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COVERING PROPERTIES CHARACTERIZED BY ORTHOCOMPACTNESS AND SUBNORMALITY OF PRODUCTS

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All spaces are assumed to be T_1, but compact spaces and paracompact spaces are assumed to be Hausdorff.

A space X is assumed to be Tychonoff when we consider the product $X \times \gamma X$, where γX denotes a compactification of X. An infinite cardinal κ is assumed to be no less than $L(X)$ when we consider the product $X \times 2^\kappa$ or the product $X \times (\kappa + 1)$, where $L(X)$ denotes the Lindelöf number of the space X.

The main purpose of this note is to give some partial answers to Problems A and C stated in Section 1.

1. CHARACTERIZATIONS OF COVERING PROPERTIES BY PRODUCTS

Let us begin with a classical result of Dowker [D].

Theorem 1.1 [D]. *For a normal space X, the following are equivalent.*

(a) *X is countably paracompact.*
(b) *$X \times (\omega + 1)$ is normal.*
(c) *$X \times [0,1]$ is normal.*

Theorem 1.1 is the first result which indicated an important implication between covering properties and products. Moreover, this led up to a beautiful characterization of paracompactness in terms of products.

Theorem 1.2 [T,M]. *For a Hausdorff space X, the following are equivalent.*

(a) *X is paracompact.*
(b) *$X \times \gamma X$ is normal.*
(c) *$X \times 2^\kappa$ is normal.*
(d) *$X \times (\kappa + 1)$ is normal.*

Remark. The equivalence (a) and (d) in Theorem 1.2 was proved by Kunen. It is found in [P, Corollary 3.7].

An open cover $\mathcal{V}$ of a space X is *interior-preserving* if $\bigcap \mathcal{V}'$ is open in X for each $\mathcal{V}' \subset \mathcal{V}$. A space X is *orthocompact* if every open cover of X has an interior-preserving open refinement.

Subsequently, as a nice analogue of Theorem 1.2, a characterization of metacompactness was obtained as follows.

Theorem 1.3 [Ju1,S]. *For a space X, the following are equivalent.*

(a) *X is metacompact.*
(b) *$X \times \gamma X$ is orthocompact.*
(c) *$X \times 2^\kappa$ is orthocompact.*

This means that there are some closed relations between normality and orthocompactness of products (see [S,KY]). Moreover, as an analogue of Theorem 1.3, we proved a characterization of submetacompactness as follows.

Theorem 1.4 [Y1]. *For a space X, the following are equivalent.*

(a) *X is submetacompact.*
(b) *$X \times \gamma X$ is suborthocompact.*
(c) *$X \times 2^\kappa$ is suborthocompact.*

Seeing Theorems 1.2 and 1.3, it is natural to raise the following problem.

Problem A [Y2]. If $X \times (\kappa + 1)$ is orthocompact, is X metacompact ?

Moreover, it is natural to ask whether there is an analogical characterization of subparacompactness in terms of products.

Recall that a space X is *subnormal* [C, Kr] *(normal)* if for any disjoint closed sets A and B in X, there are disjoint G_δ-sets (open sets) G and H such that $A \subset G$ and $B \subset H$. Note that a space X is subnormal (normal) if and only if every binary open cover of X has a countable (finite) closed refinement.

Problem B [Ju3]. If $X \times \gamma X$ is subnormal, is X subparacompact ?

Problem C [Y2]. If $X \times 2^\kappa$ is subnormal, is X subparacompact ?

Remark. As is shown later, it suffices for these three problems to prove that X is submetacompact. In fact, this follows from Lemma 2.9 and Theorem 3.3 (or Corollary 3.5) below.

2. Metacompactness and submetacompactness of β-spaces

In this section, we give an affirmative answer to our Problem A under the assumption of X being a β-space.

A space X is called a *β-space* if there is a function $g\colon X\times\omega\to \mathrm{Top}(X)$, satisfying

(i) $x\in\bigcap_{n\in\omega} g(x,n)$,

(ii) if $x\in g(x_n,n)$ for each $n\in\omega$, then $\{x_n\}$ has a cluster point in X.

Since the class of β-spaces contains the classes of Σ-spaces and semi-stratifiable spaces, it is very broad as a class of generalized metric spaces.

A well-ordered sequence $\{y_\alpha\colon \alpha\in\kappa\}$ of length κ in a space Y is a *free sequence* if $\mathrm{Cl}\{y_\beta\colon \beta<\alpha\}\cap\mathrm{Cl}\{y_\gamma\colon \alpha\leq\gamma<\kappa\}=\emptyset$ for each $\alpha\in\kappa$.

Theorem 2.1. *Let X be a β-space and C a compact space with a free sequence of length $\geq L(X)$. Then X is metacompact if and only if $X\times C$ is orthocompact.*

Since $\kappa+1$ has a free sequence of length κ, Theorem 2.1 yields a partial answer to Problem A.

Corollary 2.2. *A β-space X is metacompact if and only if $X\times(\kappa+1)$ is orthocompact.*

Moreover, Arhangel'skii's theorem in [A] and Theorem 2.1 yield

Corollary 2.3. *Let X be a β-space and C a compact space with tightness $> L(X)$. Then X is metacompact if and only if $X\times C$ is orthocompact.*

Now, we will give only a course of the proof of Theorem 2.1. On the way, we will obtain a characterization of submetacompactness of β-spaces.

A well-ordered open cover $\{U_\alpha\colon \alpha\in\kappa\}$ of a space X is *well-monotone* if $\beta<\alpha$ implies $U_\beta\subset U_\alpha$.

Lemma 2.4. *Let X be a space and C a compact space with a free sequence of length $\geq L(X)$. If $X\times C$ is orthocompact, then every well-monotone open cover of X has a closure-preserving closed refinement.*

By this, it seems to be effective to consider well-monotone open covers and their closure-preserving closed refinements. So we think of the following Junnila's theorem.

Theorem 2.5 [Ju1, Ju2]. *The following are equivalent for a space X.*

(a) *X is metacompact (submetacompact).*

(b) *Every well-monotone open cover of X has a point-finite open refinement (θ-sequence of open refinements).*

(c) *Every interior-preserving directed open cover of X has a (σ-)closure-preserving closed refinement.*

Seeing Lemma 2.4 and Theorem 2.5, we raise the following problem.

Problem D. If every well-monotone open cover of a space X has a σ-closure-preserving closed refinement, when is X submetacompact ?

Lemma 2.6 [Ji]. *Let X be a β-space and $\mathcal{U}$ a well-monotone open cover of X. If $\mathcal{H}$ is an open refinement of $\mathcal{U}$, then there is a sequence $\{\mathcal{G}_{\mathcal{H},s}\colon s \in \omega^{<\omega}\}$ of partial refinements by open sets in X, satisfying*

(1) $\mathcal{G}_{\mathcal{H},s} \subset \mathcal{G}_{\mathcal{H},s'}$ *for* $s \subset s'$,
(2) *if* $x \in X$ *with* $\mathrm{ord}(x, \mathcal{H}) \leq n$, *then* $x \in \bigcup \mathcal{G}_{\mathcal{H},s}$ *for each* $s \in \omega^{n+1}$,
(3) *for each* $x \in X$, *there is some* $\sigma \in \omega^\omega$ *such that* $\mathrm{ord}(x, \mathcal{G}_{\mathcal{H},(\sigma \upharpoonright n)}) < \omega$ *for each* $n \in \omega$.

Making use of this, we prove the following lemma. A basic idea for the proof is also due to Jiang [Ji].

Lemma 2.7 (main). *Let X be a β-space and $\mathcal{U}$ a well-monotone open cover of X. If $\mathcal{U}$ has a closure-preserving closed refinement, then it has a θ-sequence of open refinements.*

By Lemma 2.7, we can easily obtain an answer to our Problem D.

Theorem 2.8. *A β-space X is submetacompact if and only if every well-monotone open cover of X has a σ-closure-preserving closed refinement.*

Now, let us return the proof of Theorem 2.1.

Let X be a space and $\mathcal{F}$ a collection of subsets of X. A collection $\{G(F)\colon F \in \mathcal{F}\}$ of subsets in X is an *open expansion* (a *G_δ-expansion*) if $G(F)$ is an open set (a G_δ-set) in X such that $F \subset G(F)$ for each $F \in \mathcal{F}$.

A space X is *almost expandable* [SK] if every locally finite collection of closed sets in X has a point-finite open expansin.

A well-ordered sequence $\{y_\alpha\colon \alpha \in \kappa\}$ of length κ in a space Y is *right separated* if $y_\alpha \notin \mathrm{Cl}\{y_\delta\colon \delta > \alpha\}$ for each $\alpha \in \kappa$. Note that each free sequence is right sparated.

Lemma 2.9. *Let X be a space and C a compact space with a right separated sequence of length $\geq L(X)$. If $X \times C$ is orthocompact, then X is almost expandable.*

Since submetacompact, almost expandable spaces are metacompact (see [SK]), Theorem 2.1 follows from Lemmas 2.4 and 2.9, and Theorem 2.8. □

As a similar problem to Problem D, we raise

Problem D′. If every well-monotone open cover of an orthocompact space X has a closure-preserving closed refinement, is X metacompact ?

If problem D′ would be affirmatively solved, it follows from Lemma 2.4 that Problem A would be affirmative.

Concerning Problem D′, we get an additional result.

Lemma 2.10 [HV, Theorem 3.1]. *For a (an orthocompact) space X, the following are equivalent.*

(a) *For every well-monotone open cover $\{U_\alpha : \alpha \in \kappa\}$ of X, there is a well-monotone closed cover $\{F_\alpha : \alpha \in \kappa\}$ of X such that $F_\alpha \subset U_\alpha$ for each $\alpha \in \kappa$.*

(b) *Every well-monotone open cover of X has a cushioned (closure-preserving) closed refinement.*

(c) *Every infinite open cover $\mathcal{U}$ of X has an open refinement $\mathcal{V}$ with $\mathrm{ord}(x, \mathcal{V}) < |\mathcal{U}|$ for each $x \in X$.*

Let $(\lambda+1)_\lambda$ denote the space $\lambda+1$ with the topology such that the point λ has a neighborhood base in the usual order topology and that all other points are isolated.

Using Lemma 2.10, we obtain

Theorem 2.11. *For an orthocompact space X, every well-monotone open cover of X has a closure-preserving closed refinement if and only if $X \times (\lambda+1)_\lambda$ is orthocompact for each λ ($\leq L(X)$).*

We close this section with the following two unsolved problems, which seem to be related to Problems D and D′.

Problem E [Ka,Y1]. If every directed open cover of a (suborthocompact) space X has a σ-cushioned closed refinement, is X submetacompact ?

Problem E′ [Ka,Ju3]. If every directed open cover of a space X has a cushioned closed refinement, is X metacompact ?

Problem E′ was affirmatively solved under the assumption of X being suborthocompact (see [Y1]).

3. Countable subparacompactness

In this section, we give some partial answers to our Problem C.

A space X is ***countably subparacompact*** [Kr] if every countable open cover of X has a countable closed refinement. Note that countably subparacompact spaces are, equivalently, countably metacompact and subnormal (see [Kr, Theorem 2.5]).

Recently, a list of analogues of Theorem 1.1 was given in [GT, p.118]. Here we can add another analogue, answering to Problem C in the case of $\kappa = \omega$.

Theorem 3.1. *For a space X, the following are equivalent.*

(a) *X is countably subparacompact.*

(b) *$X \times 2^\omega$ is subnormal.*

(c) *$X \times [0,1]$ is subnormal.*

Remark 1. The equivalence of (a) and (c) in Theorem 3.1 was stated in [GT, p.127] without proof. However, at the 10th Summer Conference on General Topology and Application (Amsterdam, August 1994), Good and Tree kindly informed the author that this equivalence had *not* been proved yet, because they misunderstood the proof.

Theorem 3.1 immediately yields a generalization of Theorem 1.1.

Corollary 3.2. *For a normal space X, the following are equivalent.*

(a) *X is countably paracompact.*
(b) *$X \times (\omega + 1)$ is normal.*
(c) *$X \times [0, 1]$ is subnormal.*

Remark 2. It should be noticed that Theorem 3.1 and Corollary 3.2 are essentially different from all the analogues in the list of [GT, p.118]. Because we can replace $[0, 1]$ with $\omega + 1$ in all of them, but we cannot do in Theorem 3.1 and Corollary 3.2. In fact, consider a Dowker space Y, whose existence is assured by Rudin [R1]. Since the product of a subnormal space and a countable space is subnormal, $Y \times (\omega + 1)$ is subnormal. On the other hand, Y is normal, but not countably metacompact.

A space X is *collectionwise δ-normal* [Ju3] if every discrete collection of closed sets in X has a disjoint G_δ-expansion.

Theorem 3.3 [R2]**.** *Let X be a space and C a compact space with weight $\geq L(X)$. If $X \times C$ is subnormal, then X is collectionwise δ-normal.*

A space X is *collectionwise subnormal* [C, Kr] if for each discrete collection $\mathcal{F}$ of closed sets in X, there is a sequence $\{\mathcal{U}_n\}$ of open expansions of $\mathcal{F}$ such that for each $x \in X$, there is some $n \in \omega$ such that at most one member of $\mathcal{U}_n$ contains x. Note

"subparacompact $\Rightarrow$ collectionwise subnormal $\Rightarrow$ collectionwise δ-normal".

Now, we get another partial answer to Problem C.

Theorem 3.4. *If $X \times 2^\kappa$ is subnormal, then X is collectionwise subnormal.*

Since collectionwise δ-normal and submetacompact spaces are subparacompact [Ju3], Theorems 1.4 and 3.3 yields a partial answer to Problems B and C.

Corollary 3.5. *For a space X, the following are equivalent.*

(a) *X is subparacompact.*
(b) *$X \times \gamma X$ is subnormal and suborthocompact.*
(c) *$X \times 2^\kappa$ is subnormal and suborthocompact.*

4. Lindelöf spaces

Recall that a space X is *ω_1-compact* if every closed discrete subset in X is at most countable. Note that Lindelöf spaces are ω_1-compact.

Lemma 4.1. *Let C be a countably compact space and X a subspace of C. If the subspace $(X \times C) \cup (C \times X)$ of the square C^2 is subnormal, then X is ω_1-compact.*

Using this, we can obtain an analoguou characterization of Lindelöf spaces to Tamano's theorem for paracompactness (see Theorem 1.2).

Theorem 4.2. *For a Tychonoff space X, the following are equivalent.*

(a) *X is Lindelöf.*
(b) *The subspace $(X \times \gamma X) \cup (\gamma X \times X)$ of the square $(\gamma X)^2$ is normal.*
(c) *X is submetacompact and the subspace $(X \times \gamma X) \cup (\gamma X \times X)$ of the square $(\gamma X)^2$ is subnormal.*

In Theorem 4.2, we can find a kind of similarity to the form of Corollary 3.2.

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