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Introduction to vertex operator algebras I

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1 Introduction

The theory of vertex (operator) algebras has developed rapidly in the last few years. These rich algebraic structures provide the proper formulation for the moonshine module construction for the Monster group ([B1-B2], [FLM1], [FLM3]) and also give a lot of new insight into the representation theory of the Virasoro algebra and affine Kac-Moody algebras (see for instance [DL3], [DMZ], [FZ], [W]). The modern notion of chiral algebra in conformal field theory [BPZ] in physics essentially corresponds to the mathematical notion of vertex operator algebra; see e.g. [MS].

This is the first part of three consecutive lectures by Huang, Li and myself. In this part we are mainly concerned with the definitions of vertex operator algebras, twisted modules and examples. The second part by Li is about the duality and local systems and the third part by Huang is devoted to the contragradient modules and geometric interpretations of vertex operator algebras. (We refer the reader to Li and Huang's lecture notes for the related topics.) So many exciting topics are not covered in these three lectures. The book [FHL] is an excellent introduction to the subject. There are also existing papers [H1], [Ge] and [P] which review the axiomatic definition of vertex operator algebras, geometric interpretation of vertex operator algebras, the connection with conformal field theory, Borcherds algebras and the monster Lie algebra.

Most work on vertex operator algebras has been concentrated on the concrete examples of vertex operator algebras and the representation theory. In particular, the repre-

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sentation theory for the concrete vertex operator algebras, which include the moonshine vertex operator algebra $V^\natural$ ([FLM3],[D3]), the vertex operator algebras based on even positive definite lattices [D1], the vertex operator algebras associated with the integrable representations of affine Lie algebras, Virasoro algebra and W-algebras ([DMZ], [DL3], [FKRW], [FZ], [KW] and [W]), have been studied extensively. There are also abstract approaches such as one to one correspondence between the set of inequivalent irreducible (twisted) modules for a given vertex operator algebra and the set of inequivalent irreducible modules for an associative algebra associated with the vertex operator algebra and an automorphism of the algebra (see [DLM] and [Z]), the theory of local system [L1-L2], induced modules [DLi], the tensor products of modules ([HL1-HL4], [H3] and [L3]); See also [FHL] for the results concerning intertwining operators and contragredient modules. Many of these results are analogues of the corresponding results in the classical Lie algebra theory.

This paper is organized as follows: In Section 2, we present the definitions of vertex operator algebras and twisted modules. We also make some remarks. The Section 3 is about the examples of vertex operator algebras. In particular, we discuss the vertex operator algebras associated with Heisenberg algebras, positive definite even lattices, affine Lie algebras and Virasoro algebra. We emphasize how a vertex operator algebras with finitely many generators can be constructed by using the Jacobi identity. In Section 5 we finally mention the orbifold theory which is not treated in the introductory text and various generalizations of the notion of vertex operator algebras.

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2 Vertex operator algebras and modules

The notion of vertex (operator) algebra arose naturally from the problem of realizing the monster simple group as a symmetry group of certain algebraic structure. In this section we shall review the definition of vertex operator algebras and their (twisted) modules (see [B1], [D2], [FFR], [FHL] and [FLM3]). We shall also discuss some consequences of the definitions and basic properties.

First we introduce some notation. We shall use commuting formal variables z, z_0, z_1, z_2, etc., and the basic generating function

$$\delta(z) = \sum_{n\in\mathbb{Z}} z^n, \tag{2.1}$$

formally the expansion of the δ-function at $z = 1$. The fundamental (and elementary) properties of the δ-function can be found in [FLM3], [FHL] and [DL3].

For a vector space V (we work over $\mathbb{C}$) and a positive integer r we denote the vector space of formal Laurent series in $z^{1/r}$ with coefficients in V by

$$V[[z^{1/r}, z^{-1/r}]] = \left\{ \sum_{n\in\frac{1}{r}\mathbb{Z}} v_n z^n | v_n \in V \right\}. \tag{2.2}$$

We also define a formal residue notation:

$$\mathrm{Res}_z \sum_{n\in\frac{1}{r}\mathbb{Z}} v_n z^n = v_{-1}.$$

A *vertex operator algebra* is a $\mathbb{Z}$-graded vector space:

$$V = \coprod_{n\in\mathbb{Z}} V_n; \quad \text{for} \quad v \in V_n, \quad n = \mathrm{wt}\, v; \tag{2.3}$$

such that $\dim V_n < \infty$ for all $n \in \mathbb{Z}$ and $V_n = 0$ if n is sufficiently small; equipped with a linear map

$$\begin{aligned} &V \to (\mathrm{End}\, V)[[z, z^{-1}]] \\ &v \mapsto Y(v, z) = \sum_{n\in\mathbb{Z}} v_n z^{-n-1} \quad (v_n \in \mathrm{End}\, V) \end{aligned} \tag{2.4}$$

and with two distinguished vectors $\mathbf{1} \in V_0$, $\omega \in V_2$ satisfying the following conditions for $u, v \in V$:

$$u_n v = 0 \quad \text{for } n \text{ sufficiently large;} \tag{2.5}$$

$$Y(\mathbf{1}, z) = 1; \tag{2.6}$$

$$Y(v, z)\mathbf{1} \in V[[z]] \quad \text{and} \quad \lim_{z \to 0} Y(v, z)\mathbf{1} = v; \tag{2.7}$$

$$\begin{aligned} \langle z_0^{-1}\delta\left(\frac{z_1 - z_2}{z_0}\right) Y(u, z_1)Y(v, z_2) - z_0^{-1}\delta\left(\frac{z_2 - z_1}{-z_0}\right) Y(v, z_2)Y(u, z_1) \\ \langle = z_2^{-1}\delta\left(\frac{z_1 - z_0}{z_2}\right) Y(Y(u, z_0)v, z_2) \end{aligned} \tag{2.8}$$

(Jacobi identity) where all binomial expressions, for instance, $(z_1 - z_2)^n$ $(n \in \mathbb{Z})$ are to be expanded in nonnegative integral powers of second variable z_2: This identity is interpreted algebraically as follows: if this identity is applied to a single vector of V then the coefficient of each monomial in z_0, z_1, z_2 is a finite sum in V;

$$[L(m), L(n)] = (m - n)L(m + n) + \frac{1}{12}(m^3 - m)\delta_{m+n,0}(\operatorname{rank} V) \tag{2.9}$$

for $m, n \in \mathbb{Z}$, where

$$L(n) = \omega_{n+1} \quad \text{for} \quad n \in \mathbb{Z}, \quad \text{i.e.,} \quad Y(\omega, z) = \sum_{n \in \mathbb{Z}} L(n) z^{-n-2} \tag{2.10}$$

and

$$\operatorname{rank} V \in \mathbb{Q}; \tag{2.11}$$

$$L(0)v = nv = (\operatorname{wt} v)v \quad \text{for} \quad v \in V_n \tag{2.12}$$

$$\operatorname{rank} V \in \mathbb{Q}; \tag{2.13}$$

$$L(0)v = nv = (\operatorname{wt} v)v \quad \text{for} \quad v \in V_n \ (n \in \mathbb{Z}); \tag{2.14}$$

$$\frac{d}{dz} Y(v, z) = Y(L(-1)v, z). \tag{2.15}$$

This completes the definition. We denote the vertex operator algebra just defined by

$$(V, Y, \mathbf{1}, \omega) \tag{2.16}$$

(or briefly, by V). The series $Y(v, z)$ are called vertex operators.

The following are consequences of the definitions: For $u, v \in V$,

$$[L(-1), Y(v,z)] = Y(L(-1)v, z) \tag{2.17}$$

$$[L(0), Y(v,z)] = Y(L(0)v, z) + zY(L(-1)v, z) \tag{2.18}$$

$$Y(e^{z_0 L(-1)}v, z) = Y(v, z+z_0) \tag{2.19}$$

$$e^{z_0 L(-1)}Y(v,z)e^{-z_0 L(-1)} = Y(e^{z_0 L(-1)}v, z) \tag{2.20}$$

$$Y(v,z)\mathbf{1} = e^{zL(-1)}v \tag{2.21}$$

$$Y(u,z)v = e^{zL(-1)}Y(v,-z)u. \tag{2.22}$$

Let S be a subset of V. The subalgebra $\langle S\rangle$ *generated* by S, defined as the smallest subalgebra containing S, is given by

$$\{v^1_{n_2} \cdots v^k_{n_k} \cdot \mathbf{1} | v^i \in S \cup \{\mathbf{1}, \omega\}, n_i \in \mathbb{Z}\}.$$

A vertex operator algebra V is generated by S if $V = \langle S\rangle$. If we further assume that S is a finite set we say that V is finitely generated.

An *ideal* I of V is a subspace of V such that $\mathbf{1} \notin I$ and $u_n v \in I$ for any $u \in V$ and $v \in I$. Using the skew symmetry (2.22) one can verify that the quotient space V/I has a structure of vertex operator algebra.

An *automorphism* g of the vertex algebra V is a linear automorphism of V preserving $\mathbf{1}$ and ω such that the actions of g and $Y(v,z)$ on V are compatible in the sense that

$$gY(v,z)g^{-1} = Y(gv, z) \tag{2.23}$$

for $v \in V$. Then $gV_n \subset V_n$ for $n \in \mathbb{Z}$ and V is a direct sum of the eigenspaces of g :

$$V = \coprod_{j \in \mathbb{Z}/r\mathbb{Z}} V^j \tag{2.24}$$

where r is the order of g, $\eta = e^{2\pi i/r}$ and $V^j = \{v \in V | gv = \eta^j v\}$.

We also have the notion of g-twisted module (see [D2], [FFR], [Le] and [FLM2]). Let $(V, Y, \mathbf{1}, \omega)$ be a vertex algebra and let g be an automorphism of V of order r. A g-twisted *module* M for $(Y, V, \mathbf{1}, \omega)$ is a $\mathbb{C}$-graded vector space:

$$M = \coprod_{n \in \mathbb{C}} M_n; \quad \text{for} \quad w \in M_n, \quad n = \mathrm{wt}\, w; \tag{2.25}$$

such that for any fixed $\lambda \in \mathbb{C}$, $M_{\lambda+n} = 0$ if $n \in \frac{1}{r}\mathbb{Z}$ is sufficiently small and $\dim M_c < \infty$ for all $c \in \mathbb{C}$; equipped with a linear map

$$\begin{aligned} &V \to (\operatorname{End} M)[[z^{1/r}, z^{-1/r}]] \\ &v \mapsto Y(v,z) = \sum_{n\in\frac{1}{r}\mathbb{Z}} v_n z^{-n-1} \quad (v_n \in \operatorname{End} M) \end{aligned} \tag{2.26}$$

satisfying the following conditions for $u, v \in V$ and $w \in M$ and $l \in \mathbb{Z}$:

$$Y(v,z) = \sum_{n\in l/r+\mathbb{Z}} v_n z^{-n-1} \quad \text{for} \quad v \in V^l; \tag{2.27}$$

$$u_n w = 0 \quad \text{for} \quad n \quad \text{sufficiently large;} \tag{2.28}$$

$$Y(\mathbf{1}, z) = 1; \tag{2.29}$$

$$\begin{aligned} &z_0^{-1}\delta\left(\frac{z_1 - z_2}{z_0}\right) Y(u,z_1)Y(v,z_2) - z_0^{-1}\delta\left(\frac{z_2 - z_1}{-z_0}\right) Y(v,z_2)Y(u,z_1) \\ &= z_2^{-1}\left(\frac{z_1 - z_0}{z_2}\right)^{-i/r} \delta\left(\frac{z_1 - z_0}{z_2}\right) Y(Y(u,z_0)v,z_2) \end{aligned} \tag{2.30}$$

where $u \in V^i$ and $Y(u, z_0)$ is an operator on V;

$$[L(m), L(n)] = (m-n)L(m+n) + \frac{1}{12}(m^3 - m)\delta_{m+n,0}(\operatorname{rank} V) \tag{2.31}$$

for $m, n \in \mathbb{Z}$, where

$$L(n) = \omega_{n+1} \quad \text{for} \quad n \in \mathbb{Z}, \quad \text{i.e.,} \quad Y(\omega, z) = \sum_{n\in\mathbb{Z}} L(n)z^{-n-2}; \tag{2.32}$$

$$L(0)w = nw = (\operatorname{wt} w)w \quad \text{for} \quad w \in M_n \ (n \in \mathbb{Q}); \tag{2.33}$$

$$\frac{d}{dz}Y(v,z) = Y(L(-1)v, z). \tag{2.34}$$

This completes the definition. We denote this module by (M, Y) (or briefly by M).

Remark 2.1 *If $g = 1$, then M is an ordinary module in the precise sense of [FLM3]. Note that any g-twisted V-module is a V^0-module.*

Remark 2.2 *In the definition of twisted module if removing the finiteness condition $\dim M_c < \infty$ we get a* weak *g-twisted module.*

Remark 2.3 *Taking* Res_{z_0} *in (2.30), we get the commutator formula:*

$$[Y(u,z_1),Y(v,z_2)] = \mathrm{Res}_{z_0}\left\{z_2^{-1}\left(\frac{z_1-z_0}{z_2}\right)^{-i/r}\delta\left(\frac{z_1-z_0}{z_2}\right)Y(Y(u,z_0)v,z_2)\right\}. \quad (2.35)$$

Note that the factor $\left(\frac{z_1-z_0}{z_2}\right)^{-i/r}\delta\left(\frac{z_1-z_0}{z_2}\right)$ *only involves the nonnegative powers of* z_0. *Thus in the computation of commutator* $[Y(u,z_1),Y(v,z_2)]$ *we only use the "singular terms" in* $Y(u,z_0)v$, *namely,* $\sum_{n\geq 0} u_n v z_0^{-n-1}$. *This fact was well-known in the physics literature. Moreover, for* $s,t\in\mathbb{Q}$, *we can compare the coefficients of* $z_1^{-s-1}z_2^{-t-1}$ *on the both sides of (2.35) to get the commutator* $[u_s,v_n]$:

$$[u_s,v_t] = u_sv_t - v_tu_s = \sum_{m\geq 0}\binom{s}{m}(u_mv)_{s+t-m}. \quad (2.36)$$

Remark 2.4 *Acting on* M *the operator* $Y(Y(u,z_0)v,z_2)$ *is determined uniquely by the operators* $Y(u,z_1)$ *and* $Y(v,z_2)$. *In order to see this, we first recall a relation on* δ*-functions from [FHL]:*

$$z_1^{-1}\left(\frac{z_2+z_0}{z_1}\right)^{i/r}\delta\left(\frac{z_2+z_0}{z_1}\right) = z_2^{-1}\left(\frac{z_1-z_0}{z_2}\right)^{-i/r}\delta\left(\frac{z_1-z_0}{z_2}\right).$$

Now multiplying (2.8) by $z_1^{i/r}$ *and taking* Res_{z_1} *we see that*

$$\begin{aligned} Y(Y(u,z_0)v,z_2) &= \mathrm{Res}_{z_1} z_1^{-i/r} z_0^{-1}\delta\left(\frac{z_1-z_2}{z_0}\right)Y(u,z_1)Y(v,z_2) \\ &\quad - \mathrm{Res}_{z_1} z_1^{i/r} z_0^{-1}\delta\left(\frac{z_2-z_1}{-z_0}\right)Y(v,z_2)Y(u,z_1). \end{aligned} \quad (2.37)$$

If $g=1$ *noting that* u_nv *is the coefficient of* z_0^{-n-1} *in* $Y(u,z_0)v$, *multiplying (2.37) by* z_0^n *and taking* Res_{z_0} *we find that*

$$\begin{aligned} &Y(u_nv,z_2) \\ &= \mathrm{Res}_{z_0}\left\{z_0^{n-1}\mathrm{Res}_{z_1}\left\{\delta\left(\frac{z_1-z_2}{z_0}\right)Y(u,z_1)Y(v,z_2)\right.\right. \\ &\quad \left.\left. -\delta\left(\frac{z_2-z_1}{-z_0}\right)Y(v,z_2)Y(u,z_1)\right\}\right\} \\ &= \mathrm{Res}_{z_1}\left\{(z_1-z_2)^nY(u,z_1)Y(v,z_2) - (-z_2+z_1)^nY(v,z_2)Y(u,z_1)\right\}. \end{aligned} \quad (2.38)$$

In fact, this suggests a way to construct vertex operator algebras and their modules for finitely generated vertex operator algebras. The vertex operator algebras associated with *highest weight representations for affine Lie algebras and Virasoro algebra were constructed in this spirit [FZ]. For a different construction by using so called "the local systems" see [L1-L2].*

Here we give an easy consequence of (2.38). In (2.38), if $n = -1$, we have

$$Y(u_{-1}v, z_2) = \sum_{m<0} u_m z_2^{-m-1} Y(v, z_2) + Y(v, z_2) \sum_{m \geq 0} u_m z_2^{-m-1}.$$

This suggests defining a "normal ordering" operation by: For $u, v \in V$, $m, n \in \mathbb{Z}$,

$$ {}^{\times}_{\times} u_m v_n {}^{\times}_{\times} = \begin{cases} \langle u_m v_n & \langle \text{if } \ m < 0 \\ \langle v_n u_m & \langle \text{if } \ m \geq 0. \end{cases} \tag{2.39}$$

Then (since $v = v_{-1}\mathbf{1}$)

$$Y(u_{-1}v_{-1}\mathbf{1}, z_2) = Y(u_{-1}v, z_2) = {}^{\times}_{\times} Y(u, z_2) Y(v, z_2) {}^{\times}_{\times}. \tag{2.40}$$

However, this normal ordering is not in general a commutative operation, since if m and n are both negative or both nonnegative, ${}^{\times}_{\times} u_m v_n {}^{\times}_{\times}$ and ${}^{\times}_{\times} v_n u_m {}^{\times}_{\times}$ differ by $\pm[u_m, v_n]$. The following proposition explains that the nilpotent property of vertex operators which holds in the algebra will also hold for modules under a mild assumption [DL3]. In particular, this is true for the vertex operator algebras associated with the integrable highest weight representations of affine Lie algebras.

Proposition 2.5 *Let V be any vertex operator algebra and let (W, Y_W) be any V-module. Let $v \in V$ be such that the component operators v_n $(n \in \mathbb{Z})$ of $Y_W(v, z)$ all commute with one another, so that $Y_W(v, z)^N$ is well defined on W for $N \in \mathbb{N}$. Then*

$$Y_W((v_{-1})^N \mathbf{1}, z) = Y_W(v, z)^N. \tag{2.41}$$

In particular, if $(v_{-1})^N \mathbf{1} = 0$ for a fixed N, then

$$Y_W(v, z)^N = 0. \qquad \square \tag{2.42}$$

Let V be a vertex operator algebra and g a finite order automorphism. We say that V is g-rational if V has only finitely many irreducible g-twisted modules and if any g-twisted V-module is completely reducible. V is rational if V is id-rational. V is holomorphic if V is rational and V is the only irreducible module for itself.

3 Examples

In this section we will present some well-known examples of vertex operator algebras and their modules. We refer the reader to [D2], [DL4], [FFR], [FLM1-FLM3], [Le], [L2], [DM2-DM3] for various examples of twisted modules.

Vertex operator algebras associated with Heisenberg algebras. Let $\mathbf{h}$ be a vector space equipped with a symmetric nondegenerate bilinear form $\langle\cdot,\cdot\rangle$. So we can identify $\mathbf{h}$ with its dual $\mathbf{h}^*$ naturally.

Viewing $\mathbf{h}$ as an abelian Lie algebra, we consider the corresponding affine Lie algebra

$$\hat{\mathbf{h}} = \mathbf{h} \otimes \mathbb{C}[t,t^{-1}] \oplus \mathbb{C}c \tag{3.1}$$

with structure defined by

$$[x\otimes t^m, y\otimes t^n] = \langle x,y\rangle m\delta_{m+n,0}c \ \text{ for } x,y\in\mathbf{h},\ m,n\in\mathbb{Z}, \tag{3.2}$$

$$[c,\hat{\mathbf{h}}] = 0. \tag{3.3}$$

Set

$$\hat{\mathbf{h}}^+ = \mathbf{h}\otimes t\mathbb{C}[t],\ \ \hat{\mathbf{h}}^- = \mathbf{h}\otimes t^{-1}\mathbb{C}[t^{-1}]. \tag{3.4}$$

The subalgebra

$$\hat{\mathbf{h}}_{\mathbb{Z}} = \hat{\mathbf{h}}^+ \oplus \hat{\mathbf{h}}^- \oplus \mathbb{C}c \tag{3.5}$$

of $\hat{\mathbf{h}}$ is a Heisenberg algebra, in the sense that its commutator subalgebra coincides with its center, which is one-dimensional. Let $\lambda\in\mathbf{h}$ and consider the induced $\hat{\mathbf{h}}$-module

$$M(1,\lambda) = U(\hat{\mathbf{h}}) \otimes_{U(\mathbf{h}\otimes\mathbb{C}[t]\oplus\mathbb{C}c)} \mathbb{C}_\lambda \simeq S(\hat{\mathbf{h}}^-) \quad \text{(linearly)}, \tag{3.6}$$

$\mathbf{h}\otimes t\mathbb{C}[t]$ acting trivially on $\mathbb{C}$, $\mathbf{h}$ acting as $\langle h,\lambda\rangle$ for $h\in\mathbf{h}$ and c acting as 1. We shall write $M(1)$ for $M(1,0)$. For $\alpha\in\mathbf{h}$ and $n\in\mathbb{Z}$ we write $\alpha(n)$ for the operator $\alpha\otimes t^n$ and set

$$\alpha(z) = \sum_{n\in\mathbb{Z}} \alpha(n) z^{-n-1}$$

and define

$$Y(v,z) = {\circ \atop \circ} \left(\frac{1}{(n_1-1)!} \left(\frac{d}{dz} \right)^{n_1-1} \alpha_1(z) \right) \cdots \left(\frac{1}{(n_k-1)!} \left(\frac{d}{dz} \right)^{n_k-1} \alpha_k(z) \right) {\circ \atop \circ},$$

where

$$v = \alpha_1(-n_1) \cdots \alpha_k(-n_k) \in M(1)$$

for $\alpha_1, ..., \alpha_k \in \mathbf{h},\ n_1, ..., n_k \in \mathbb{Z}\ (n_i > 0)$ and where we use a normal ordering procedure, indicated by open colons, which signify that the expression is to be reordered if necessary so that all the operators $\alpha(n)$ $(\alpha \in \mathbf{h},\ n < 0)$, are to be placed to the left of all the operators $\alpha(n)$ $(\alpha \in \mathbf{h},\ n \geq 0)$ before the expression is evaluated. Extend this definition to all $v \in M(1)$ by linearity. Set $\mathbf{1} = 1$, $\omega = \frac{1}{2}\sum_{i=1}^{d} \alpha_i(-1)^2 \in M(1)$ where $\{\alpha_i\}$ is an orthonomal basis of $\mathbf{h}$. The following result can be found in [Gu].

Proposition 3.1 *The space $M(1) = (M(1), Y, \mathbf{1}, \omega)$ is a vertex operator algebra and $M(1,\lambda) = (M(1,\lambda), Y)$ is a complete list of inequivalent irreducible module for $M(1)$ for $\lambda \in \mathbf{h}$.*

Remark 3.2 *It is easy to see that the vertex operator algebra $M(1)$ is generated by $S = \{\alpha_i(-1)\}$. If we identify $M(1)$ with a symmetric algebra $\mathbb{C}[x_{i,n} | i = 1, ..., d, n = 1, 2, ...]$. Then it is easy to see that all the vertex operators are built from the operators $x_{i,n}$ which is a multiplication operator on $M(1)$ and $\frac{\partial}{\partial x_{i,n}}$.*

Vertex operator algebras associated with even positive definite lattices. Let L be an even lattice, i.e., a finite-rank free abelian group equipped with a symmetric nondegenerate $\mathbb{Q}$-valued $\mathbb{Z}$-bilinear form $\langle \cdot, \cdot \rangle$, not necessarily positive definite such that $\langle \alpha, \alpha \rangle \in 2\mathbb{Z}$ for all $\alpha \in L$. Set vector space $\mathbf{h} = \mathbb{C} \otimes_{\mathbb{Z}} L$ and extend the form $\langle \cdot, \cdot \rangle$ from L to $\mathbf{h}$ by $\mathbb{C}$-linearity.

The dual lattice L° of L is defined to be

$$L^\circ = \{\beta \in \mathbf{h} | \langle \beta, L \rangle \subset \mathbb{Z}\}. \tag{3.7}$$

Then L° is a rational lattice whose rank is equal to the rank of L. Let $\hat{L}^\circ$ be a central extension of L°:

$$1 \to \langle \omega_q \rangle \to \hat{L}^\circ \bar{\to} L^\circ \to 1, \tag{3.8}$$

with the commutator map $c(\alpha,\beta)$ for $\alpha,\beta \in L^\circ$ such that $c(\alpha,\beta) = (-1)^{\langle\alpha,\beta\rangle}$ if $\alpha,\beta \in L$, where $\langle\omega_q\rangle$ is the cyclic group generated by a primitive q^{th} root of unity $\omega_q \in \mathbb{C}^\times$ and q is a positive even integer.

Form the induced $\hat{L}^\circ$-module and $\mathbb{C}$-algebra

$$\langle\mathbb{C}\{L^\circ\} = \mathbb{C}[\hat{L}^\circ] \otimes_{\mathbb{C}[\langle\omega_q\rangle]} \mathbb{C} \simeq \mathbb{C}[L^\circ] \quad \text{(linearly)}, \tag{3.9}$$

where $\mathbb{C}[\cdot]$ denotes group algebra, ω_q acts on $\mathbb{C}$ as multiplication by ω_q. For $a \in \hat{L}^\circ$, write $\iota(a)$ for the image of a in $\mathbb{C}\{L^\circ\}$. Then the action of $\hat{L}^\circ$ on $\mathbb{C}\{L^\circ\}$ is given by:

$$a \cdot \iota(b) = \iota(ab), \tag{3.10}$$

$$\omega_q \cdot \iota(b) = \omega_q \iota(b) \tag{3.11}$$

for $a, b \in \hat{L}^\circ$. We give $\mathbb{C}\{L^\circ\}$ the $\mathbb{C}$-gradation determined by:

$$\mathrm{wt}\, \iota(a) = \frac{1}{2}\langle\bar{a},\bar{a}\rangle \quad \text{for } a \in \hat{L}^\circ. \tag{3.12}$$

Also define an action of $\mathbf{h}$ on $\mathbb{C}\{L^\circ\}$ by:

$$h \cdot \iota(a) = \langle h, \bar{a}\rangle \iota(a) \tag{3.13}$$

for $h \in \mathbf{h}$. Define

$$z^h \cdot \iota(a) = z^{\langle h,\bar{a}\rangle} \iota(a) \tag{3.14}$$

for $h \in \mathbf{h}$. Set

$$V_{L^\circ} = M(1) \otimes_{\mathbb{C}} \mathbb{C}\{L^\circ\} \simeq S(\hat{\mathbf{h}}^-) \otimes \mathbb{C}[L^\circ] \quad \text{(linearly)} \tag{3.15}$$

Then $\hat{L}^\circ$, $\hat{\mathbf{h}}_{\mathbb{Z}}$, h, z^h ($h \in \mathbf{h}$) act naturally on V_{L° by acting on either $M(1)$ or $\mathbb{C}\{L^\circ\}$ as indicated above. In particular, c acts as 1.

For a subset M of L° (not necessarily a sublattice), we write

$$\hat{M} = \{b \in \hat{L}^\circ \mid \bar{b} \in M\} \tag{3.16}$$

and

$$\mathbb{C}\{M\} = \mathrm{span}\{\iota(b) \mid b \in \hat{M}\} \subset \mathbf{C}\{L^\circ\}, \tag{3.17}$$

$$V_M = M(1) \otimes \mathbb{C}\{M\} \subset V_{L^\circ}. \tag{3.18}$$

(Later we shall take M to be a coset of a lattice.) We may apply the considerations of the present section to $\hat{L}$ in place of $\hat{L}^\circ$. In particular, we have

$$\mathbb{C}\{L\} = \mathbb{C}[\hat{L}] \otimes_{\mathbb{C}[\omega_p]} \mathbb{C} \tag{3.19}$$

and

$$V_L = M(1) \otimes \mathbb{C}\{L\}. \tag{3.20}$$

Let

$$L^\circ = \bigcup_{i \in L^\circ/L} (L + \lambda_i) \tag{3.21}$$

be the coset decomposition such that $\lambda_0 = 0$. Let $\Lambda_i \in \hat{L}^\circ$ so that $\bar{\Lambda}_i = \lambda_i$. Then

$$\hat{L}^\circ = \bigcup_{i \in L^\circ/L} \hat{L}\Lambda_i \tag{3.22}$$

is the coset decomposition of $\hat{L}^\circ$. For $i \in L^\circ/L$, set

$$V(i) = V_{L+\lambda_i} = M(1) \otimes \mathbb{C}\{L + \lambda_i\} \simeq S(\hat{\mathbf{h}}^-) \otimes \mathbb{C}[L + \lambda_i] \quad \text{(linearly)}. \tag{3.23}$$

Then we have

$$V_{L^\circ} = \coprod_{i \in L^\circ/L} V(i). \tag{3.24}$$

We shall next define the untwisted vertex operators $Y(v, z)$ for $v \in V_{L^\circ}$. For $\alpha \in \mathbf{h}$ and $n \in \mathbb{Z}$ we write $\alpha(n)$ for the operator $\alpha \otimes t^n$ and set

$$\alpha(z) = \sum_{n \in \mathbb{Z}} \alpha(n) z^{-n-1}. \tag{3.25}$$

As before we use a normal order notation $\,{}^\circ_\circ \cdot {}^\circ_\circ\,$ to signify that the enclosed expression is to be reordered if necessary so that all the operators $\alpha(n)$ $(\alpha \in \mathbf{h},\ n < 0)$, $a \in \hat{L}^\circ$ are to be placed to the left of all the operators $\alpha(n)$, z^α $(\alpha \in \mathbf{h},\ n \geq 0)$ before the expression is evaluated. For $a \in \hat{L}^\circ$, set

$$Y(\iota(a), z) = Y(a, z) = {}^\circ_\circ e^{\int (\bar{a}(z) - \bar{a}(0) z^{-1})} a z^{\bar{a}} {}^\circ_\circ, \tag{3.26}$$

using an obvious formal integration notation. Let

$$a \in \hat{L},\ \alpha_1, ..., \alpha_k \in \mathbf{h},\ n_1, ..., n_k \in \mathbb{Z}\ \ (n_i > 0)$$

and set

$$\begin{aligned} v &= \alpha_1(-n_1) \cdots \alpha_k(-n_k) \otimes \iota(a) \\ &= \alpha_1(-n_1) \cdots \alpha_k(-n_k) \cdot \iota(a) \in V_{L^\circ}. \end{aligned} \tag{3.27}$$

We define

$$Y(v,z) = {\circ \atop \circ} \left(\frac{1}{(n_1-1)!} \left(\frac{d}{dz} \right)^{n_1-1} \alpha_1(z) \right) \cdots \left(\frac{1}{(n_k-1)!} \left(\frac{d}{dz} \right)^{n_k-1} \alpha_k(z) \right) Y(a,z) {\circ \atop \circ}. \tag{3.28}$$

This gives us a well-defined linear map

$$\begin{aligned} &V_{L^\circ} \to (\operatorname{End} V_{L^\circ})\{z\} \\ &v \mapsto Y(v,z) = \sum_{n \in \mathbb{C}} v_n z^{-n-1}, \quad (v_n \in \operatorname{End} V_{L^\circ}), \end{aligned} \tag{3.29}$$

where for any vector space W, we define $W\{z\}$ to be the vector space of W-valued formal series in z, with arbitrary complex powers of z allowed:

$$W\{z\} = \left\{ \sum_{n \in \mathbb{C}} w_n z^n \mid w_n \in W \right\}. \tag{3.30}$$

We call $Y(v,z)$ the *untwisted vertex operator associated with* v.

Theorem 3.3 *1.* $(V(0), Y, \mathbf{1}, \omega)$ *is a simple vertex operator algebra (see [B1],[FLM3]).*

2. $V(i)$ *for* $i \in L^\circ/L$ *is a complete list of inequivalent irreducible modules for* $V(0)$ *(see [D1], [FLM3]).*

3. Any $V(0)$*-module is completely reducible, that is,* $V(0)$ *is rational (see [Gu]).*

Note that from this theorem, V_L is holomorphic if and only if L is self dual: $L = L^\circ$. So a holomorphic vertex operator algebra is also called a self dual vertex operator algebra [Go].

Vertex operator algebras associated with the highest weight representations of affine Lie algebras. Let $\mathbf{g}$ be a simple Lie algebra over $\mathbb{C}$, $\mathbf{h}$ its Cartan subalgebra and Δ the corresponding root system. We fix a set of positive root Δ_+ and a nondegenerate symmetric invariant bilinear form $(\cdot,\cdot)$ of $\mathbf{g}$ such that the square length of a long root is 2. The affine Lie algebra $\hat{\mathbf{g}}$ is defined as

$$\hat{\mathbf{g}} = g \otimes \mathbb{C}[t,t^{-1}] \oplus \mathbb{C}c$$

with Lie algebra structure

$$[x \otimes t^m, y \otimes t^n] = [x,y] \otimes t^{m+n} + (x,y)m\delta_{m+n,0}c$$

for $x, y \in \mathbf{h}$ and $m, n \in \mathbb{Z}$ where c is an center element. For convenience we shall write $x(n)$ for $x \times t^n$. Then $\hat{\mathbf{g}}_{\pm} = \hat{\mathbf{g}} \otimes t^{\pm}\mathbb{C}[t^{\pm}]$ and $\mathbf{g}$ (identified with with $\mathbf{g} \otimes 1$) are subalgebras.

Let V be a $\mathbf{g}$-module which is extended to a $\hat{\mathbf{g}}_+ \oplus \mathbf{g} \oplus \mathbb{C}c$-module by letting $\hat{\mathbf{g}}_+$ act as 0 and c act as a fixed scalar $k \in \mathbb{C}$. Let $\hat{V}_k = U(\hat{\mathbf{g}}) \otimes_{U(\hat{\mathbf{g}}_+ \oplus \mathbf{g} \oplus \mathbb{C}c)} V$ be the induced $\hat{\mathbf{g}}$-module of level k. If $V = V(\lambda)$ is an irreducible highest weight module for $\mathbf{g}$ with highest weight $\lambda \in \mathbf{h}^*$ we denote the corresponding $\hat{V}_k$ by $\hat{V}_{k,\lambda}$. Then $\hat{V}_{k,\lambda}$ is a Verma module. Note that $\hat{V}_k = \hat{V}_{k,0}$ is linearly isomorphic to $U(\hat{\mathbf{g}}_-)$ as vector spaces. Our goal next is to define a vertex operator algebra structure on $\hat{V}_k$ if k is not equal to the dual Coxeter number and also define an action of $\hat{V}_k$ on $\hat{V}_{k,\lambda}$.

First for $u(-1) \in \hat{V}_k$ we define

$$Y(u(-1),z) = u(z) \sum_{n\in\mathbb{Z}} u(n)z^{-n-1}. \tag{3.31}$$

Then using the $L(-1)$-derivation property (2.15) to define

$$Y(u(-n-1),z) = \frac{1}{n!}\frac{d^n}{dz^n}Y(u(-1),z). \tag{3.32}$$

In general if $Y(v,z)$ has been defined, we use (2.38) define $Y(u(-n)v,z)$ for $u \in \mathbf{g}$ and $0 < n \in \mathbb{Z}$ as

$$Y(u(-n)v,z_2) = \mathrm{Res}_{z_1}\left\{(z_1-z_2)^{-n}Y(u,z_1)Y(v,z_2) - (-z_2+z_1)^{-n}Y(v,z_2)Y(u,z_1)\right\}. \tag{3.33}$$

Then we get a linear map Y from $\hat{V}_k$ to $(\mathrm{End}\hat{V}_k)[[z,z^{-1}]]$. Set $\mathbf{1} = 1 \times 1$ and $\omega = \frac{1}{2(k+h^\vee)}\sum_i v_i(-1)^2$ where $h^\vee$ is the dual Coxeter number and $\{v_i\}$ is an orthonomal basis of $\mathbf{g}$ with respect the form $(\cdot,\cdot)$. The following theorem can be found in [FZ] (also see [L1]):

Theorem 3.4 *(1) If $k \neq -h^\vee$ then $(\hat{V}_k, Y, \mathbf{1}, \omega)$ is a vertex operator algebra.*

(2) Define an action of $\hat{V}_k$ on $\hat{V}_{k,\lambda}$ by the the same formulas. Then $(\hat{V}_{k,\lambda}, Y)$ is a module for $\hat{V}_k$.

Now we turn our attention to the irreducible quotients. It is well-known that $\hat{V}_{k,\lambda}$ has a unique maximal submodule $I(k,\lambda)$ whose intersection with $V(\lambda)$ (which can be identified with $1 \otimes V(\lambda)$) is 0. Let $L(k,\lambda)$ be the corresponding irreducible quotient. Note that if $k \neq 0$ and $k \neq h^\vee$ $I(k,0)$ is an ideal and $L(k,0)$ is a quotient vertex operator algebra. The following theorem can be also found in [FZ] (see [DL3] for a different approach):

Theorem 3.5 *If k is a positive integer, $L(k,0)$ is rational vertex operator algebra whose irreducible modules are $\{L(k,\lambda)|(\lambda,\theta) \leq k\}$ where θ is the longest positive root in Δ.*

Vertex operator algebras associated with Virasoro algebras. Recall that the Virasoro algebra Vir has a basis $\{L_n|n \in \mathbb{Z}\} \cup \{c\}$ with relation:

$$[L_m, L_n] = (m-n)L_{m+n} + \frac{m-m^3}{12}\delta_{m+n,0}c$$

and c is in the center. Define two subalgebras

$$\mathrm{Vir}^{\geq 0} = \oplus_{n=0}^{\infty}\mathbb{C}L_n, \quad \mathrm{Vir}^{<0} = \oplus_{n=1}^{\infty}\mathbb{C}L_{-n}.$$

Given a pair of complex numbers (k,h), the Verma module $V(k,h)$ is an induced module

$$V(k,h) = U(\mathrm{Vir}) \otimes_{U(\mathrm{Vir}^{\geq 0})} \mathbb{C}_{k,h} \simeq U(\mathrm{Vir}^{<0})$$

where $\mathbb{C}_{k,h} = \mathbb{C}$ is a module for $\mathrm{Vir}^{\geq 0}$ such that $L_n 1 = 0$ for $n > 0$ and $L_0 1 = h,\ c1 = k$.

As in the case of affine algebra, we expect $V(k,0)$ is a vertex operator algebra. But this is not quite true. The relations $Y(L_{-1}\mathbf{1}, z) = \frac{d}{dz}Y(\mathbf{1}, z) = 0$ and $\lim_{z\mapsto 0} Y(v,z)\mathbf{1} = v$ force $L_{-1}\mathbf{1} = 0$. So it is natural to consider the quotient module $V_k = V(k,0)/U(\mathrm{Vir})L_{-1}1$ instead of $V(k,0)$ where $1 = 1 \otimes 1$. Set $\mathbf{1} = 1 + U(\mathrm{Vir})L_{-1}1$. Then V_k has a basis

$$\{L{-}n_1 \cdots L_{-n_s}\mathbf{1}|n_i \in \mathbb{N}, n_1 \geq n_2 \geq \cdots \geq n_2 \geq 2, s \geq 0\}.$$

Define operator

$$Y(L_{-2}\mathbf{1},z)=L(z)=\sum_{n\in\mathbb{Z}}L_nz^{-n-2}. \tag{3.34}$$

Again using the $L(-1)$-derivation property (2.15) to define

$$Y(L_{-n-2}\mathbf{1},z)=\frac{1}{n!}\frac{d^n}{dz^n}L(z). \tag{3.35}$$

and use (2.38)

$$Y(L(-n)v,z_2)=\mathrm{Res}_{z_1}\left\{(z_1-z_2)^{-n+1}L(z_1)Y(v,z_2)-(-z_2+z_1)^{-n+1}Y(v,z_2)L(z_1)\right\} \tag{3.36}$$

if $Y(v,z)$ has been defined. This yields a linear map Y from V_k to $(\mathrm{End}V_k))[[z,z^{-1}]]$. Set $\omega=L_{-2}\mathbf{1}$.

Theorem 3.6 *(1)* $(V_k,Y,\mathbf{1},\omega)$ *is vertex operator algebra.*

(2) Define an action of V_k *on* $V(k,h)$ *by the the same formulas. Then* $(V(k,h),Y)$ *is a module for* V_k*.*

This theorem was proved in [FZ]. Also see [H1]. Now it is easy corollary of the general theory on local systems [L1].

Now we denote by $W(k,h)$ the irreducible quotient of $V(k,h)$ module the unique maximal submodule whose intersection with $C_{k,h}$ is 0. Then as before $W(k,0)$ is a vertex operator algebra. Next we shall discuss the rationality of $W(k,0)$. This is related to the discrete series of unitary representations of the Virasoro algebra ([FQS] and [GKO]).

The work in [FQS] and [GKO] gives a complete classification of unitary highest weight representations of the Virasoro algebra. The highest weight representation $W(k,h)$ is unitary if and only if either (k,h) satisfies $k\geq 1$ and $h\geq 0$, or else (k,h) is among the following list:

$$k=c_m=1-\frac{6}{(m+2)(m+3)}\quad(m=0,1,2\cdots),$$

$$h=h^m_{r,s}=\frac{[(m+3)r-(m+2)s]^2-1}{4(m+2)(m+3)}\quad(r,s\in\mathbb{N},1\leq s\leq r\leq m+1).$$

The unitary representations $L(c_m,h^m_{r,s})$ for $(c_m,h^m_{r,s})$ in the discrete series as above are called the discrete series of the Virasoro algebra. If $m=0$, the only unitary representation is the trivial representation $L(0,0)$. If $m=1$, there are three unitary representations $L(\frac{1}{2},0)$, $L(\frac{1}{2},\frac{1}{2})$ and $L(\frac{1}{2},\frac{1}{16})$.

Theorem 3.7 *$W(k,0)$ is a rational vertex operator algebra if and only if $k=c_m$. In this case all irreducible modules are given by $L(c_m,h^m_{r,s})$.*

The rationality of $L(c_m,0)$ was proved and all its irreducible modules were found in [DMZ]. This theorem was proved completely later in [W].

4 Twisted modules and associative algebras

In [Z], Zhu introduced an associative algebra $A(V)$ associated to a VOA V which is extremely useful in study the representation theory of V. These are analogues for g-twisted modules in [DLM1]. We shall review these results form [DLM1].

Fix a VOA V and an automorphism g of finite r. For $u,v\in V$ with u homogeneous, define a product $u*v$ as follows:

$$u*v=\mathrm{Res}_z(Y(u,z)\frac{(z+1)^{\mathrm{wt}\,u}}{z}v)=\sum_{i=0}\infty\binom{\mathrm{wt}\,u}{i}u_{i-1}v. \tag{4.1}$$

Then extends (4.1) to a linear product $*$ on V.

Recall from (2.24) that V^j is the eigenspace of g with eigenvalue $e^{2\pi ij/r}$ where $0\le j<r$. Define a subspace $O_g(V)$ of V to be the linear span of all elements $u\circ_g v$ of the following type if u is homogeneous and $u\in V^j$ and $v\in V$,

$$u\circ_g v=\begin{cases}\langle\mathrm{Res}_z(Y(u,z)\frac{(z+1)^{\mathrm{wt}\,u}}{z^2}v) & \langle\text{if } j=0\\ \langle\mathrm{Res}_z(Y(u,z)\frac{(z+1)^{\mathrm{wt}\,u+j/r-1}}{z}v) & \langle\text{if } j>0.\end{cases} \tag{4.2}$$

Note that if $0<j<r$, $u\circ_g\mathbf{1}=u$. Thus we have

Lemma 4.1 *If $V^{\langle g\rangle}$ is the sub VOA of g-invariants of V then $V=V^{\langle g\rangle}+O_g(V)$.*

Let (M,Y_g) be a weak g-twisted V-module. Consider for each $\lambda\in\mathbb{C}$ the subspace $M(\lambda)=\sum_{i\in\mathbb{Z}}M_{\lambda+\frac{i}{K}}$ which, in fact, is a g-twisted submodule of M by (2.27). Then we have $M = \oplus_{\lambda \in \mathbb{C}/\frac{\mathbb{Z}}{K}} M(\lambda)$. Thus it is enough to study the twisted module of type:

$$M = \coprod_{n \in \frac{1}{K}\mathbb{Z}, n \geq 0} M_{c+n}$$

where $c \in \mathbb{C}$ is a fixed number and $M_c \neq 0$. We will call M_c the "top level" of M. For homogeneous $u \in V$, the component operator $u_{\mathrm{wt}\,u-1}$ preserves each homogeneous subspace of M and in particular acts on the top level M_c of M. Let $o_g(u)$ be the restriction of $u_{\mathrm{wt}\,u-1}$ to M_c, so that we have a linear map

$$\begin{aligned} V &\to \mathrm{End}(M_c) \\ u &\mapsto o_g(u). \end{aligned} \tag{4.3}$$

Note that if $u \in V^{j/r}$ with $0 < j < r$, $o_g(u) = 0$ from (2.27). Set $A_g(V) = V/O_g(V)$. Then we have [DLM1]:

Theorem 4.2 *(i) $A_g(V)$ is an associative algebra with multiplication $*$ and the centralizer $C(g)$ of g in* Aut(V) *induces a group of algebra automorphisms of $A_g(V)$.*

(ii) $u \mapsto o_g(u)$ gives a representation of $A_g(V)$ on M_c. Moreover, if any weak g-twisted module is completely reducible, $A_g(V)$ is semisimple.

(iii) $M \to M_c$ gives a bijection between the set of equivalence classes of simple weak g-twisted V-modules and the set of equivalence classes of simple $A_g(V)$-modules.

The associative algebra $A(V) = A_1(V)$ was discovered in [Z] and the theorem above was also established in this case. It was proved in [FZ] that for the vertex operator algebra $\hat{V}_k$, the associative algebra $A(\hat{V}_k)$ is canonically isomorphic to $U(\mathbf{g})$ and $A(L(k,0))$ is isomorphic to $U(\mathbf{g})/\langle e_\theta^k \rangle$ where $0 \neq e_\theta \in \mathbf{g}_\theta$ and $\langle e_\theta \rangle$ is the two-sided ideal generated by e_θ^{k+1}. Moreover, $A(V_k)$ is isomorphic to $\mathbb{C}[x]$. The reader can verify that $A(M(1))$ is isomorphic to $\mathbb{C}[x_1, ..., x_d]$.

5 Some remarks

Finally we want to comment briefly on orbifold theory and on the generalizations of vertex operator algebras.

It is believed that the module category for a rational VOA is equivalent to the module category of a Hopf algebra associated with the vertex operator algebra. In the case of VOAs associated with the level k highest weight unitary representations for affine Lie algebras $\hat{\mathbf{g}}$, the Hopf algebras are essentially the quantum group $U_q(\mathbf{g})$ where $q = e^{2\pi i/h+k}$ where h is the dual Coxeter number. If V is holomorphic the corresponding Hopf algebra is $\mathbb{C}$. It is natural next to determine the Hopf algebra for V^G where V is a holomorphic VOA, G is a finite automorphism group and V^G is the G-invariants which is a vertex operator subalgebra of V. This is the so-called orbifold theory in the physical literature. In fact, the moonshine module is an $\mathbb{Z}_2$-orbifold theory constructed from the vertex operator algebra associated with Leech lattice and an order 2 automorphism induced from -1 isometry of the Leech lattice (a $\mathbb{Z}_p$-orbifold construction of the moonshine module have been studied in [DM1] and [Mon]).

The main new feature of the orbifold theory is the introduction of twisted modules because any g-twisted module is an ordinary module for V^G for $g \in G$. It is conjectured that for holomorphic vertex operator algebra V and finite automorphism group G, the corresponding Hopf algebra is the quasi quantum double $D^c(G)$ where $c \in H^3(G, \mathbb{C})$ is uniquely determined by V (see [DVVV], [DPR], [DM2-DM5]). There have been a lot progress in this direction in [DM4-DM5] for nilpotent group G. The results concerning the modular invariance of trace functions (or correlation functions on the torus) developed in [Z] and [DM6] play important roles in the orbifold theory. (These results also explain mysterious relations among affine Lie algebras, Virasoro algebra, monster group and the modular group.)

In [DL1-DL3], the theory of vertex operator algebras has been generalized in a systematic way at three successively more general levels, all of which incorporate one-dimensional braid group representations intrinsically into the algebraic structure: First, the notion of "generalized vertex operator algebra" incorporates such structure as Z-algebras, parafermion algebras, and the vertex operator superalgebras. Next, what they term "generalized vertex algebras" further encompass the algebras of vertex operators associated with rational lattices. Finally, the most general of three notions, that of "abelian intertwining algebras," also intertwining operators for certain classes of vertex operator algebras. See [L4] and [DLM2] for more examples of abelian intertwining algebras related to the simple currents. The notion of generalized vertex algebra has been also independently introduced and studied in [FFR] with different motivations, examples (involving spinor constructions) and axiomatic approach from ours, and also see [Mos]. See also [LZ] and [H2] for the notion of topological vertex algebra.

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