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A final coalgebra theorem for concurrent computation

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Abstract

This paper presents an elementary and self-contained proof of an existence theorem of final coalgebras for endofunctors on the category of sets and functions.

1 Introduction

Graphs are fundamental algebraic structures in computer science. Recently labelled transition systems, namely, labelled directed graphs have been considered an appropriate model for concurrent computations. It is known that graph structures are often represented by coalgebra structures [4, 7, 8]. Many kinds of coalgebras have been considered as objects with circularity in programming semantics, knowledge dynamics and situation theory (Cf. [9]). In 1988 Peter Aczel [1] pointed out that the axiom of anti-foundation (AFA) on axiomatic set theory claims that the universal class of all sets with the membership relation is the final graph structure on classes. Moreover Peter Aczel and Nax Mendler [2] proved a final coalgebra theorem for set-based endofunctors. In fact Michael Barr [3] showed the theorem of Aczel and Mendler [2] on the existence of final coalgebras for accessible endofunctors on the category **Set** of (well-founded) sets and functions.

On the other hand, R. Milner's CCS is the language for communicating concurrent processes, which has the equationally axiomatic system. Its semantics is given as labelled transition systems and bisimulation equivalences. The labelled transition systems is expressed as coalgebra structures with respect to the endofunctor $\Phi X = \wp(A \times X)$ on **Set**, for reason of the nondeterministic property of concurrency. In general, considering the category of coalgebras and their homomorphisms with respect to endofunctor on **Set**, the final coalgebra does not always exist.In the case of powerset functor $\wp$ it is well-known that nonexistence of the final coalgebra can be proved by the Cantor's diagonal method. From a point of view for concurrent computation, however, one may restrict the range of transition to some cardinality.

In this paper we will also discuss with the same small final coalgebra theorem as in [3]. Some detailed analysis on trees (in other words, the subcoalgebras generated by single elements) and congruences [2] (or, bisimulation equivalences) on coalgebras are essential for our results. The discussion of the paper is elementary and self-contained. The main theorem of the paper is as follows:

Theorem 1.1 *If an endofunctor Φ : **Set** $\rightarrow$ **Set** preserves intersections and there is a set M such that all trees of Φ-coalgebras are M-bounded, then the category* **Set**(Φ) *of Φ-coalgebras has a final Φ-coalgebra.*

The paper is as follows. In Section 2 we review the definition of coalgebras for endofunctors on **Set**, and only note that the class of all coalgebras defined on subsets of a given set forms a set. In Section 3 we recall some basic properties of subcoalgebras for endofunctors on **Set**. In particular, it turns out that, when the involved endofunctor preserves intersections of subsets, the notion of trees of coalgebras, which are the smallest subcoalgebras containing singleton sets, can be considered. In Section 4 we discuss congruences on coalgebras initiated by Aczel and Mendler [2]. The notion of

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congruences is a modification of bisimulation equivalence relations on labelled transition systems due to D. Park. The aim of this section is to show a usual fact (Cf. [1, theorem 2.4] and [2, Lemma 4.3]) that every coalgebra has the maximum . The fact indicates that the quotient coalgebra of a weak final coalgebra with respect to the maximum congruence is a final coalgebra (Cf. **4.8**). Thus the category of coalgebras has a final object if and only if it has a weak final object. In Section 5 we state the main result of the paper. First we introduce tree congruences on coalgebras using the notion of trees. Then we show that, whenever all trees of coalgebras are bounded to a set, there is a weak final coalgebra. Thus by the similar fashion to Aczel and Mendler [2] an existence theorem of final coalgebras is proved. In section 6 a few examples of coalgebras which satisfy the main theorem are listed.

2 Coalgebras

This section defines the notion of coalgebras for endofunctors on the category **Set** of sets and functions. Let $\Phi : \mathbf{Set} \to \mathbf{Set}$ denote an endofunctor throughout the paper. A *Φ-coalgebra* (A, a) is a pair of a set A and a function $a : A \to \Phi A$. A *homomorphism* $f : (A, a) \to (B, b)$ of a Φ-coalgebra (A, a) into another Φ-coalgebra (B, b) is a function $f : A \to B$ rendering the following square commutative:

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "a"'] & B \arrow[d, "b"] \\ \Phi A \arrow[r, "\Phi(f)"'] & \Phi B. \end{tikzcd}$$

All Φ-coalgebras and all their homomorphisms form a category $\mathbf{Set}(\Phi)$ which is called the category of Φ-coalgebras.

Proposition 2.1 *The category* $\mathbf{Set}(\Phi)$ *of Φ-coalgebras has all small colimits.*

Proof. It suffices to prove the existence of coequalizers and coproducts because of [6, V **2.1**]. First let $f, g : (A, a) \to (B, b)$ be a pair of parallel homomorphisms of Φ-coalgebras. As the category **Set** has all small colimits there is a coequalizer $e : B \to Q$ of a pair of functions f and g in **Set**. Noticing that $\Phi(e)bf = \Phi(e)\Phi(f)a = \Phi(e)\Phi(g)a = \Phi(e)bg$ there is a unique function $q : Q \to \Phi Q$ such that $qe = \Phi(e)b$. It is an elementary exercise to show that $e : (B, b) \to (Q, q)$ is a coequalizer of f and g in $\mathbf{Set}(\Phi)$. Next suppose that $\{(A_\lambda, a_\lambda)\}_{\lambda \in \Lambda}$ is a family of Φ-coalgebras indexed by a set Λ. Let A be a coproduct (or disjoint union) of $\{A_\lambda\}_{\lambda \in \Lambda}$ and $i_\lambda : A_\lambda \to A$ the inclusion of coproducts for $\lambda \in \Lambda$. Define a function $a : A \to \Phi A$ to be a unique function such that a square

$$\begin{tikzcd} A_\lambda \arrow[r, "i_\lambda"] \arrow[d, "a_\lambda"'] & A \arrow[d, "a"] \\ \Phi A_\lambda \arrow[r, "\Phi(i_\lambda)"'] & \Phi A \end{tikzcd}$$

commutes for every $\lambda \in \Lambda$. It is also a routine work to show that a Φ-coalgebra (A, a) is a coproduct of $\{(A_\lambda, a_\lambda)\}_{\lambda \in \Lambda}$.□

The following lemma is a basic fact of functors on **Set**.

Lemma 2.2 *If $f : X \to Y$ is an injection and X is a nonempty set, then $\Phi(f) : \Phi X \to \Phi Y$ is an injection.*

Proof. Choose $x_0 \in X$ and define a function $g : Y \to X$ by $g(y) = x$ if $y = f(x)$ for $x \in X$ and $g(y) = x_0$ if there is no $x \in X$ such that $y = f(x)$. Then it is clear that $gf = \mathrm{id}_X$ and $\Phi(g)\Phi(f) = \mathrm{id}_{\Phi X}$, which shows that $\Phi(f) : \Phi X \to \Phi Y$ is injective.□

Given a set M the class of all Φ-coalgebras (A, a) such that A is a nonempty subset of M is denoted by $\mathbf{Set}_M(\Phi)$. The following proposition points out that $\mathbf{Set}_M(\Phi)$ constitutes a set.

Proposition 2.3 *For every set M the class $\mathbf{Set}_M(\Phi)$ is a subset of $\wp(M) \times \wp(M \times \Phi M)$, that is,*

$$\mathbf{Set}_M(\Phi) \subseteq \wp(M) \times \wp(M \times \Phi M).$$

Proof. Let (A, a) be a Φ-coalgebras in $\mathbf{Set}_M(\Phi)$ and $i : A \to M$ the inclusion. Then it is immediate that $A \in \wp(M)$ and $a \in \wp(M \times \Phi M)$ since a function $a : A \to \Phi A$ can be identified with a subset $\hat{a} = \{(x, \Phi(i)a(x)) | x \in A\}$ of $M \times \Phi M$ by the last lemma. □

In a category of coalgebras a *final* coalgebra is a coalgebra such that there is a unique homomorphism from each coalgebra into it. A *weak final* coalgebra is a coalgebra such that there is at least one homomorphism from each coalgebra into it. The purpose of the paper is to show an existence theorem of final coalgebras for endofunctors on **Set**.

3 Subcoalgebras

This section is devoted to state the notion and the basic properties of subcoalgebras. Trees, that is, the smallest subcoalgebras containing singleton sets, play an important role to prove the main theorem of the paper.

Let (A, a) be a Φ-coalgebra. A subset H of A is called a *subcoalgebra* of (A, a) if there is a function $h : H \to \Phi H$ which makes the inclusion $i : H \to A$ a homomorphism $i : (H, h) \to (A, a)$ of Φ-coalgebras. (By the definition the empty set $\emptyset$ is always a subcoalgebra.)

Lemma 3.1 *Let (A, a) be a Φ-coalgebra. A subset H of A is a subcoalgebra of (A, a) if and only if for each $x \in H$ there is $z \in \Phi H$ such that $a(x) = \Phi(i)(z)$, where $i : H \to A$ is the inclusion of H into A.*

Proof. Only if part is trivial from the definition of subcoalgebras. We have to show its converse. Assume that for each $x \in H$ there is $z \in \Phi H$ such that $a(x) = \Phi(i)(z)$. When $H = \emptyset$ the assertion is in the case. So we can assume that H is nonempty. A function $h : H \to \Phi H$ can be defined by:

$$\text{For each } x \in H : h(x) = z \text{ if } a(x) = \Phi(i)(z) \text{ for some } z \in \Phi H.$$

Since $\Phi(i)$ is injective by **2.2** h is uniquely defined and $\Phi(i)h = ai$ is immediate. □

Let (A, a) be a Φ-coalgebra and H a subcoalgebra of (A, a). Then it is easy to see that a subset S of H is a subcoalgebra of H if and only if S is a subcoalgebra of (A, a).

Proposition 3.2 *Let $f : (A, a) \to (B, b)$ be a homomorphism of Φ-coalgebras.*

(a) *If H is a subcoalgebra of (A, a), then $fH = \{f(x) | x \in H\}$ is a subcoalgebra of (B, b).*

(b) *If H is a subcoalgebra of (A, a) and $f : (A, a) \to (B, b)$ is an injective homomorphism, then fH is isomorphic to H as Φ-coalgebras.*

Proof. (a) Let $i : H \to A$ and $j : fH \to B$ be inclusions. A function $f' : H \to fH$ is defined by $f'(x) = f(x)$ for each $x \in H$. Then $fi = jf'$. Hence for $x \in H$ there is $z \in \Phi H$ such that $a(x) = \Phi(i)(z)$ and so $b(f(x)) = \Phi(f)a(x) = \Phi(f)\Phi(i)(z) = \Phi(j)\Phi(f')(z)$. This means that for each $y \in fH$ there is $w \in \Phi(fH)$ such that $b(y) = \Phi(j)(w)$, which completes the proof. (b) It suffices to see that a bijective homomorphism of Φ-coalgebras is an isomorphism. Assume that $f : (A, a) \to (B, b)$ is a bijective homomorphism and let $g : B \to A$ be the inverse set function of f. Then $gf = \mathrm{id}_A$ and $fg = \mathrm{id}_B$ and $\Phi(g)b = \Phi(g)bfg = \Phi(g)\Phi(f)ag = ag$ by $bf = \Phi(f)a$. □

An endofunctor $\Phi : \mathbf{Set} \to \mathbf{Set}$ *preserves intersections* if $\Phi(i)\Phi(\cap_\lambda H_\lambda) = \cap_\lambda \Phi(i_\lambda)\Phi H_\lambda$ for all families $\{H_\lambda\}_\lambda$ of subsets of a set A, where $i_\lambda : H_\lambda \to A$ and $i : \cap_\lambda H_\lambda \to A$ are inclusions, respectively.

Lemma 3.3 *Let (A, a) be a Φ-coalgebra. If $\Phi : \mathbf{Set} \to \mathbf{Set}$ preserves intersections, then for every family $\{H_\lambda\}_\lambda$ of subcoalgebras of (A, a) its intersection $H = \cap_\lambda H_\lambda$ is a subcoalgebra of (A, a).*

Proof. Let $i : H \to A$ and $i_\lambda : H_\lambda \to A$ be inclusions, respectively. We have to show that for each $x \in H$ there is $z \in \Phi H$ such that $a(x) = \Phi(i)(z)$. Assume that $x \in H$. Then for each λ there is $z_\lambda \in \Phi H_\lambda$ such that $a(x) = \Phi(i_\lambda)(z_\lambda)$. Hence $a(x) \in \cap_\lambda \Phi(i_\lambda)\Phi H_\lambda = \Phi(i)\Phi H$ and there is $z \in \Phi H$ such that $a(x) = \Phi(i)(z)$. □

Let $\Phi : \mathbf{Set} \to \mathbf{Set}$ be an endofunctor preserving intersections. For a Φ-coalgebra (A, a) and $x \in A$ consider the family of all subcoalgebras of (A, a) containing x. Then by the last lemma its intersection is the smallest subcoalgebra containing x, which is called the *tree* of (A, a) with a root x and denoted by $[x]_A$.

Proposition 3.4 *Let $\Phi : \mathbf{Set} \to \mathbf{Set}$ be an endofunctor preserving intersections.*

(a) *If H is a subcoalgebra of a Φ-coalgebra (A, a), then $[x]_H = [x]_A$ for $x \in H$.*

(b) *If $f : (A, a) \to (B, b)$ is an injective homomorphism of Φ-coalgebras, then $f[x]_A = [f(x)]_B$ for $x \in A$.*

Proof. (a) First note that $[x]_H \subseteq H$. By **3.2**(a) $[x]_H$ is a subcoalgebra of (A, a) and so $[x]_A \subseteq [x]_H$. On the other hand $[x]_A$ is a subcoalgebra of H since $[x]_A \subseteq H$. Hence $[x]_H \subseteq [x]_A$. (b) Set $K = fA$. By **3.2** K is a subcoalgebra of (B, b) and isomorphic to (A, a). Then $f[x]_A = [f(x)]_K = [f(x)]_B$ by (a). This completes the proof. □

4 Congruences

This section discusses the notion of congruences on coalgebras initiated by Aczel and Mendler [2]. The notion of congruences is a modification of bisimulation equivalence relations on labelled transition systems. Clearly congruences generalize bisimulations and correspond to quotient coalgebras. The aim of this section is to show a usual fact (Cf. [1, theorem 2.4] and [2, Lemma 4.3]) that every coalgebra has the maximum congruence.

A (binary) *relation* on a set A is a subset of $A \times A$. Hence boolean operations such as union and intersections can be applied to relations. An *equivalence* relation θ on a set A is a relation on A such that $(x, x) \in \theta$ (reflexive), $(x, y) \in \theta \Rightarrow (y, x) \in \theta$ (symmetric) and $(x, y) \in \theta \wedge (y, z) \in \theta \Rightarrow (x, z) \in \theta$ (transitive) for all $x, y, z \in A$. Note that the identity relation $\mathrm{id}_A = \{(x, x) | x \in A\}$ (the diagonal set of A) is an equivalence relation on any set A. Given an equivalence relation θ on A there is a surjection of A onto a (quotient) set Q such that $(x, y) \in \theta$ if and only if $e(x) = e(y)$. We call such a surjection $e : A \to Q$ a *quotient* function with respect to θ. Since a quotient function is unique up to isomorphisms, an equivalence relation $\Phi(\theta)$ on ΦA is uniquely defined as follows:

$$(u, v) \in \Phi(\theta) \iff \Phi(e)(u) = \Phi(e)(v).$$

Moreover let $f : A_0 \to A$ be a function. An equivalence relation θ_f on A_0 is a relation on A_0 such that $(u, v) \in \theta_f \iff (f(u), f(v)) \in \theta$.

Proposition 4.1 *Let $f : A_0 \to A$ be a function and θ, θ' equivalence relations on A.*

(a) *If $\theta \subseteq \theta'$, then $\theta_f \subseteq \theta'_f$.*

(b) *If $\theta \subseteq \theta'$, then $\Phi(\theta) \subseteq \Phi(\theta')$.*

Proof. (a) It is trivial from definition. (b) Let $e : A \to Q$ and $e' : A \to Q'$ be quotient functions with respect to θ and θ', respectively. Then there is a function $k : Q \to Q'$ such that $ke = e'$. Hence

$$\begin{aligned}
& (u, v) \in \Phi(\theta) \\
\Rightarrow\ & \Phi(e)(u) = \Phi(e)(v) \\
\Rightarrow\ & \Phi(e')(u) = \Phi(k)\Phi(e)(u) = \Phi(k)\Phi(e)(v) = \Phi(e')(v) \\
\Rightarrow\ & (u, v) \in \Phi(\theta'). \quad \square
\end{aligned}$$

Definition 4.2 *Let (A, a) be a Φ-coalgebra.*

(a) *An equivalence relation θ on a set A is a congruence on (A, a) if $\theta \subseteq \Phi(\theta)_a$.*

(b) *A relation θ on A is a pre-congruence on (A, a) if the equivalence relation θ^* generated by θ (that is, reflexive, symmetric and transitive closure of θ) is a congruence on (A, a), that is, $\theta \subseteq \Phi(\theta^*)_a$.*

The condition $\theta \subseteq \Phi(\theta)_a$ in the above definition (a) is equivalent to a condition that $(x, y) \in \theta$ implies $(a(x), a(y)) \in \Phi(\theta)$.

Proposition 4.3 *If $f : (A, a) \to (B, b)$ is a homomorphism of Φ-coalgebras, then an equivalence relation $(\mathrm{id}_B)_f$ is a congruence on (A, a).*

Proof. Let $e : A \to Q$ be a quotient function with respect to $(\mathrm{id}_B)_f$. Then there is a unique injection $m : Q \to B$ such that $f = me$. Note that $\Phi(m)$ is injective by **3.1**. Hence

$$\begin{aligned} & (x, y) \in (\mathrm{id}_B)_f \\ \Rightarrow\ & f(x) = f(y) \\ \Rightarrow\ & \Phi(m)\Phi(e)a(x) = \Phi(f)a(x) = bf(x) = bf(y) = \Phi(f)a(y) = \Phi(m)\Phi(e)a(y) \\ \Rightarrow\ & \Phi(e)a(x) = \Phi(e)a(y) \\ \Rightarrow\ & (x, y) \in \Phi((\mathrm{id}_B)_f)_a. \quad \square \end{aligned}$$

Proposition 4.4 *Given a congruence θ on (A, a) and a quotient function $e : A \to Q$ with respect to θ there is a unique function $q : Q \to \Phi Q$ such that $e : (A, a) \to (Q, q)$ is a homomorphism of Φ-coalgebras.*

Proof. A function $q : Q \to \Phi Q$ can be defined as follows:

$$\text{For } w \in Q : q(w) = \Phi(e)a(x) \text{ if } w = e(x).$$

This definition is well-defined, since if $e(x) = e(y)$ then $(x, y) \in \theta$ and so $(a(x), a(y)) \in \Phi(\theta)$ by $\theta \subseteq \Phi(\theta)_a$. It is trivial that $qe = \Phi(e)a$. The uniqueness of q follows from the surjectivity of e. This completes the proof. $\square$

The Φ-coalgebra (Q, q) constructed in the above proposition is called a *quotient* Φ-coalgebra of (A, a) with respect to a congruence θ and denoted by $(A/\theta, a/\theta)$.

Lemma 4.5 (a) *If θ is a pre-congruence on (A, a), then θ^* is a congruence on (A, a).*

(b) *If θ and μ are pre-congruences on (A, a), then $\theta \cup \mu$ is a pre-congruence on (A, a).*

Proof. (a) Assume that $\theta \subseteq \Phi(\theta^*)_a$. As $\Phi(\theta^*)_a$ is an equivalence relation on A it simply follows that $\theta^* \subseteq \Phi(\theta^*)_a$. (b) By **4.1** we have

$$\theta \cup \mu \subseteq \Phi(\theta^*)_a \cup \Phi(\mu^*)_a \subseteq \Phi((\theta \cup \mu)^*)_a. \quad \square$$

Theorem 4.6 *Every Φ-coalgebra (A, a) has the maximum congruence Ξ_A.*

Proof. Define a relation Ξ_A on A to be a union (supremum) of all pre-congruences on (A, a), that is,

$$\Xi_A = \bigcup_{\theta \in S} \theta,$$

where S is the set of all pre-congruences on (A, a). Then

$$\Xi_A = \bigcup_{\theta \in S} \theta \subseteq \bigcup_{\theta \in S} \Phi(\theta^*)_a \subseteq \Phi(\Xi_A^*)_a$$

since by **4.1** $\Phi(\theta^*) \subseteq \Phi(\Xi_A^*)$ for each $\theta \in S$. This shows that Ξ_A is the maximum pre-congruence. Finally it suffices to prove that Ξ_A is an equivalence relation on A. As the identity relation id_A on A

is a congruence it is clear that $\mathrm{id}_A \subseteq \Xi_A$ (reflexive). Assume that $(x, y) \in \Xi_A$. Then there is a pre-congruence θ such that $(x, y) \in \theta$ and so $(y, x) \in \theta^*$. But by the last lemma θ^* is a (pre-)congruence and hence $(y, x) \in \Xi$ (symmetric). Finally assume that $(x, y) \in \Xi_A$ and $(y, z) \in \Xi_A$. Then $(x, y) \in \theta_0$ and $(y, z) \in \theta_1$ for some $\theta_0, \theta_1 \in S$. Hence

$$(x, y) \in \theta_0 \subseteq (\theta_0 \cup \theta_1)^* \text{ and } (y, z) \in \theta_1 \subseteq (\theta_0 \cup \theta_1)^*$$

and so $(x, z) \in (\theta_0 \cup \theta_1)^*$ because $(\theta_0 \cup \theta_1)^*$ is an equivalence relation. As $(\theta_0 \cup \theta_1)^*$ is a (pre-)congruence by the last lemma we have $(x, z) \in \Xi_A$ (transitive). □

Theorem 4.7 *For every Φ-coalgebra (A, a) there is at most one homomorphism from any Φ-coalgebra into $(A/\Xi_A, a/\Xi_A)$.*

Proof. Let $e : A \to A/\Xi_A$ be a quotient function with respect to Ξ_A. Assume that $f, g : (B, b) \to (A/\Xi_A, a/\Xi_A)$ are homomorphisms. Construct a coequalizer $d : (A/\Xi_A, a/\Xi_A) \to (R, r)$ of f and g (which does exist by **2.1**). Then for any $u \in B$ there is $x, y \in A$ such that $f(u) = e(x)$ and $g(u) = e(y)$. Moreover $de(x) = df(u) = dg(u) = de(y)$, which means that $(x, y) \in (\mathrm{id}_R)_{de}$. As $(\mathrm{id}_R)_{de} \subseteq \Xi_A$ by **4.3** it follows that $(x, y) \in \Xi_A$ and $e(x) = e(y)$. Hence $f(u) = e(x) = e(y) = g(u)$, which proves that $f = g$. □

The following corollary is an immediate consequence from the last theorem.

Corollary 4.8 *If the category* **Set**(Φ) *of Φ-coalgebras has a weak final coalgebra, then it has a final coalgebra.* □

5 Main Theorem

This section proves the main theorem of the paper. To treat freely with trees of coalgebras we assume that an endofunctor $\Phi : \mathbf{Set} \to \mathbf{Set}$ preserves intersections throughout this section.
First we introduce tree congruences on coalgebras using the notion of trees. Then we show that, whenever all trees of coalgebras are bounded to a set, there is a weak final coalgebra. Thus by the similar fashion to Aczel and Mendler [2] an existence theorem of final coalgebras is proved.

Let (A, a) be a Φ-coalgebra. Define a relation ξ_A on A as follows: $(x, y) \in \xi_A$ for $x, y \in A$ if and only if there is an isomorphism $f : [x]_A \to [y]_A$ of Φ-coalgebras such that $f(x) = y$. Obviously $\xi_A : A \to A$ is an equivalence relation on A, which we call the *tree congruence* on (A, a) by virtue of the following

Theorem 5.1 *For each Φ-coalgebra (A, a) the equivalence relation ξ_A on A is a congruence on (A, a).*

Proof. Let $e : A \to Q$ be a quotient function with respect to ξ_A. It suffices to show that $(x, y) \in \xi_A$ implies $\Phi(e)a(x) = \Phi(e)a(y)$. Assume that $(x, y) \in \xi_A$. Let $i : [x]_A \to A$ and $j : [y]_A \to A$ be inclusions, respectively. There is an isomorphism $k : [x]_A \to [y]_A$ with $k(x) = y$.

$$\begin{tikzcd} A \arrow[d, "a"'] & {[x]_A} \arrow[l, "i"'] \arrow[r, "jk"] \arrow[d, "h_x"] & A \arrow[d, "a"] \\ \Phi A & \Phi[x]_A \arrow[l, "\Phi(i)"] \arrow[r, "\Phi(jk)"'] & \Phi A. \end{tikzcd}$$

First note that $ei = ejk$. For each $z \in [x]_A (= H)$ we have

$$\begin{aligned} [i(z)]_A &= [z]_H && (3.4(\text{a})) \\ &\cong jk[z]_H && (3.2(\text{b})) \\ &= [jk(z)]_A && (3.4(\text{b})), \end{aligned}$$

which indicates that $(i(z), jk(z)) \in \xi_A$ and so $ei(z) = ejk(z)$. Therefore it follows that

$$\begin{aligned} \Phi(e)a(x) &= \Phi(e)ai(x) \\ &= \Phi(e)\Phi(i)h_x(x) && (i \text{ is a homomorphism.}) \\ &= \Phi(e)\Phi(jk)(x) && (ei = ejk) \\ &= \Phi(e)ajk(x) && (jk \text{ is a homomorphism.}) \\ &= \Phi(e)a(y) && (y = jk(x)). \end{aligned}$$

The proof is completed. □

Theorem 5.2 *If every tree of a Φ-coalgebra (A,a) is isomorphic to a subcoalgebra of a Φ-coalgebra (T,t), then there is at least one homomorphism $f:(A,a)\to(T/\xi_T, t/\xi_T)$.*

Proof. Let $e:(T,t)\to(T/\xi_T,t/\xi_T)$ be a quotient homomorphism by ξ_T. For every $x\in A$ there is an injective homomorphism $k:[x]_A\to(T,t)$ by the assumption. Define a function $f:A\to T/\xi_T$ by $f(x)=ek(x)$.

$$\begin{tikzcd} A \arrow[d, "a"'] & {[x]_A} \arrow[l, "i"'] \arrow[r, "k"] \arrow[d, "h_x"] & T \arrow[r, "e"] \arrow[d, "t"] & T/\xi_T \arrow[d, "t/\xi_T"] \\ \Phi(A) & \Phi[x]_A \arrow[l, "\Phi(i)"] \arrow[r, "\Phi(k)"'] & \Phi T \arrow[r, "\Phi(e)"'] & \Phi(T/\xi_T). \end{tikzcd}$$

Note that this definition of $f(x)$ is independent on the choice of an injective homomorphism k. (Let $k':[x]_A\to T$ be another injective homomorphism. Then by **3.2**(b) and **3.4**(b) it is trivial that $[k(x)]_R\cong[x]_A\cong[k'(x)]_R$. Hence $ek(x)=ek'(x)$.) Next we show that $fi=ek$. For each $z\in[x]_A$ the composite mk of the inclusion $m:[z]_A\to[x]_A$ followed by k is an injective homomorphism into T and so $f(z)=ekm(z)$. Hence $fi(z)=f(z)=ekm(z)=ek(z)$, which shows that $fi=ek$. Finally we show that $f:A\to T/\xi_T$ is a homomorphism, that is, $a\Phi(f)=f(t/\xi_T)$. But we have

$$\begin{aligned} \Phi(f)a(x) &= \Phi(f)ai(x) \\ &= \Phi(i)\Phi(f)h_x(x) && (i \text{ is a homomorphism.}) \\ &= \Phi(ek)h_x(x) && (fi=ek) \\ &= (t/\xi_T)ek(x) && (ek \text{ is a homomorphism.}) \\ &= (t/\xi_T)f(x) && (f(x)=ek(x)). \quad \square \end{aligned}$$

For a set M the coproduct of all coalgebras in $\mathbf{Set}_M(\Phi)$ will be denoted by (T_M,t_M), that is,

$$(T_M,t_M)=\coprod_{(A,a)\in\mathbf{Set}_M(\Phi)}(A,a)$$

and $i_A:(A,a)\to(T_M,t_M)$ denotes the inclusion of the coproduct for a Φ-coalgebra $(A,a)\in\mathbf{Set}_M(\Phi)$. A Φ-coalgebra (A,a) is called *M-bounded* if there is an injection of A into M. It is obvious that for an M-bounded Φ-coalgebra (A,a) there is an injective homomorphism $k:(A,a)\to(T_M,t_M)$, that is, $card(A)\le card(M)$. Hence we have the following

Corollary 5.3 *If all trees of Φ-coalgebras are M-bounded for a set M, then for each Φ-coalgebra (A,a) there is at least one homomorphism $f:(A,a)\to(T_M/\xi_{T_M},t_M/\xi_{T_M})$, that is, the quotient coalgebra $(T_M/\xi_{T_M},t_M/\xi_{T_M})$ of (T_M,t_M) is a weak final coalgebra in* $\mathbf{Set}(\Phi)$. $\square$

In a category of coalgebras a final coalgebra is a coalgebra such that there is a unique homomorphism from each coalgebra into it. Combining with **4.8** and the last corollary we have the following

Theorem 5.4 *If there is a set M such that all trees of Φ-coalgebras are M-bounded, then the category* $\mathbf{Set}(\Phi)$ *of Φ-coalgebras has a final coalgebra.* $\square$

6 Examples

This section illustrates a few examples of coalgebras which satisfy the main theorem **5.4** and so have a final coalgebra.

Let M be a set. The M-bounded power set functor $\wp_M:\mathbf{Set}\to\mathbf{Set}$ is a functor such that

$$\wp_M(A)=\{S|S\subseteq A\wedge card(S)\le card(M)\}$$

for all sets A, where $card(M)$ denotes the cardinality of M. For a set M n-th product M^n is defined by $M^0=1$ (a singleton set) and $M^{n+1}=M^n\times M$ for $n\ge 0$. The set M^* of all finite strings of elements in M is formally defined by $M^*=\cup_{n\ge 0}M^n$.

Theorem 6.1 *All trees of $\wp_M$-coalgebras are M^*-bounded.*

Proof. Let (A, a) be a $\wp_M$-coalgebra and $x \in A$. Define a subset $[x]_n$ of A by $[x]_0 = \{x\}$ and $[x]_{n+1} = \cup_{y \in [x]_n} a(y)$ for $n \geq 0$. Set $[x]_\infty = \cup_{n \geq 0}[x]_n$. From $card([x]_{n+1}) \leq card([x]_n \times M)$ it follows that

$$card([x]_\infty) \leq card(\cup_{n \geq 0} M^n) = card(M^*).$$

Finally it suffices to see that $[x]_A = [x]_\infty$. By induction we have $[x]_n \subseteq [x]_A$ for all $n \geq 0$ and so $[x]_\infty \subseteq [x]_A$. Because $[x]_0 \subseteq [x]_A$ and if $[x]_n \subseteq [x]_A$ then $[x]_{n+1} = \cup_{y \in [x]_n} a(y) \subseteq [x]_A$. Finally note that $[x]_\infty$ is a subcoalgebra of (A, a) since $a(y) \subseteq [x]_{n+1} \subseteq [x]_\infty$ (i.e. $a(y) \in \wp_M([x]_\infty)$) for $y \in [x]_n$. Hence $[x]_A \subseteq [x]_\infty$. □

Combining with **5.4** and the last theorem we have the following

Corollary 6.2 *The category* **Set**$(\wp_X)$ *has a final coalgebra.* □

Note that $\wp_1(X) = 1 + X$ for a singleton set $1 (= \{\emptyset\})$.

Let Ψ and Φ be endofunctors on **Set**. A natural transformation $\nu : \Psi \to \Phi$ is *strict* if for every injection $f : X \to Y$ a naturality square

$$\begin{array}{ccc} \Psi X & \xrightarrow{\Psi(f)} & \Psi Y \\ {\scriptstyle \nu_X}\downarrow & & \downarrow{\scriptstyle \nu_Y} \\ \Phi X & \xrightarrow[\Phi(f)]{} & \Phi Y \end{array}$$

is a pullback.

Proposition 6.3 *Let $\nu : \Psi \to \Phi$ be a natural transformation between endofunctors Ψ and Φ on* **Set**. *If Φ preserves intersections and $\nu : \Psi \to \Phi$ is strict, then Ψ also preserves intersections.*

Proof. It follows from easy diagram chasing. □

Lemma 6.4 *Let $\nu : \Psi \to \Phi$ be a strict natural transformation and (B, b) a Ψ-coalgebra. Then a subset H of B is a subcoalgebra of (B, b) if and only if H is a subcoalgebra of a Φ-coalgebra $(B, \nu_B b)$.*

Proof. Let $i : H \to B$ be the inclusion and consider a diagram

$$\begin{array}{ccc} H & \xrightarrow{i} & B \\ & & \downarrow{\scriptstyle b} \\ \Psi H & \xrightarrow{\Psi(i)} & \Psi B \\ {\scriptstyle \nu_H}\downarrow & & \downarrow{\scriptstyle \nu_B} \\ \Phi H & \xrightarrow[\Phi(i)]{} & \Phi B, \end{array}$$

in which the square is a pullback by the strictness of ν. Then it is trivial that a function $h : H \to \Psi H$ with $bi = \Psi(i)h$ bijectively corresponds to a function $h' : H \to \Phi H$ with $\nu_B bi = \Phi(i)h'$. This completes the proof. □

As a direct result from the above lemma we have the following

Corollary 6.5 *Let Φ, Ψ :* **Set** $\to$ **Set** *be endofunctors preserving intersections and $\nu : \Psi \to \Phi$ a strict natural transformation.*

(a) *If (B, b) is a Ψ-coalgebra, then $[x]_{(B,b)} = [x]_{(B,\nu_B b)}$ for all $x \in B$.*

(b) *If all trees of Φ-coalgebras are M-bounded for a set M, then so are those of Ψ-coalgebras.* □

By **6.1** and **5.4** we have the following

Example 6.6 *All categories of coalgebras for the following endofunctors have final coalgebra.*

(a) *The finite powerset functor* $\wp_{\rm fin}$: **Set** $\to$ **Set**.

(b) *The Kleene functor* X^* : **Set** $\to$ **Set**.

(c) *A polynomial functor* $\Phi X = A_0 + A_1 \times X + \cdots + A_n \times X^n + \cdots$: **Set** $\to$ **Set** *(where* $A_0, A_1, \cdots$ *are fixed sets).*

(d) *A functor* $\wp_M(A \times X)$: **Set** $\to$ **Set**.

(e) *A functor* $(A \times X)^*$: **Set** $\to$ **Set**

Proof. (a) Let ω denote the set of all natural numbers. A natural inclusion $\wp_{\rm fin}(X) \to \wp_\omega(X)$ is a strict natural transformation. (b) A natural transformation $X^* \to \wp_\omega(X)$ assigning $\{\sigma_1, \sigma_2, \cdots, \sigma_k\} \in \wp_\omega(X)$ to $\sigma_1\sigma_2\cdots\sigma_k \in X^*$ is strict. (c) A natural transformation $\Phi X \to \wp_\omega(X)$ assigning $\{\sigma_1, \sigma_2, \cdots, \sigma_k\} \in \wp_\omega(X)$ to $(a, \sigma_1\sigma_2\cdots\sigma_k) \in A_k \times X^k$ $(k \geq 0)$ is strict. (d) A natural transformation $\wp_M(A \times X) \to \wp_M(X)$ induced by the projection $A \times X \to X$ is strict. (e) A natural transformation $(A \times X)^* \to X^*$ assigning $\{\sigma_1, \sigma_2, \cdots, \sigma_k\} \in \wp_\omega(X)$ to $(a_1, \sigma_1)(a_2, \sigma_2)\cdots(a_k, \sigma_k) \in (A \times X)^*$ is strict.

References

[1] Peter Aczel, Non-well-founded sets, CSLI Lecture Notes, No. **14**, Stanford University, 1988.

[2] Peter Aczel and Nax Mendler, A final coalgebra theorem, Lecture Notes in Computer Science, Vol. **389** (1989), 357-365.

[3] Michael Barr, Terminal coalgebras in well-founded set theory, Theoretical Computer Science **114**(3) (1993), 299 - 315.

[4] Yasuo Kawahara and Yoshihiro Mizoguchi, Relational structures and their partial morphisms in the view of single pushout rewriting, To appear in Lecture Notes in Computer Science, **776** (1994), 218 – 233.

[5] Yasuo Kawahara and Masao Mori. A small final coalgebra theorem. Technical Report RIFIS-TR-CS-90, Research Institute of Fundamental Information Science, Kyushu University, 1994.

[6] Saunders Mac Lane, Categories for the working mathematicians, Springer-Verlag, 1971.

[7] Yoshihiro Mizoguchi, A graph structure over the category of sets and partial functions, Cahiers de topologie et géométrie différentielle catégoriques, **34** (1993), 2 – 12.

[8] J.C. Raoult, *On graph rewritings*, Theoret. Comput. Sci. **32**(1984), 1–24.

[9] Jan J.M.M. Rutten and Daniel Turi, *On the foundations of Final Semantics : Non-Standard Sets, Metric Spaces, Partial Orders*, Lecture Notes in Computer Science, **666** (1992), 477 – 530.