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SHIFT MAPS AND ATTRACTORS

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1 Introduction.

All spaces considered here are assumed to be separable metric spaces. *Maps* are continuous functions. By a *compactum* we mean a compact metric space. A *continuum* is connected, nondegenerate compactum. Let R be the real line and R^n the Euclidean n-dimensional space. Let S be the unit circle in the plane R^2. For a manifold M, ∂M denotes the manifold boundary. Let $F: Y \to Y$ be a homeomorphism of a space Y (onto itself) with metric d and let Λ be a compact subset of Y. Then Λ is said to be an *attractor* of F provided that there exists an open neighborhood of U of Λ in Y such that

$$F(\mathrm{Cl}(U)) \subset U \text{ and } \Lambda = \cap_{n\geq 0} F^n(U).$$

Note that $F(\Lambda) = \Lambda$. Moreover, if for each $y \in Y$ $\lim_{n\to\infty} d(F^n(y), \Lambda) = 0$, then we say that Λ is a *global attractor* of F, where $d(A, B) = \inf\{d(a, b) | a \in A, b \in B\}$ for sets A, B. Let $f: X \to X$ and $g: Y \to Y$ be maps. Then f is *topologically conjugate to* g if there is a homeomorphism $\phi: X \to Y$ such that $\phi \cdot f = g \cdot \phi$.

The notion of shift maps is very convenient for dynamical systems. Let $\mathbf{X} = \{X_n, p_{i,i+1} | i = 1, 2, \ldots\}$ be an inverse sequence of compacta X_i and maps $p_{i,i+1}: X_{i+1} \to X_i (i = 1, 2, \ldots)$ and let

$$\mathrm{invlim}\, \mathbf{X} = \{(x_i)_{i=1}^{\infty} | x_i \in X_i, p_{i,i+1}(x_{i+1}) = x_i \text{ for each } i\} \subset \prod_{i=1}^{\infty} X_i.$$

Then invlim $\mathbf{X}$ is a topological space as a subspace of the product space $\prod_{i=1}^{\infty} X_i$. Then invlim $\mathbf{X}$ is a compactum. Let $f : X \to X$ be a map of a compactum X. Consider the following special inverse limit space:

$$(X, f) = \{(x_i)_{i=1}^{\infty} | \, x_i \in X \text{ and } f(x_{i+1}) = x_i \text{ for each } i \geq 1\}.$$

Define a map $\tilde{f} : (X, f) \to (X, f)$ by $\tilde{f}(x_1, x_2, \ldots,) = (f(x_1), x_1, \ldots,)$. Then $\tilde{f}$ is a homeomorphism and it is called the *shift map* of f. A map $f : X \to Y$ of compacta is a *near homeomorphism* if f can be approximated arbitrarily closely by homeomorphisms from X onto Y.

Isbell proved that if $X = \text{invlim}\{X_i, p_{i,i+1}\}$ where each X_i is a compactum which can be embedded into R^n (n fixed), then X can be embedded into R^{2n}. Barge and Martin proved that if $f : I \to I$ is any map of the unit interval $I = [0, 1]$, then there is a homeomorphism $F : R^2 \to R^2$ such that (I, f) is contained in R^2, F is an extension of the shift map $\tilde{f} : (I, f) \to (I, f)$, and (I, f) is a global attractor of F.

2 Shift maps of compact polyhedra in R^n.

In this section, we obtain the following theorem which is a generalization of Barge-Martin's theorem, and which is related to Isbell's theorem.

Theorem 2.1 *If P is a compact polyhedron in R^n and $f : P \to P$ is any map, then there is a homeomorphism $F : R^{2n} \to R^{2n}$ such that (P, f) is contained in R^{2n}, F is an extension of the shift map $\tilde{f} : (P, f) \to (P, f)$ of f, and (P, f) is an attractor of F. Moreover, if P is collapsible, then F can be chosen so that (P, f) is a global attractor of F.*

To prove the above theorem, we need the following lemma which was proved by Brown.

Lemma 2.2 *Let $\mathbf{X} = \text{invlim}\{X_i, p_{i,i+1}\}$ be an inverse sequence of compacta X_i. If each $p_{i,i+1} : X_{i+1} \to X_i$ is a near homeomorphism, then* invlim $\mathbf{X}$ *is homeomorphic to X_i for each i.*

By using the above lemma, we can easily obtain the following.

Lemma 2.3 *Suppose that X is a compact subset of a compactum Y and $f : X \to X$ is a map of X. If there is an extension $h : Y \to Y$ of f such that h is a near homeomorphism and there is a neighborhood N of X in Y such that $h(N) \subset X$, then there is a homeomorphism $F : Y \to Y$ such that F is topologically conjugate to $\tilde{h} : (Y, h) \to (Y, h)$, (X, f) is contained in Y, F is an extension of the shift map $\tilde{f} : (X, f) \to (X, f)$ of f, and (X, f) is an attractor of F.*

3 Shift maps of the unit circle S.

In this section, for the special case $P = S$ we obtain the following.

Theorem 3.1 *Let $f : S \to S$ be any map of the unit circle S, then there is a homeomorphism $F : R^3 \to R^3$ such that (S, f) is contained in R^3, F is an extension of the shift map $\tilde{f} : (S, f) \to (S, f)$, and (S, f) is an attractor of F.*

Corollary 3.2 *Let $f : S \to S$ be a map of the unit circle S with $|\deg(f)| \geq 1$, then there is a homeomorphism $F : S^3 \to S^3$ of the 3-sphere S^3 such that $(S, f) \subset S^3$, F is an extension of $\tilde{f}$, (S, f) is an attractor of F and if X is the attractor of F^{-1}, then $F^{-1}|X : X \to X$ is topologically conjugate to the shift map $\tilde{g} : (S, g) \to (S, g)$, where $g : S \to S$ is the natural covering map with* $\deg(g) = \deg(f)$.

Note that there is a finite graph G which is naturally embedded into R^3 and a homeomorphism $f : G \to G$ such that there is no near homeomorphism $F : R^3 \to R^3$ which in an extension of f. Naturally, we have the following problem.

Problem 3.3 *If $f : G \to G$ is a map of any finite graph G, does there exist a homeomorphism $F : R^3 \to R^3$ such that $(G, f) \in R^3$, F is an extension of the shift map $\tilde{f} : (G, f) \to (G, f)$, and (G, f) is an attractor of F ?*

4 Everywhere chaotic homeomorphisms in the sense of Li-Yorke on manifolds and k-dimensional Menger manifold.

In this section, we deal with everywhere chaotic homeomorphisms in the sense of Li-Yorke. By using the notions of attractor and shift map, we can show that every manifold and k-dimensional Menger manifold admit such chaotic homeomorphisms.

A map $f : X \to X$ is *sensitive* if there is $\tau > 0$ such that for each $x \in X$ and each neighborhood U of x in X, there is a point $y \in U$ and a natural number $n \geq 0$ such that $d(f^n(x), f^n(y)) > \tau$. A map $f : X \to X$ is *accessible* if for any nonempty open sets U, V of X and each $\epsilon > 0$, there are two points $x \in U$, $y \in V$ and a natural number $n \geq 0$ such that $d(f^n(x), f^n(y)) < \epsilon$.

Let $f : Y \to Y$ be a map and $\tau > 0$. A subset S of Y is called a τ-*scrambled set* of f if the next three conditions are satisfied: For each $x, y \in S$ with $x \neq y$,

1. $\limsup_{n\to\infty} d(f^n(x), f^n(y)) > \tau$,
2. $\liminf_{n\to\infty} d(f^n(x), f^n(y)) = 0$, and
3. $\limsup_{n\to\infty} d(f^n(x), f^n(p)) > \tau$ for any periodic point p of f.

If there is an uncountable τ-scrambled set S of f, then we say that f is τ-*chaotic (in the sense of Li-Yorke)* on S. A map $f : Y \to Y$ is *everywhere chaotic* if there is $\tau > 0$ such that f is τ-chaotic on almost all Cantor sets in Y, i.e., for any closed subset A of Y and $\epsilon > 0$, there is an Cantor set C in Y such that $d_H(A, C) < \epsilon$ and f is τ-chaotic on C, where d_H denotes the Hausdorff metric.

Then we have the following characterization of everywhere chaotic homeomorphism.

Theorem 4.1 *Let $f : X \to X$ be a map of a compactum X. Then f is everywhere chaotic if and only if f is sensitive and accessible.*

Then we have the following theorem.

Theorem 4.2 *Every compact n-manifold ($n \geq 2$) admits an everywhere chaotic homeomorphism.*

For the case of Menger manifolds, we obtain the following theorem.

Theorem 4.3 *If P is a compact connected polyhedron with $\dim P \leq k$, then there is a k-dimensional compact Menger manifold M^k such that M^k is $(k-1)$-homotopy equivalent to P satisfying the following property; if a map $f : P \to P$ is $(k-1)$-homotopic to id_P, then there is a Z-set P' such that P' is homeomorphic to P, (P, f) is contained in $M^k - P'$ and there is a homeomorphism $F : M^k \to M^k$ such that $F|P' = id_{P'}$, (P, f) is a global attractor of $F|M^k - P'$ and F is an extension of $\tilde{f} : (P, f) \to (P, f)$. In particular, if P is $(k-1)$-connected compact polyhedron with $\dim P \leq k$, then for any map $f : P \to P$, there is a homeomorphism $F : \mu^k \to \mu^k$ of the k-dimensional Menger compactum μ^k such that (P, f) is contained in $\mu^k - \{*\}$ $(* \in \mu^k)$, $F(*) = *$, (P, f) is a global attractor of $F|\mu^k - \{*\}$, and F is an extension of $\tilde{f}$.*

Corollary 4.4 *Let $f : G \to G$ be any map of a compact connected graph G. Then there is a homeomorphism $F : \mu^1 \to \mu^1$ of the Menger curve μ^1 such that $(G, f) \subset \mu^1 - \{*\}, F(*) = *, (G, f)$ is a global attractor of $F|\mu^1 - \{*\}$ and F is an extension of $\tilde{f}$.*

By using the above theorem, we obtain the following.

Theorem 4.5 *Every compact Menger manifold admits an everywhere chaotic homeomorphism. In particular, every compact Menger manifold admits a sensitive homeomorphism.*

Remark 4.6 *There is a Z-set X in μ^k $(k \geq 1)$ such that for any homeomorphism $h : X \to X$, there is no homeomorphism $F : \mu^k \to \mu^k$ so that F is an extension of h and X is an attractor of F.*

For the case of chaos of Devaney, the following problem remains open.

Problem 4.7 *Do compact Menger manifolds admit chaotic homeomorphisms in the sense of Devaney ?*