数理解析研究所講究録
901 巻 1995 年 58-61

次元論における局所化について

横井 勝弥 (KATSUYA YOKOI)

筑波大学数学系

1. Introduction. Dennis Sullivan [S] pointed out the availability and applicability of localization methods in homotopy theory. We shall apply the methods to dimension theory.

Throughout this paper, we shall denote by $\mathcal{P}$ the set of all prime numbers. The full subcategory of the category $\mathcal{G}$ of all groups consisting of all nilpotent groups is denoted by $\mathcal{N}$. Let $P \subseteq \mathcal{P}$. A homomorphism $e\colon G \to G_P$ in $\mathcal{N}$ is said to be a *P-localizing map* if G_P is P-local (i.e., $x \mapsto x^n$, $x \in G_P$, is bijective for all $n \in P'$, where $n \in P'$ means that n is a product of primes in the complementary collection P' of primes with respect to P) and if $e^*\colon \mathrm{Hom}(G_P, K) \approx \mathrm{Hom}(G, K)$ provided $K \in \mathcal{N}$, with K P-local. We know that there exists a P-localization theory on the category $\mathcal{N}$ [H-M-R].

Definition. A connected CW-complex X is *nilpotent* if $\pi_1(X)$ is nilpotent group and operates nilpotently on $\pi_n(X)$ for every $n \geq 2$.

Let $\mathcal{NH}$ be the homotopy category of nilpotent CW-complexes. $\mathcal{NH}$ contains the homotopy category of simply connected CW-complexes. Moreover, the *simple* CW-complexes are plainly in $\mathcal{NH}$; in particular, $\mathcal{NH}$ contains all connected Hopf spaces.

Definition. Let $X \in \mathcal{NH}$ and $P \subseteq \mathcal{P}$. Then X is *P-local* if $\pi_n(X)$ is P-local for all $n \geq 1$. A map $f\colon X \to Y$ in $\mathcal{NH}$ *P-localizes* if Y is P-local and

$$f^*\colon [Y, Z]_* \approx [X, Z]_*$$

for all P-local Z in $\mathcal{NH}$, where $[A, B]_*$ means the set of pointed homotopy classes of maps from A to B.

The following results ([Su], [H-M-R]) are very useful in this paper.

Theorem A. *Every X in $\mathcal{NH}$ admits a P-localization.*

Theorem B. *Let $f\colon X \to Y$ in $\mathcal{NH}$. Then the following statements are equivalent:*

(i) *f P-localizes,*
(ii) *$\pi_n f\colon \pi_n X \to \pi_n Y$ P-localizes for all $n \geq 1$, and*
(iii) *$H_n f\colon H_n X \to H_n Y$ P-localizes for all $n \geq 1$.*

2. Results. In this paper, we define the P-local dimension as follows: the *P-local dimension of a space X is at most n* (denoted by $\dim_P X \leq n$) provided that every map $f\colon A \to S^n_P$ of a closed subset A of X into a P-local n-dimensional sphere S^n_P admits a continuous extension over X. Recall that a space X is said to have cohomological dimension with respect to a coefficient group $G \leq n$, written $c\text{-}\dim_G X \leq n$, provided, for every map $f\colon A \to K(G,n)$ of a closed subset A of X into an Eilenberg-MacLane space $K(G,n)$ of type (G,n) there is an extension to a map $F\colon X \to K(G,n)$. By the dimension of a space X (denoted by $\dim X$) we mean the *covering dimension* of X. $\mathbf{Z}$ is the additive group of all integers and $\mathbf{Q}$ is the additive group of all rational numbers. $\mathbf{Z}_{(P)}$ is the ring of integers localized at P, that is, the subring $\mathbf{Q}$ consisting of rationals expressible as fractions k/l with $l \in P'$. We denote by $\mathbf{Z}_p$ and $\mathbf{Z}_{p^\infty}$ the cyclic group of order p and the quasicyclic group of type $\mathbf{Z}_{p^\infty}$, respectively.

First we shall see the following basic properties:

Proposition 1. *If $\dim_P X \leq n$, then $\dim_P X \leq n+1$.*

Proposition 2. *Let X be a metrizable space. We have the following inequality:*

$$c\text{-}\dim_{\mathbf{Z}_{(P)}} X \leq \dim_P X.$$

In particular, if X is finite dimensional or ANR's, the equality holds.

Proposition 3. *Let X be a metrizable space. We have the following equality:*

$$c\text{-}\dim_{\mathbf{Q}} X = \dim_{\mathbf{Q}} X.$$

Theorem 4. *Let $P_1 \subseteq P_2 \subseteq \mathcal{P}$. Then we have the following inequality:*

$$\dim_{P_1} X \leq \dim_{P_2} X.$$

We shall illustrate by the following example the essential differences between the theory of cohomological dimension and the theory of P-local dimension.

Example 5. There exists a compactum X such that $\dim_2 X \geq 3$ and $c\text{-}\dim_{\mathbf{Z}_{(2)}} X \leq 2$.

Remark 6. The above example follows that $\mathrm{Tor}(\mathbf{Z}_2, \pi_q(S^2)) \neq *$ for infinitely many q [Se]. Note that Serre's proof used Poincaré series and methods of analytic number theory.

The remainder is devoted to developing the main results.

A finite collection $P_1, \cdots, P_s$ of subsets of $\mathcal{P}$ is called a *partition of* $\mathcal{P}$ if $P_1 \cup \cdots \cup P_s = \mathcal{P}$ (we do not assume that P_i are pairwise disjoint).

Theorem 7. *Let X be a compactum. Then the following conditions are equivalent:*

(1) $\dim X < \infty$,
(2) *for some partition* $P_1, \cdots, P_s$ *of* $\mathcal{P}$, $\max\{\dim_{P_i} X : i = 1, \cdots, s\} < \infty$,
(3) *for any partition* $P_1, \cdots, P_s$ *of* $\mathcal{P}$, $\max\{\dim_{P_i} X : i = 1, \cdots, s\} < \infty$.

Remark 8. There exists an infinite dimensional compactum X such that for any partition $P_1, \cdots, P_s$ of $\mathcal{P}$, $\max\{c\text{-}\dim_{\mathbf{Z}_{(P_i)}} X : i = 1, \cdots, s\} < \infty$.

Remark 9. Let $\mathcal{P} = \{p_1, p_2, \cdots\}$. There is an infinite dimensional compactum Y such that $\dim_{p_i} T = i$ for $i \in \mathbf{N}$.

Remark 10. By Theorem 7 and an argument of cohomological dimension, we have

$$\dim X = \sup\{\dim_{P_i} X : i = 1, \cdots, s\}$$

for any partition $P_1, \cdots, P_s$ of $\mathcal{P}$.

We note that the above does not hold for non-compact spaces [Dr-R-S].

Corollary 11. *Let X be a compactum and $P_1, \cdots, P_s$ be a partition of $\mathcal{P}$. Then if $\dim_{P_i} X = c\text{-}\dim_{\mathbf{Z}_{(P_i)}} X$ for $i \in \{1, \cdots, s\}$, $\dim X = c\text{-}\dim_{\mathbf{Z}} X$.*

Remark 12. There is a compactum such that $\dim X = c\text{-}\dim_{\mathbf{Z}} X = \infty$, $\dim_2 X \geq 3$ and $c\text{-}\dim_{\mathbf{Z}_{(2)}} X \leq 2$.

We can get the following Menger-Urysohn's type sum formula for the local dimension.

Theorem 13. *Let $X = A \cup B$ be a metrizable space. Then we have the following inequality:*

$$\dim_P X \leq \dim_P A + \dim_P B + 1.$$

Corollary 14. *Let $X = A \cup B$ be a metrizable space. Then we have the following inequality:*

$$c\text{-}\dim_{\mathbf{Q}} X \leq c\text{-}\dim_{\mathbf{Q}} A + c\text{-}\dim_{\mathbf{Q}} B + 1.$$

In particular, if X is finite dimensional, then the inequality with respect to $\mathbf{Z}_{(p)}$ holds.

Next, we shall develop the relation between localization and cohomological dimension.

Theorem 15. *Let G be an abelian group. We have the following equality:*

$$c\text{-}\dim_G X = \sup\{c\text{-}\dim_{G_p} X : p \in \mathcal{P}\}.$$

Corollary 16. *Let X be a finite dimensional compactum and K be a simply connected CW-complex. The following are equivalent:*

(1) $K \in AE(X)$,

(2) $K_p \in AE(X)$ *for each prime* $p \in \mathcal{P}$.

References

[Bo] B. F. Bockstein, *Homological invariants of topological spaces, I,* (English translation in Amer.Math. Soc.Transl. 11:3 (1050)), Trudy Moskov. Mat. Obshch. **5** (1956), 3–80. (Russian)

[Dr₁] A. N. Dranishnikov, *Homological dimension theory*, Russian Math. Surveys **43:4** (1988), 11–62.

[Dr-R-S] ———, D. Repovš and E. Schepin, *Dimension of products with continua*, preprint.

[D-T] A.Dold and R.Thom, *Quasifaserungen und Unendliche Symmetrische Produkte*, Annals of Math. **67** (1958), 239–281. (German)

[D₁] J.Dydak, *Cohomological dimension and metrizable spaces*, Trans. of the Amer.Math.Soc. **337** (1993), 219–234.

[D₂] ———, *Cohomological dimension and metrizable spaces II*, preprint.

[D-W] ——— and J.J.Walsh, *Infinite dimensional compacta having cohomological dimension two: an application of the Sullivan conjecture*, Topology **32** (1993), 93–104.

[H-R-M] P. Hilton, G. Mislin and J. Roitberg, *Localization of nilpotent groups and spaces*, North-Holland Mathematical Studied 15, Amsterdam, 1975.

[H-W] W. Hurewicz and H. Wallman, *Dimension theory*, Princeton University Press, Princeton, 1941.

[Ko] Y.Kodama, *Appendix to* K. Nagami, *Dimension theory*, Academic Press, New York, 1970.

[Ku] W. I. Kuzminov, *Homological dimension theory*, Russian Math. Surveys **23** (1968), 1–45.

[R] J. Roitberg, *Note on nilpotent spaces and localization*, Math. Z. **137** (1974), 67–74.

[Se] J. P. Serre, *Cohomologie modulo 2 des complexes d'Eilenberg-MacLane*, Comm. Math. Helv. **27** (1953), 198–232.

[Sp] E.Spanier, *Algebraic topology*, McGraw-Hill, New York, 1966.

[Su] D.Sullivan, *Geometric Topology, Part I: Localization, Periodicity, and Galois Symmetry*, M.I.T. Press, 1970.

[Wa] J.J.Walsh, *Dimension, cohomological dimension, and cell-like mappings*, Lecture Notes in Math. 870, 1981, pp. 105-118.

[Wh] George W.Whitehead, *Elements of homotopy theory*, Springer-Verlag, 1978.

[Z] A. Zabrodsky, *On phantom maps and a theorem of H. Miller*, Israel J. Math. **58** (1987), 129–143.

Institute of Mathematics, University of Tsukuba, Tsukuba-shi, Ibaraki, 305, Japan

E-mail address: yokoi@sakura.cc.tsukuba.ac.jp