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A REMARK ON TANAKA'S QUESTION

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In the proceedings of General Topology Symposium(Dec. 1993, Saitama Univ.), Yoshio Tanaka posed the following question.

Question Is every space with a locally countable k-network a σ-space ?

It is known that every k-space with a locally countable k-network is the topological sum of $\aleph_0$-spaces(hence, a σ-space), and there is a space with a locally countable k-network which is not an $\aleph$-space.

In this note, we remark that we can find counterexamples for the question under some set theoretic axioms. The author does not know any counterexample in ZFC.

For terminology and notions, see the article[2] of Tanaka.

We have only to find a space X with the following:

(1) locally countable(i.e. every point of X has a countable neighborhood),
(2) every compact subset of X is a finite set,
(3) not perfect(i.e. there is an open subset of X which is not an F_σ-set).

From (1) and (2), the family $\{\{x\} : x \in X\}$ is a locally countable k-network of X, and (3) means that X is not a σ-space.

For cardinals α, β, we set $[\alpha]^\beta = \{A : A \subset \alpha, | A |= \beta\}$. We endow ω_1 with the discrete topology.

Let $\mathcal{P} = \{P_\alpha : \alpha < \tau\} \subset [\omega_1]^\omega$ be an almost disjoint family, and choose any $(p_\alpha) \in \prod\{P^*_\alpha : \alpha < \tau\}$, where $P^*_\alpha = Cl_{\beta\omega_1}P_\alpha - P_\alpha$. Then the subspace $X = \omega_1 \cup \{p_\alpha : \alpha < \tau\}$ of $\beta\omega_1$ obviously satisfies (1) and (2). Moreover, if (p_α) satisfies the following (*), then X is not perfect.

(*) For every $A \in [\omega_1]^{\omega_1}$, there is an $\alpha < \tau$ such that $A \in p_\alpha$.

Thus we have only to find an almost disjoint family $\mathcal{P} = \{P_\alpha : \alpha < \tau\} \subset [\omega_1]^\omega$ and

$(p_\alpha) \in \prod\{P^*_\alpha : \alpha < \tau\}$ satisfying (*).

First we assume MA + $\neg$CH(Martin's Axiom plus the negation of the Continuum Hypothesis).

Remark [1] MA + $\neg$CH implies:

(a) if $\mathcal{P} \subset [\omega_1]^\omega$ is a maximal almost disjoint family, then $|\mathcal{P}| = 2^\omega$,
(b) $2^\omega = 2^{\omega_1}$.

Fact 1 [MA + $\neg$CH] If $\mathcal{P} \subset [\omega_1]^\omega$ is a maximal almost disjoint family with $|\mathcal{P}| = 2^\omega$, then for every $A \in [\omega_1]^{\omega_1}$ $|\{P \in \mathcal{P} : |P \cap A| = \omega\}| = 2^\omega$.

Proof. We set $I(A) = \{P \in \mathcal{P} : |P \cap A| = \omega\}$. Choose a countable infinite subset $\{P_i : i \in \omega\} \subset I(A)$ and set $B = \cup\{P_i \cap A : i \in \omega\}$. Then B is an infinite countable subset of A. Let $\mathcal{G}$ be a maximal almost disjoint family in B which contains $\{P \cap B : P \in \mathcal{P}, |P \cap B| = \omega\}$. From Remark (a), $|\mathcal{G}| = 2^\omega$. If we assume $|I(A)| < 2^\omega$, then there is a $G \in \mathcal{G} - \{P \cap B : P \in \mathcal{P}, |P \cap B| = \omega\}$. The set G does not belong to $\mathcal{G}$ and $\mathcal{P} \cup \{G\}$ is almost disjoint. This is a contradiction. □

Fact 2 [MA + $\neg$CH] For every maximal almost disjoint family $\mathcal{P} \subset [\omega_1]^\omega$ with $|\mathcal{P}| = 2^\omega$, there is a $(p_\alpha) \in \prod\{P^* : P \in \mathcal{P}\}$ satisfying (*).

Proof. By Remark (b), we may set $[\omega_1]^{\omega_1} = \{A_\alpha : \alpha < 2^\omega\}$. Fix $\gamma < 2^\omega$. We assume that for every $\alpha < \gamma$ we chose $P_\alpha \in \mathcal{P}$ such that $|A_\alpha \cap P_\alpha| = \omega$. By Fact 1 there is a $P_\gamma \in \mathcal{P} - \{P_\alpha : \alpha < \gamma\}$ such that $|A_\gamma \cap P_\gamma| = \omega$. We choose $p_\alpha \in A^*_\alpha \cap P^*_\alpha$ for every $\alpha < 2^\omega$. Then $\{p_\alpha : \alpha < 2^\omega\}$ is a desired one. □

At General Topology Symposium(June, 1994, Tsukuba), the author asked Professor M.E. Rudin whether there is a space satisfying (1), (2) and (3). She told me that the axiom ♣ was enough, where ♣ is the following principle:

(♣) For every countable limit ordinal α, there is a subset P_α of α such that (a) each P_α is cofinal in α, (b) if $A \in [\omega_1]^{\omega_1}$, then $P_\alpha \subset A$ for some α.

It is known that MA + $\neg$CH implies the negation of ♣. From the definition of ♣, the following fact is obvious.

Fact 3 [♣] There is an almost disjoint family $\mathcal{P} = \{P_\alpha : \alpha < \omega_1\} \subset [\omega_1]^\omega$ such that every

$(p_\alpha) \in \prod\{P^*_\alpha : \alpha < \omega_1\}$ satisfies (*).

References

1. K. Kunen, Set Theory, North-Holland, 1980.
2. Y. Tanaka, k-space と k-network, in the proceedings of General Topology Symposium at Saitama University, 1993, 16-32.

Addendum

At the conference, Set Theoretic Topology and its Applications(Dec. 1994), the author asked Professor Dow almost disjoint families we needed. He suggested to see the chapter of Balcar and Simon in [3].

In fact, the following is well known.

Fact [3, Example 4.2] Let X be a set of size 2^ω. Then there is an almost disjoint family $\mathcal{D}$ of countable infinite subsets of X such that for every uncountable $A \subset X$ there is some $D \in \mathcal{D}$ with $D \subset A$.

Hence we obtain a counterexample for the question in ZFC. In addition, we may think that our counterexample is not even countably metacompact. Recall that every perfect space is countably metacompact.

3. B. Balcar and P. Simon, Disjoint refinement, in: Handbook of Boolean Algebras, North-Holland, 1989, vol. 2, 332-386.