega\}$, and then reformulate (1.1) and (1.2) as an abstract functional differential equation

$$\frac{dv}{dt} = Av(t) + G(t, v_t), \quad t > 0, \tag{1.3}$$

on X, where $v(t) = u(t,\cdot) - K$ and $G(t, v_t)(x) = f(t, x, u_t(\cdot, x))$. Moreover, following an idea in [2] and [14] we consider a functional differential equation with a parameter k together with an associated map H corresponding to (1.3). Next, observing that a fixed point of the map H for $k = 1$ yields a periodic solution of (1.3), we deduce the existence of positive periodic solutions of (1.3) from an a priori bound on all possible positive fixed point of the map H for $0 < k \le 1$. Consequently, one can obtain positive periodic solutions of (1.1) and (1.2) by assuming a $C(\bar{\Omega})$-bound for all possible positive periodic solutions of the parametrized diffusive functional differential equation corresponding to (1.1) and (1.2) (Theorem 3.2). We provide also two examples to illustrate how our theorem is effectively applicable (Theorems 3.3 and 3.4). In the examples, we derive a $C(\bar{\Omega})$-bound for possible positive periodic solutions of the parametrized equation by applying the maximum principle. Our approach in this paper would be advantageous in several ways. Among others, it should be noted that the intermediate space defined by the fractional power $(-A)^{1/2}$ is not needed in the analysis of concrete problems (Theorems 3.3 and 3.4), while it played an important role in [14]. We emphasize that the structure of the intermediate space is well-known in case $X = L^2(\Omega)$, but it is not so in case $X = C(\bar{\Omega})$.

§2. Abstract results.

Let X be a Banach space with norm $\|\cdot\|$, and let A be a (unbounded) linear operator which generates an analytic compact semigroup $T(t)$ of bounded linear operators on X with $\sup_{t\geq 0}\|T(t)\|<\infty$. We consider the (abstract) functional differential equation

$$\frac{du}{dt}=A(u(t)-a)+F(t,u_t), \qquad t>0, \tag{2.1}$$

where a is a fixed element in X and F is a function mapping $R\times BC(R_-;X), R_-:=(-\infty,0]$, into X. Here and hereafter, for any topological space O and any Banach space Y we denote by $C(O;Y)$ the space of all continuous functions mapping O into Y, and by $BC(O;Y)$ the space of all $\varphi\in C(O;Y)$ whose supremum norm $\|\varphi\|:=\sup\{\|\varphi(\theta)\|:\theta\in O\}$ is finite. Moreover, for any function $u\in BC(R;Y)$ and any $t\in R$, u_t denotes the element in $BC(R_-;Y)$ defined by $u_t(\theta)=u(t+\theta)$ for $\theta\in R_-$. We impose the following condition on F.

(H1) (i) For some $\omega>0, F$ is $\omega-$periodic in t; that is, $F(t+\omega,\varphi)=F(t,\varphi)$ for all $(t,\varphi)\in R\times BC(R_-;X)$;

(ii) for any $\varphi\in BC(R;X), F(t,\varphi_t)$ is continuous in $t\in R$;

(iii) for any $r>0$ there exist constants $L>0$ and $\vartheta\in(0,1]$ such that

$$\|F(t,\varphi)-F(s,\psi)\|\leq L\{|t-s|^{\vartheta}+\|\varphi-\psi\|^{\vartheta}\}$$

for all $(t,\varphi),(s,\psi)\in[0,r]\times BC(R_-;X)$ with $\|\varphi\|\leq r$ and $\|\psi\|\leq r$.

For any $c>0$, we set $A_c=A-cI$, where I is the identity operator on X. Clearly A_c generates the analytic compact semigroup $T_c(t):=T(t)e^{-ct}$. Since the semigroup $T(t)$ is uniformly bounded by the assumption, one can derive that for each $z\in \mathcal{C}$ with $\Re z>0$, the bounded inverse $(A-zI)^{-1}$ exists and it is given by the formula $(A-zI)^{-1}x=-\int_0^{\infty}e^{-zt}T(t)x\,dt,\ x\in X$. Combining this with the fact that the semigroup $T(t)$ is analytic, one can see that there exist $M > 0$ and $\pi/2 < \eta < \pi$ such that $A_c - \lambda I$ is invertible for $\lambda \in \Sigma := \{\lambda : |\arg \lambda| < \eta\} \cup \{0\}$ and

$$\|(A_c - \lambda I)^{-1}\| \leq \frac{M}{1 + |\lambda|}, \quad \lambda \in \Sigma.$$

Therefore the fractional powers $(-A_c)^\alpha$ of $-A_c$ is defined for $\alpha \geq 0$ (e.g., [8, Section 2.6]), and the estimate

$$\|(-A_c)^\alpha T_c(t)\| \leq C_\alpha t^{-\alpha} e^{-\delta t}, \qquad t > 0, \tag{2.2}$$

holds, here δ and C_α are some positive constants (independent of t) (e.g., [8, Theorem 2.6.13 (c)]).

Now we consider the space

$$\mathcal{X} = C_\omega(R; X) = \{\varphi \in C(R; X) : \varphi(t + \omega) = \varphi(t) \text{ on } R\}.$$

Clearly $\mathcal{X}$ endowed with the norm

$$\|\varphi\| = \sup\{\|\varphi(t)\| : 0 \leq t \leq \omega\}$$

is a Banach space. For any $\varphi \in \mathcal{X}$, we set

$$(\mathcal{H}\varphi)(t) = a + \int_{-\infty}^{t} T_c(t - \theta)[c(\varphi(\theta) - a) + F(\theta, \varphi_\theta)]d\theta, \quad t \in R. \tag{2.3}$$

By the periodicity of the function $F(\theta, \varphi_\theta)$ and (2.2) with $\alpha = 0$, one can see that $\mathcal{H}$ is a well-defined mapping from $\mathcal{X}$ into $\mathcal{X}$.

Lemma 2.1. *The map $\mathcal{H} : \mathcal{X} \mapsto \mathcal{X}$ is compact.*

Proof. First we establish the continuity of $\mathcal{H}$. Let φ and ψ with $\|\varphi\| \leq r$ and $\|\psi\| \leq r$ be given. By (H1-iii) and (2.2), we have

$$\begin{aligned}
\|\mathcal{H}\varphi - \mathcal{H}\psi\| &= \sup_{0 \leq t \leq \omega} \left\| \int_{-\infty}^{t} T_c(t - \theta)[c(\varphi(\theta) - \psi(\theta)) + F(\theta, \varphi_\theta) - F(\theta, \psi_\theta)]d\theta \right\| \\
&\leq \sup_{0 \leq t \leq \omega} \int_{-\infty}^{t} C_0 e^{-\delta(t-\theta)} d\theta \{c\|\varphi - \psi\| + L\|\varphi - \psi\|^\vartheta\} \\
&\leq (C_0/\delta)\{c\|\varphi - \psi\| + L\|\varphi - \psi\|^\vartheta\}.
\end{aligned}$$

This shows the continuity of $\mathcal{H}$. Next we prove that for any bounded set S in $\mathcal{X}$ the set $\mathcal{L}S$ is relatively compact in $\mathcal{X}$, where $\mathcal{L}\varphi := \mathcal{H}\varphi - a$. By (H1-iii), we get

$$\sup\{\|c(\varphi(\theta) - a) + F(\theta, \varphi_\theta)\| : \varphi \in S, \theta \in R\}(=: Q) < \infty.$$

Then (2.2) yields

$$\begin{aligned} \|(-A_c)^{1/2}(\mathcal{L}\varphi)(t)\| &\leq Q\int_{-\infty}^{t} \|(-A_c)^{1/2}T_c(t-\tau)\|d\tau \\ &\leq QC_{1/2}\int_{-\infty}^{t} (t-\tau)^{-1/2}e^{-\delta(t-\tau)}d\tau \\ &= QC_{1/2}\int_0^{\infty} s^{-1/2}e^{-\delta s}ds < \infty \end{aligned}$$

for all $\varphi \in S$, which shows that the set $[((-A_c)^{1/2}\mathcal{L})S](t)$ is bounded in X for all $t \in R$. Therefore the set $(\mathcal{L}S)(t)$ is relatively compact in X, because the operator $(-A_c)^{-1/2} = \dfrac{1}{\Gamma(1/2)}\displaystyle\int_0^{\infty} t^{-1/2}T_c(t)dt : X \mapsto X$ is compact. We claim that the family of functions $\{(\mathcal{L}\varphi)(\cdot) : \varphi \in S\}$ is equicontinuous on R. If the claim holds true, then the set $\mathcal{L}S$ is relatively compact in $\mathcal{X}$ by the Ascoli-Arzéla theorem, as required. Let $h > 0$ and $\varphi \in S$ be given, and set $g(t) = c(\varphi(t) - a) + F(t, \varphi_t)$. Then

$$\begin{aligned} &\|(\mathcal{L}\varphi)(t+h) - (\mathcal{L}\varphi)(t)\| \\ &= \|\int_t^{t+h} T_c(t+h-\tau)g(\tau)d\tau + \int_{-\infty}^{t}(T_c(t+h-\tau) - T_c(t-\tau))g(\tau)d\tau\| \\ &\leq C_0Qh + Q\int_{-\infty}^{t} \|T_c(t+h-\tau) - T_c(t-\tau)\|d\tau \end{aligned}$$

by (2.2). Since $\|T_c(t+h-\tau) - T_c(t-\tau)\| \leq 2C_0e^{-\delta(t-\tau)}$ and

$$\begin{aligned} \|T_c(t+h-\tau) - T_c(t-\tau)\| &= \|\int_{t-\tau}^{t-\tau+h} A_cT_c(\theta)d\theta\| \\ &\leq C_1\int_{t-\tau}^{t-\tau+h} \theta^{-1}e^{-\delta\theta}d\theta \\ &\leq C_1h(t-\tau)^{-1}e^{-\delta(t-\tau)} \end{aligned}$$

for $\tau < t$, we get

$$\|(\mathcal{L}\varphi)(t+h) - (\mathcal{L}\varphi)(t)\|$$

$$
\begin{aligned}
&\leq C_0Qh + Q\{\int_{-\infty}^{t-h} \|T_c(t+h-\tau) - T_c(t-\tau)\| d\tau + \int_{t-h}^{t} \|T_c(t+h-\tau) - T_c(t-\tau)\| d\tau\} \\
&\leq C_0Qh + Q\{\int_{-\infty}^{t-h} C_1 h(t-\tau)^{-1} e^{-\delta(t-\tau)} d\tau + \int_{t-h}^{t} 2C_0 e^{-\delta(t-\tau)} d\tau\} \\
&\leq C_0Qh + Q\{C_1 h \int_{h}^{\infty} \theta^{-1} e^{-\delta\theta} d\theta + 2C_0 h\} \\
&\leq 3C_0Qh + QC_1 h(\delta^{-1} + |\log h|).
\end{aligned}
$$

Thus

$$
\sup\{\|(\mathcal{L}\varphi)(t+h) - (\mathcal{L}\varphi)(t)\| : \varphi \in S, t \in R\} \leq 3C_0Qh + QC_1h(\delta^{-1} + |\log h|)
$$

for $h > 0$, and consequently $\sup\{\|(\mathcal{L}\varphi)(t+h) - (\mathcal{L}\varphi)(t)\| : \varphi \in S, t \in R\} \to 0$ as $h \to 0^+$ This proves the equicontinuity, as required.

For any $k \in R$ we consider the equation

$$
\frac{du}{dt} = A_c(u(t) - a) + k[c(u(t) - a) + F(t, u_t)], \quad t > 0, \tag{2.4}
$$

and moreover, we define the mapping $H : R \times \mathcal{X} \mapsto \mathcal{X}$ by

$$
[H(k, \varphi)](t) = a + k \int_{-\infty}^{t} T_c(t-\theta)\{c(\varphi(\theta) - a) + F(\theta, \varphi_\theta)\} d\theta \tag{2.5}
$$

for any $(k, \varphi) \in R \times \mathcal{X}$. In case of $k = 1$, (2.4) and $H(k, \cdot)$ are identical with (2.1) and $\mathcal{H}$, respectively. $u \in C(R; X)$ is called a solution of (2.4) if u is continuously differentiable on $(0, \infty)$ and it satisfies (2.4) together with $u(t) - a \in \mathcal{D}(A_c) = \mathcal{D}(A)$ for $t > 0$.

Lemma 2.2. *Let $\varphi \in \mathcal{X}$. Then φ is an ω-periodic solution of (2.4) if and only if $H(k, \varphi) = \varphi$.*

Proof. In order to prove the "if " part, we suppose $H(k, \varphi) = \varphi$. We first assert that

$$
\varphi(t) - a = T_c(t)(\varphi(0) - a) + k \int_0^t T_c(t-s) g(s) ds, \quad t \geq 0, \tag{2.6}
$$

where $g(t) := c(\varphi(t)-a)+F(t,\varphi_t)$. Indeed, since $g \in C_\omega(R;X)$, one can choose a sequence of continuously differentiable functions $\{g_n\} \subset C_\omega(R;X)$ such that $\|g_n\| \le \|g\|$ and $\|g_n(t)-g(t)\| < 1/n$ on R. Set $v_n(t) = \int_{-\infty}^{t} T_c(t-s)g_n(s)ds$ for $t \in R$. Then

$$\begin{aligned}(1/h)\{T_c(h)-I\}v_n(t) &= (1/h)\{\int_{-\infty}^{t} T_c(t+h-s)g_n(s)ds - \int_{-\infty}^{t} T_c(t-s)g_n(s)ds\} \\ &= \int_{-\infty}^{t} T_c(t-\theta)\frac{g_n(\theta+h)-g_n(\theta)}{h}d\theta - \frac{1}{h}\int_{t-h}^{t} T_c(t-\theta)g_n(\theta+h)d\theta\end{aligned}$$

for $h>0$, and hence

$$\begin{aligned}\lim_{h\to 0+}(1/h)\{T_c(h)-I\}v_n(t) &= \int_{-\infty}^{t} T_c(t-\theta)g_n'(\theta)d\theta - g_n(t) \\ &= v_n'(t)-g_n(t)\end{aligned}$$

by the convergence theorem. We thus get

$$v_n(t) \in \mathcal{D}(A_c) \text{ and } A_c v_n(t) = v_n'(t) - g_n(t). \tag{2.7}$$

Making use of this, one can derive the relation

$$v_n(t) = T_c(t)v_n(0) + \int_0^t T_c(t-s)g_n(s)ds, \quad t \ge 0. \tag{2.8}$$

Since $\lim_{n\to\infty}[kv_n(t)] = k\int_{-\infty}^{t} T_c(t-s)g(s)ds = [H(k,\varphi)](t) - a = \varphi(t)-a$, (2.6) follows from (2.8). Now, by [8, Theorem 4.3.1], (2.6) implies that φ is locally Hölder continuous on $(0,\infty)$, and in particular, it is Hölder continuous on $[\omega, 2\omega]$. Therefore φ is uniformly Hölder continuous on R because of the periodicity. From this observation and (H1-iii) it follows that the function $F(t,\varphi_t)$ of t is Hölder continuous on R, and so is the function $g(t)$. Thus, by (2.6) and [8, Theorem 4.3.2], we see that φ is a solution of (2.4).

Next we prove the "only if " part. Suppose that $\varphi \in \mathcal{X}$ is a solution of (2.4). Then $(d/dt)(\varphi(t)-a) = (d/dt)\varphi(t) = A_c(\varphi(t)-a)+kg(t)$, and hence

$$\varphi(t)-a = T_c(t)(\varphi(0)-a) + k\int_0^t T_c(t-\theta)g(\theta)d\theta$$

for $t \geq 0$, where $g(t) = c(\varphi(t) - a) + F(t, \varphi_t)$. Since φ is ω-periodic, so is the function g. Then

$$\begin{aligned} \varphi(t) &= a + T_c(t+n\omega)(\varphi(0) - a) + k\int_0^{t+n\omega} T_c(t+n\omega-\theta)g(\theta)d\theta \\ &= a + T_c(t+n\omega)(\varphi(0) - a) + k\int_{-n\omega}^{t} T_c(t-\theta)g(\theta)d\theta. \end{aligned}$$

Note that $\|T_c(t+n\omega)(\varphi(0)-a)\| \to 0$ as $n \to \infty$. Letting $n \to \infty$ in the above, we get $\varphi(t) = a + k\int_{-\infty}^{t} T_c(t-\theta)g(\theta)d\theta = [H(k,\varphi)](t)$, as required.

Proposition 2.3. *Assume (H1). Moreover, let G be a bounded open set in $\mathcal{X}$ with $a \in G$, and suppose that*

$$\varphi \notin \partial G \quad (:= \text{the boundary of } G)$$

whenever $\varphi \in \bar{G}$ is a solution of (2.4) with $k \in (0,1]$. Then the equation (2.1) has an ω-periodic solution which belongs to G.

Proof. Consider the operator $\mathcal{T} = \mathcal{H}\,|_{\bar{G}}: \bar{G} \mapsto \mathcal{X}$, where $\mathcal{H}$ is the one defined by (2.3). We assert that $\mathcal{T}\varphi \neq a + \tau(\varphi - a)$ for all $\tau > 1$ and $\varphi \in \partial G$. Indeed, if this is false, then there exist $\varphi \in \partial G$ and $\tau > 1$ such that $\mathcal{T}\varphi = a + \tau(\varphi - a)$, and hence

$$\begin{aligned} \varphi(t) = a + \frac{1}{\tau}[(\mathcal{H}\varphi)(t) - a] &= a + \frac{1}{\tau}\int_{-\infty}^{t} T_c(t-\theta)[c(\varphi(\theta) - a) + F(\theta, \varphi_\theta)]d\theta \\ &= [H(1/\tau, \varphi)](t). \end{aligned}$$

Then $\varphi \in \partial G$ is a solution of (2.4) with $k = 1/\tau$ by Lemma 2.2, which contradicts our assumption. Thus the assertion must be true. Now the operator $\mathcal{T}$ is compact by Lemma 2.1. Therefore, by the fixed point principle of omitted rays (e.g., [13, Theorem 13.A]), there exists a $\varphi \in \bar{G}$ such that $\mathcal{T}\varphi = \varphi$. Such φ is a solution of (2.4) with $k = 1$ by Lemma 2.2, and hence $\varphi \in G$ by our assumption. Thus φ is the desired one.

§3. Periodic solutions of some diffusive functional differential equations.

Throughout this section, we will employ the following notation. Let R^N be the N-dimensional Euclidean space with norm $|\cdot|$. For any diagonal matrix $B = diag(b_1, \cdots, b_N)$ and any vector $u = (u_1, \cdots, u_N)$, we denote by Bu the vector $(b_1u_1, \cdots, b_Nu_N)$. For any vectors $u = (u_1, \cdots, u_N)$ and $v = (v_1, \cdots, v_N)$, we write as $u \leq v$ (resp. $u < v$) whenever $u_i \leq v_i$ (resp. $u_i < v_i$) for all $i = 1, \cdots, N$. If $u, v \in R^N$ with $u \leq v$, we set $[u, v] = \{w \in R^N : u \leq w \leq v\}$, and call it an interval in R^N. Also, we denote by R^N_+ the set $\{u \in R^N : 0 := (0, \cdots, 0) \leq u\}$. Let Ω be a bounded domain in R^ℓ with smooth boundary $\partial\Omega$ (e.g., $\partial\Omega \in C^{2+\alpha}$ for some $\alpha \in (0,1)$), and denote by $\partial/\partial n$ the exterior normal derivative at $\partial\Omega$. Furthermore, Δ denotes the Laplacian operator in R^ℓ, and Δv and $\partial v/\partial n$ denote $\Delta v = (\Delta v_1, \cdots, \Delta v_N)$ and $\partial v/\partial n = (\partial v_1/\partial n, \cdots, \partial v_N/\partial n)$, respectively, for any (smooth) mapping $v = (v_1, \cdots, v_N) : \bar{\Omega} \mapsto R^N$.

In this section, we discuss the existence of periodic solutions of the diffusive functional differential equation

$$\frac{\partial u}{\partial t}(t, x) = D(x)\Delta u(t, x) + f(t, x, u_t(\cdot, x)) \quad \text{in } (0, \infty) \times \Omega \tag{3.1}$$

satisfying the boundary condition

$$\frac{\partial u}{\partial n}(t, x) = \kappa(x)(K - u(t, x)) \quad \text{on } (0, \infty) \times \partial\Omega. \tag{3.2}$$

Here $D(x) = diag(d_1(x), \cdots, d_N(x))$ with $d_i \in C^\alpha(\bar{\Omega})$ with $d_i(x) > 0$ on $\bar{\Omega}, \kappa(x) = diag(\kappa_1(x), \cdots, \kappa_N(x))$ with $\kappa_i \in C^{1+\alpha}(\partial\Omega)$ and $\kappa_i(x) \geq 0$ on $\partial\Omega$, and $K \in R^N$ is a (fixed) constant vector such that $0 < K$. We assume the following condition on the mapping $f : R \times \bar{\Omega} \times BC(R_-; R^N) \mapsto R^N$.

(H2) (i) $f(t, x, \xi)$ is ω-periodic in t;

(ii) for each $\psi \in BC(R \times \bar{\Omega}; R^N)$, the function $f(t, x, \psi_t(\cdot, x))$ is continuous in

$(t,x) \in R \times \bar{\Omega}$;

(iii) for any $r > 0$ there exist constants $L > 0$ and $\vartheta \in (0,1]$ such that

$$|f(t,x,\xi) - f(s,y,\chi)| \leq L\{|t-s|^{\vartheta} + |x-y|^{\vartheta} + \|\xi - \chi\|^{\vartheta}\}$$

for all $(t,x,\xi),(s,y,\chi) \in [0,r] \times \bar{\Omega} \times BC(R_-;R^N)$ with $\|\xi\| \leq r$ and $\|\chi\| \leq r$.

In order to apply the results in the previous section, we take the Banach space $C(\bar{\Omega};R^N)$ equipped with the supremum norm as X, and define the map $F : R \times BC(R_-;X) \mapsto X$ by

$$F(t,\varphi)(x) = f(t,x,\varphi(\cdot,x)), \quad t \in R, x \in \bar{\Omega}.$$

Clearly F satisfies the condition (H1). For each $i = 1,\cdots,N$, we next consider the (unbounded) linear operator $\tilde{A}_i$ in $\tilde{X} = C(\bar{\Omega};R)$ which is the closed extension of the operator $d_i\Delta$ with the domain $\mathcal{D}(d_i\Delta) = \{\xi \in C^2(\bar{\Omega};R) : \partial\xi/\partial n + \kappa_i\xi = 0 \text{ on } \partial\Omega\}$. By virtue of [11, Theorem 2], $\tilde{A}_i$ generates an analytic semigroup $\tilde{T}_i(t)$ on $\tilde{X}$. Moreover, by the estimate

$$|z|^{-1/2}\|\xi\|_{C^1(\bar{\Omega})} \leq C\|(\tilde{A}_i - z)\xi\|_{C(\bar{\Omega})}$$

for all complex z in a truncated sector $|\arg z| \leq \frac{1}{2}\pi + \varepsilon, |z| \geq \lambda_0$ (cf. (1.1) in [11]), we see that each resolvent operators of $\tilde{A}_i$ is compact on $\tilde{X}$, and hence the semigroup $\tilde{T}_i(t)$ is compact. It is a direct consequence of the maximum principle that $\tilde{T}_i(t)$ is nonexpansive; that is, $\|\tilde{T}_i\| \leq 1$. For any $\varphi = (\varphi_1,\cdots,\varphi_N) \in \mathcal{D}(\tilde{A}_1) \times \cdots \times \mathcal{D}(\tilde{A}_N) =: \mathcal{D}(A)$, we set

$$A\varphi = (\tilde{A}_1\varphi_1,\cdots,\tilde{A}_N\varphi_N).$$

Then A generates the analytic compact semigroup $T(t) := (\tilde{T}_1(t),\cdots,\tilde{T}_N(t))$ of nonexpansive bounded linear operators on X. Take any (small) constant $c > 0$, and consider the operator $H(k,\varphi)$ defined by (2.5) with $a = K$. By Lemma 2.2, $\varphi = H(k,\varphi)$ means that $\varphi(t)$ is an ω-periodic solution of the (abstract) equation (2.4) on $X = C(\bar{\Omega};R^N)$.

Now we certify that $u(t,x) := [\varphi(t)](x)$ satisfies the diffusive functional differential equation

$$\frac{\partial u}{\partial t}(t,x) = D(x)\Delta u(t,x) + c(1-k)(K-u(t,x)) + kf(t,x,u_t(\cdot,x)) \tag{3.3}$$

in $(0,\infty)\times\Omega$, together with the boundary condition (3.2), whenever $\varphi = H(k,\varphi)$.

Lemma 3.1. *Let (H2) hold, and suppose that $H(k,\varphi)=\varphi$ for some $\varphi\in\mathcal{X}$. Then the function $u(t,x) := [\varphi(t)](x)$ is an ω-periodic (classical) solution of (3.3) and (3.2).*

Proof. The lemma can be proved by the standard regularity argument (e.g., [4, pp.75-76]). For completeness we contain the proof. Set $v(t) = \varphi(t) - a$ and $g(t) = cv(t) + F(t,\varphi_t)$. Since v is ω-periodic, from the fact that $v\in C^1((0,\infty);X)$ it follows that $v(t,x) := [v(t)](x)$ is continuously differentiable with respect to $t\in R$ uniformly for $x\in\bar\Omega$, together with $\sup_{t\in R}\|(d/dt)v(t)\| = \sup\{|(\partial/\partial t)v(t,x)| : t\in R,\ x\in\bar\Omega\} < \infty$. Then (H2-iii) yields that $\|g(t)-g(s)\| \le C|t-s|^{\vartheta}$, $t,\ s\in R$, for some constant C. Let $0<\delta<\beta<\vartheta$, and take $p>0$ so large that $\delta + (N/p) < \min\{2\beta, 1\}$. Since $(d/dt)v(t) = A_c v(t) + kg(t)$, it follows from [4, Lemma 3.5.1] that the function $t\in R\mapsto A_c^{\beta}(dv/dt)\in X$ $(\subset L^p(\Omega))$ is locally Hölder continuous, and consequently the function $t\in R\mapsto (\partial v/\partial t)(t,\cdot)\in C^{\delta}(\bar\Omega)$ also is locally Hölder continuous by the standard argument in L^p-theory (e.g., [4, p.75], [8, Chapter 8]). Also, from the fact that the function $t\in R\mapsto A_c v(t)\in X$ is continuous, it follows that the function $t\in R\mapsto v(t,\cdot)\in C^{1+\delta}(\bar\Omega)$ is continuous. Thus $g(t)(\cdot)\in C^{\vartheta}(\bar\Omega)$ by (H2-iii), and consequently $A_c v(t) = dv/dt - kg(t)\in C^{\delta}(\bar\Omega)$. Hence $v(t,\cdot)\in C^{2+\delta}(\bar\Omega)$ by a classical regularity theorem for elliptic equations (cf. [4, p.10]), and consequently $g\in C^{\delta/2,\delta}(R\times\bar\Omega)$ by (H2-iii). Also, by the standard argument in L^p-theory (e.g., [4, p.75]) it follows that v satisfies $\partial v/\partial n + \kappa v = 0$ on $(0,\infty)\times\partial\Omega$. Consequently, $\partial u/\partial n = \partial v/\partial n = -\kappa v = \kappa(K-u)$ on $(0,\infty)\times\partial\Omega$, and hence u satisfies (3.2). Since the compatibility condition of order 0 is satisfied for $v(0,\cdot)\in C^{2+\delta}(\bar\Omega)$, by [5, Theorem 5.3, p.320] there exists a unique function

$\bar{v} \in C^{1+\delta/2,2+\delta}(R\times\bar{\Omega})$ satisfying $\partial\bar{v}/\partial t = D\Delta\bar{v} - c\bar{v} + kg$ in $(0,\infty)\times\Omega$, $\partial\bar{v}/\partial n + \kappa\bar{v} = 0$ on $(0,\infty)\times\partial\Omega$ and $\bar{v}(0,x) = v(0,x)$ on $\bar{\Omega}$. Then $(d/dt)\bar{v}(t) = A_c\bar{v}(t) + kg(t)$ and $\bar{v}(0) = v(0)$ in X, and hence one gets $\bar{v}(t) = T_c(t)v(0) + k\int_0^t T_c(t-s)g(s)ds = v(t)$ or $\bar{v}(t,x) \equiv v(t,x)$. Consequently, the function $u(t,x)$ is continuously differentiable in t, twice continuously differentiable in x, and satisfies (3.3) on $R\times\Omega$. This completes the proof.

Combining Lemma 3.1 with Proposition 2.3, we obtain the following result:

Theorem 3.2. *Let (H2) hold, and assume that there exist some constant vectors μ_1, μ_2, ν_1 and ν_2 in R^N such that $\mu_1 < \mu_2 \le K \le \nu_2 < \nu_1$ and that*

$$\mu_2 \le u(t,x) \le \nu_2 \quad \text{on } R\times\bar{\Omega}$$

whenever $u(t,x)$ is an ω-periodic solution of (3.3) with $k\in(0,1]$ satisfying $\mu_1 \le u(t,x) \le \nu_1$ on $R\times\bar{\Omega}$ together with (3.2). Then there exists an ω-periodic solution of (3.1) and (3.2) of which the range is contained in the interval $[\mu_2,\nu_2]$.

Now we provide two examples which show how Theorem 3.2 is effectively applicable.

Example 1. Together with the boundary condtion (3.2) (with $N=1$), we consider the scalar diffusive functional differential equation

$$\frac{\partial u}{\partial t}(t,x) = d(x)\Delta u(t,x) - \lambda(t,x)u(t,x) + g(t,x,u_t(\cdot,x)) \quad \text{in } (0,\infty)\times\Omega, \tag{3.4}$$

where $d\in C^\alpha(\bar{\Omega})$ with $d(x)>0$ on $\bar{\Omega}$. Assume that:

(H3) (H2) is satisfied for the function $f(t,x,\xi) = -\lambda(t,x)\xi(0) + g(t,x,\xi)$ with $N=1$.

(H4) (i) $0 < \underline{\lambda} := \inf_{t,x}\lambda(t,x) \le \sup_{t,x}\lambda(t,x) =: \bar{\lambda} < \infty$;

(ii) $\xi \in BC(R_-;R_+)$, $R_+ := [0,\infty)$, implies $\inf_{t,x} g(t,x,\xi) > 0$. Moreover, $\xi,\chi \in BC(R_-;R_+)$ with $\xi(\theta)\le\chi(\theta)$ on R_- implies $g(t,x,\xi)\ge g(t,x,\chi)$ on $R\times\bar{\Omega}$;

(iii) there exists a constant $\nu_1 > 0$ such that $\nu_1 > \nu_2 := (1/\underline{\lambda})\sup_{t,x} g(t,x,0)$

and that

$$g(t,x,\nu_1) \le K\lambda(t,x) \le g(t,x,0) \quad \text{on } R \times \bar{\Omega},$$

where K is the one in (3.2).

Equation (3.4) describes a mathematical model for the survival of red blood cells in an animal (cf. [6, 12]). It is easy to see that (H4-ii) is satisfied whenever g is given by $g(t,x,\xi) = \int_0^\infty e^{-\gamma(\tau)\xi(-\tau)}dp(\tau)$, where $\gamma \in BC(R_+;R_+)$ and $p : R_+ \mapsto R$ is bounded and nondecreasing. We set

$$\mu_2 = (1/\bar{\lambda}) \inf_{t,x} g(t,x,\nu_1).$$

Then $0 < \mu_2 \le K \le \nu_2$ by (H4).

Theorem 3.3. *Assume (H3) and (H4), and let μ_2 and ν_2 be the constants cited above. Then there exists an ω-periodic solution of (3.4) and (3.2) of which the range is contained in the interval $[\mu_2, \nu_2]$.*

Proof. Take a $\mu_1 \in (0,\mu_2)$. Then $0 < \mu_1 < \mu_2 \le K \le \nu_2 < \nu_1$. For any $k \in (0,1]$, let $u(t,x)$ be an ω-periodic solution of the equation

$$\frac{\partial u}{\partial t}(t,x) = d(x)\Delta u(t,x) + c(1-k)(K - u(t,x)) + k[-\lambda(t,x)u(t,x) + g(t,x,u_t(\cdot,x))]$$

in $(0,\infty) \times \Omega$ satisfying $\mu_1 \le u(t,x) \le \nu_1$ on $R \times \bar{\Omega}$ together with the condition (3.2). In order to establish the theorem, it suffices to prove that

$$\mu_2 \le u(t,x) \le \nu_2 \quad \text{on } R \times \bar{\Omega}.$$

Now, let $\varepsilon > 0$ be any number such that $\varepsilon < \min\{\mu_1, \bar{\lambda}\mu_2\}$, and consider the solution $m(t)$ of the ordinary differential equation

$$\frac{d}{dt}m(t) = [-k(\bar{\lambda} - c) - c]m(t) + Kc + k(\bar{\lambda}\mu_2 - Kc), \quad t > 0,$$

with $m(0) = \mu_1 - \varepsilon$. Clearly $m(t)$ is given by

$$m(t) = (\mu_1 - \varepsilon)e^{-(k\bar{\lambda}-kc+c)t} + \frac{Kc + k(\bar{\lambda}\mu_2 - Kc)}{k(\bar{\lambda} - c) + c}\{1 - e^{-(k\bar{\lambda}-kc+c)t}\}. \tag{3.5}$$

It is easy to check that $0 < m(t) < K$ on R_+. We assert that

$$u(t,x) > m(t), \quad (t,x) \in R_+ \times \bar{\Omega}. \tag{3.6}$$

To establish the assertion by a contradiction, we suppose that (3.6) is false. Then there exists some $(t_0, x_0) \in (0,\infty) \times \bar{\Omega}$ such that $u(t_0,x_0) = m(t_0)$ and $u(t,x) > m(t)$ for all $(t,x) \in [0,t_0) \times \bar{\Omega}$. Set $w(t,x) = m(t) - u(t,x)$. Then $w(t_0,x_0) = 0$ and $w(t,x) < 0$ for all $(t,x) \in [0,t_0) \times \bar{\Omega}$. Moreover, we get

$$\begin{aligned} \frac{\partial w}{\partial t}(t,x) &= \frac{d}{dt}m(t) - \frac{\partial u}{\partial t}(t,x) \\ &= d(x)\Delta w(t,x) - (k\lambda(t,x) + c - kc)w(t,x) + km(t)(\lambda(t,x) - \bar{\lambda}) \\ &\qquad + k(\bar{\lambda}\mu_2 - g(t,x,u_t(\cdot,x))) \end{aligned}$$

on $(0,t_0] \times \Omega$. Since $0 < \mu_1 \leq u(t,x) \leq \nu_1$ on $R \times \bar{\Omega}$, we get $\underline{\lambda}\nu_2 \geq g(t,x,0) \geq g(t,x,u_t(\cdot,x)) \geq g(t,x,\nu_1) \geq \bar{\lambda}\mu_2$ by (H4-ii), and hence $d(x)\Delta w(t,x) - (\partial/\partial t)w(t,x) - (k(\lambda(t,x) + c - kc)w(t,x) \geq 0$ on $(0,t_0] \times \Omega$. This would lead to a contradiction. Indeed, if $x_0 \in \Omega$, then $w(t,x) \equiv 0$ on $[0,t_0] \times \bar{\Omega}$ by the strong maximum principle (e.g.,[9, Theorems 3.3.5, 3.3.6 and 3.3.7]), which is a contradiction because of $u(0,x) \geq \mu_1 > m(0)$. We thus obtain $x_0 \in \partial\Omega$ and $w(t,x) < 0$ on $[0,t_0] \times \Omega$, and hence $\partial w/\partial n > 0$ at (t_0,x_0) by the strong maximum principle, again. However, this is impossible because of $(\partial w/\partial n)(t_0,x_0) = -(\partial u/\partial n)(t_0,x_0) = \kappa(x_0)(u(t_0,x_0) - K) = \kappa(x_0)(m(t_0) - K) \leq 0$.

Now we obtain $\lim_{t\to\infty} m(t) = [Kc + k(\bar{\lambda}\mu_2 - Kc)]/[k(\bar{\lambda} - c) + c]$ by (3.5), and consequently $\lim_{t\to\infty} m(t) \geq \mu_2$ because of $k \in (0,1]$. Therefore, from (3.6) it follows that $\underline{lim}_{t\to\infty} u(t,x) \geq \lim_{t\to\infty} m(t) \geq \mu_2$ on $\bar{\Omega}$, and consequently $\min_{\bar{\Omega}} u(t,\cdot) \geq \mu_2$ on R by the periodicity of $u(t,x)$. Similarly, one can prove that $\max_{\bar{\Omega}} u(t,\cdot) \leq \nu_2$ on R. Indeed, in place of $m(t)$, we may consider the solution $M(t)$ of the ordinary differential equation

$$\frac{d}{dt}M(t) = [-(\underline{\lambda} - c)k - c]M(t) + Kc + k(\underline{\lambda}\nu_2 - Kc), \quad t > 0,$$

with $M(0) = \nu_1+\varepsilon$, and repeat the argument similar to the one for $m(t)$. This completes the proof of the theorem.

Example 2. We next consider a system of diffusive functional differential equations

$$\frac{\partial u_i}{\partial t}(t,x) = d_i(x)\Delta u_i(t,x)+u_i(t,x)\{a_i(t,x)-b_i(t,x)u_i(t.x)-\sum_{j=1}^{N} c_{ij}(t,x)h_{ij}(t,x,u_t(\cdot,x))\}, \tag{3.7}$$

$$i = 1,\cdots,N,$$

in $(0,\infty)\times\Omega$, together with the homogeneous Neumann boundary condition

$$\frac{\partial u}{\partial n}(t,x) = 0 \quad \text{on } (0,\infty)\times\partial\Omega, \tag{3.8}$$

where $u = (u_1,\cdots,u_N)$, and the functions $a_i, b_i, c_{ij} : R\times\bar{\Omega}\mapsto R, i,j = 1,\cdots,N$, are continuous and ω-periodic in t, and moreover $d_i \in C^\alpha(\bar{\Omega})$ with $d_i(x) > 0$ on $\bar{\Omega}$ for $i = 1,\cdots,N$. Note that (3.8) is a special case of (3.2) (with $\kappa = 0$). We assume that:

(H5) (H2) is satisfied for the function $f(t,x,\xi) := (f_1(t,x,\xi),\cdots,f_N(t,x,\xi))$, where

$$f_i(t,x,\xi) = \xi_i(0)\{a_i(t,x) - b_i(t,x)\xi_i(0) - \sum_{j=1}^{N} c_{ij}(t,x)h_{ij}(t,x,\xi)\}$$

for $\xi = (\xi_1,\cdots,\xi_N)$.

(H6) (i)

$$\begin{cases} 0 < \underline{a}_i := \inf_{t,x} a_i(t,x) \le \sup_{t,x} a_i(t,x) =: \bar{a}_i < \infty, \\ 0 < \underline{b}_i := \inf_{t,x} b_i(t,x) \le \sup_{t,x} b_i(t,x) =: \bar{b}_i < \infty, \\ 0 \le \underline{c}_{ij} := \inf_{t,x} c_{ij}(t,x) \le \sup_{t,x} c_{ij}(t,x) =: \bar{c}_{ij} < \infty \end{cases}$$

for $i,j = 1,\cdots,N$, and moreover

$$\underline{a}_i > \sum_{j=1}^{N} \bar{c}_{ij}(\bar{a}_j/\underline{b}_j)$$

for $i = 1, \cdots, N$;

(ii) for any $i, j = 1, \cdots, N, \xi, \chi \in BC(R_-; R_+^N)$ with $\xi(\theta) \leq \chi(\theta)$ on R_- implies $0 \leq h_{ij}(t, x, \xi) \leq h_{ij}(t, x, \chi)$, and moreover $h_{ij}(t, x, p) = p_j$ (normalized) whenever $p = (p_1, \cdots, p_N)$ is a constant.

In mathematical ecology, (3.7) describes the growth of competing N-species whose i-th population density at time t and place x is $u_i(t, x)$, and $h_{ij}(t, x, u_t(\cdot, x))$ represents the effect of the past history on the present growth rate (cf. [1, 3, 6, 10]). It is easy to see that (H6-ii) is satisfied if $h_{ij}(t, x, \xi) = \int_0^\infty K_{ij}(\theta)\xi_j(-\theta)d\theta$ for $\xi = (\xi_1, \cdots, \xi_N) \in BC(R_-; R^N)$, where $K_{ij}(\theta) \geq 0$ and $\int_0^\infty K_{ij}(\theta)d\theta = 1$. The condition (H6-i) is the one considered by Gopalsamy in [3] to derive the existence of a globally stable ω-periodic solution for integrodifferential equations (without diffusion). Ahmad and Lazer [1, Theorem 4.1] have treated the case where $N = 2$ and d_i depends on the variable t as well as x, and established the existence of an ω-periodic solution under the condition (H6-i) with $N = 2$. We remark that the case where $h_{ij}(t, x, \xi) = \xi_j(0)$; that is, (3.7) does not contain delay-terms, has been treated in [1], and the method in [1] is not applicable to delay-equations. In the following theorem we also impose the condition (H6-i) and deduce the existence of an ω-periodic solution of (3.7) and (3.8).

Theorem 3.4. *Assume (H5)and (H6). Then there exists an ω-periodic solution of (3.7) and (3.8) of which the range is contained in the product*

$$\prod_{i=1}^{N}[\frac{\underline{a}_i - \sum_{j=1}^{N} \bar{c}_{ij}(\bar{a}_j/\underline{b}_j)}{\bar{b}_i}, \frac{\bar{a}_i}{\underline{b}_i}].$$

Proof. Take a $\delta > 0$ so that

$$0 < 2\delta\bar{b}_i < \underline{a}_i - \sum_{j=1}^{N} \bar{c}_{ij}(\bar{a}_j/\underline{b}_j), \quad i = 1, \cdots, N,$$

which is possible by (H6-i), and consider the vectors $\mu_1, \mu_2, \nu_1, \nu_2$ and K in R^N whose

i-th components are defined by

$$\begin{aligned}\mu_{1,i} &= \delta, \quad \mu_{2,i} = [\underline{a}_i - \sum_{j=1}^{N} \bar{c}_{ij}(\bar{a}_j/\underline{b}_j)]/\bar{b}_i, \quad \nu_{1,i} = 2\bar{a}_i/\underline{b}_i, \\ \nu_{2,i} &= \bar{a}_i/\underline{b}_i, \quad K_i = (\bar{a}_i - 2\delta \sum_{j=1}^{N} \underline{c}_{ij})/\underline{b}_i,\end{aligned}$$

respectively. Since $2\delta < \bar{a}_i/\underline{b}_i$ for all i, we get $0 < \mu_1 < \mu_2 \leq K \leq \nu_2 < \nu_1$. Now, for any $k \in (0,1]$, let $u(t,x) = (u_1(t,x), \cdots, u_N(t,x))$ be an ω-periodic solution of a system of diffusive functional differential equations

$$\begin{aligned}\frac{\partial u_i}{\partial t}(t,x) &= d_i(x)\Delta u_i(t,x) + c(1-k)(K_i - u_i(t,x)) \\ &\quad + k u_i(t,x)\{a_i(t,x) - b_i(t,x)u_i(t,x) - \sum_{j=1}^{N} c_{ij}(t,x)h_{ij}(t,x,u_t(\cdot,x))\} \\ &\qquad i = 1, \cdots, N,\end{aligned}$$

in $(0,\infty) \times \Omega$ satisfying $\mu_1 \leq u(t,x) \leq \nu_1$ on $R \times \bar{\Omega}$ together with (3.8), where $c > 0$ is a constant. In order to establish the theorem, it suffices to certify that

$$\mu_2 \leq u(t,x) \leq \nu_2 \quad \text{on} \quad R \times \bar{\Omega}. \tag{3.9}$$

Consider the solution $M(t) = (M_1(t), \cdots, M_N(t))$ of a system of ordinary differential equations

$$\begin{aligned}\frac{d}{dt}M_i(t) &= k\{(M_i(t) - K_i)(c - \frac{c}{k}) + M_i(t)(\bar{a}_i - \underline{b}_i M_i(t) - \sum_{j=1}^{N} \underline{c}_{ij}\mu_{1,j})\}, \\ &\qquad t > 0, \quad i = 1, \cdots, N,\end{aligned}$$

with $M(0) = \nu_1 + \varepsilon$, where ε is any positive number. It is easy to see that $M(t)$ exists globally and $0 < M(t)$ on R_+. Furthermore, by employing the same manner as in the proof of (3.6), one can deduce that

$$u(t,x) < M(t), \quad (t,x) \in R_+ \times \bar{\Omega}. \tag{3.10}$$

Each component $M_i(t)$ of $M(t)$ is a positive solution of ordinary differential equation $\dot{y} = -By^2 + Ay + C(\equiv G(y))$, where $A = c(k-1) + k\bar{a}_i - k\sum_{j=1}^{N} \underline{c}_{ij}\mu_{1,j}, B = \underline{b}_i k$ and $C = K_i c(1-k)$. Since $B > 0$ and $C \geq 0$, $M_i(t)$ tends to the unique positive root of the quadratic equation $G(y) = 0$ (say, $\gamma(k,i)$), as $t \to \infty$. The root $\gamma(k,i)$ is given by the equation

$$2\underline{b}_i\gamma(k,i) = \bar{a}_i - \tau - \sum_{j=1}^{N} \underline{c}_{ij}\mu_{1,j} + \{(\bar{a}_i - \tau - \sum_{j=1}^{N} \underline{c}_{ij}\mu_{1,j})^2 + 4\underline{b}_i K_i \tau\}^{1/2}$$

with $\tau = c(1/k - 1) \geq 0$. Consider the right hand side of the above equation as a function of $\tau \geq 0$, and write it by $\Upsilon(\tau)$, simply. Then $\Upsilon(\tau)$ is nonincreasing in τ, because of

$$\begin{aligned}
&\Upsilon'(\tau)\{(\bar{a}_i - \tau - \sum_{j=1}^{N} \underline{c}_{ij}\mu_{1,j})^2 + 4\underline{b}_i K_i \tau\}^{1/2} \\
&\quad \times[\{(\bar{a}_i - \tau - \sum_{j=1}^{N} \underline{c}_{ij}\mu_{1,j})^2 + 4\underline{b}_i K_i \tau\}^{1/2} - (\bar{a}_i - \tau - \sum_{j=1}^{N} \underline{c}_{ij}\mu_{1,j}) + 2\underline{b}_i K_i] \\
&= 4\underline{b}_i^2 K_i (K_i - \frac{\bar{a}_i - \sum_{j=1}^{N} \underline{c}_{ij}\mu_{1,j}}{\underline{b}_i}) \\
&= 4\underline{b}_i K_i \sum_{j=1}^{N} \underline{c}_{ij}(\mu_{1,j} - 2\delta) \leq 0.
\end{aligned}$$

Thus we get

$$\gamma(k,i) \leq \Upsilon(0)/(2\underline{b}_i) = (\bar{a}_i - \sum_{j=1}^{N} \underline{c}_{ij}\mu_{1,j})/\underline{b}_i \leq \nu_{2,i}$$

for all $k \in (0,1]$ and all $i = 1, \cdots, N$. By the periodicity of u, this fact and (3.10) yield that $u(t,x) \leq \nu_2$ on $R \times \bar{\Omega}$, which proves the half part of (3.9). To prove the remainder of (3.9), we consider the solution $m(t) = (m_1(t), \cdots, m_N(t))$ of a system of ordinary differential equations

$$\frac{d}{dt}m_i(t) = k\{(m_i(t) - K_i)(c - \frac{c}{k}) + m_i(t)(\underline{a}_i - \bar{b}_i m_i(t) - \sum_{j=1}^{N} \bar{c}_{ij}\nu_{2,j})\},$$

$$t > 0, \quad i = 1, \cdots, N,$$

with $m(0) = \mu_1 - (\varepsilon, \cdots, \varepsilon)$, where ε is any positive number such that $\varepsilon < \mu_{1,i}$ for all $i = 1, \cdots, N$. It is easy to see that $m(t)$ exists globally and $0 < m(t)$ on R_+. Since $u(t,x) \leq \nu_2$ on $R \times \bar{\Omega}$, by the same reasoning as for $M(t)$ one can deduce that

$$m(t) < u(t,x), \quad (t,x) \in R_+ \times \bar{\Omega}. \tag{3.11}$$

Observe that each $m_i(t)$ tends to the a positive number $\tilde{\gamma}(k,i)$ as $t \to \infty$, where $\tilde{\gamma}(k,i)$ is given by the equation

$$2\bar{b}_i\tilde{\gamma}(k,i) = \underline{a}_i - \tau - \sum_{j=1}^{N} \bar{c}_{ij}\nu_{2,j} + \{(\underline{a}_i - \tau - \sum_{j=1}^{N} \bar{c}_{ij}\nu_{2,j})^2 + 4\bar{b}_i K_i \tau\}^{1/2},$$

where $\tau = c(1/k - 1) \geq 0$. It is straightforward to see that the right hand side of the above equation is nondecreasing in $\tau \geq 0$. Then we get

$$\tilde{\gamma}(k,i) \geq \frac{\underline{a}_i - \sum_{j=1}^{N} \bar{c}_{ij}\nu_{2,j}}{\bar{b}_i} = \mu_{2,i}$$

for all $k \in (0,1]$ and all $i = 1, \cdots, N$. By the periodicity of u, the above observation and (3.11) yield the remainder part of (3.9). This completes the proof.

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