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Introduction to vertex operator algebras II

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1 Introduction

This is the second of three lectures on introduction to vertex operator algebras. In this lecture, we shall continue Professor Dong's lecture to present more fundamental properties of vertex operator algebras.

¿From the mathematical point of view, a vertex operator algebra formally resembles a Lie algebra because the Jacobi identity is used as one of the main axioms. For the Lie algebra aspect of vertex operator algebras, the notion of contragredient module [FHL] and the notion of tensor product ([HL], [L4]) have been developed. On the other hand, from the physical point of view, a vertex operator algebra looks like a commutative associative algebra with identity because roughly speaking, a vertex operator algebra is a sort of quantization of the commutative associative algebra of observables in conformal field theory [BPZ]. For the associative algebra aspect of vertex operator algebras, it has been proved [FHL] that the product of any finitely many vertex operator algebras has a natural vertex operator algebra structure. In concrete examples, for a fixed level ℓ, one of the generalized Verma modules, called the vacuum representation for any affine Lie algebra $\tilde{\mathfrak{g}}$, has a natural vertex operator algebra structure ([FZ], [L2], [Lian]) and the universal enveloping algebra [FZ] of the vertex operator algebra is a certain completion of the universal enveloping algebra of $\tilde{\mathfrak{g}}$. This fact together with some facts on tensor products ([HL], [L4], [KL],..) strongly indicates that a vertex operator algebra is analogous to a quasi-Hopf algebra or a quantum group.

In this lecture, we shall discuss certain analogies between vertex operator algebras and classical algebras such as commutative associative algebras and Lie algebras. More specifically, we shall discuss commutativity and associativity and we give an analogue of the endomorphism ring for vertex operator algebras. All the materials presented here are taken from [DL], [FLM], [FHL] and [L2].

2 Commutativity and associativity

In this section, we shall present two versions of commutativity and associativity, which are given in terms of formal variables and in terms of analytic functions, respectively.

2.1 Commutativity and associativity in terms of formal variables

The version of commutativity and associativity in terms of formal variables was first presented in [DL]. Now it has been realized that the formal variable technique is powerful some time even though the Jacobi identity is such a beautiful identity.

Proposition 2.1. *Let V be a vertex operator algebra and let a and b be two elements of V. Then there is a nonnegative integer k such that*

$$(z_1 - z_2)^k Y(a, z_1)Y(b, z_2) = (z_1 - z_2)^k Y(b, z_2)Y(a, z_1). \tag{2.1}$$

Proof. In order to obtain (2.1), we need to force the term on the right hand side to vanish. For any nonnegative integer m, taking $\mathrm{Res}_{z_0} z_0^m$ of the Jacobi identity, we obtain:

$$\begin{aligned}
& (z_1 - z_2)^m [Y(a, z_1), Y(b, z_2)] \\
= \; & (z_1 - z_2)^m \left(Y(a, z_1)Y(b, z_2) - Y(b, z_2)Y(a, z_1) \right) \\
= \; & \mathrm{Res}_{z_0} z_2^{-1} \delta\left(\frac{z_1 - z_0}{z_2} \right) z_0^m Y(Y(a, z_0)b, z_2) \\
= \; & \sum_{i=0}^{\infty} \frac{1}{i!} \left(\left(\frac{\partial}{\partial z_2} \right)^i z_1^{-1} \delta\left(\frac{z_2}{z_1} \right) \right) Y(a_{m+i} b, z_2).
\end{aligned} \tag{2.2}$$

Let k be a nonnegative integer such that $a_n b = 0$ for $n \geq k$. Then setting $m = k$ we obtain the commutativity (2.1). □

Proposition 2.2. *Let V be a vertex operator algebra and let a and c be two elements of V. Then there is a nonnegative integer k such that for any $b \in V$ we have*

$$(z_0 + z_2)^k Y(Y(a, z_0)b, z_2)c = (z_0 + z_2)^k Y(a, z_0 + z_2)Y(b, z_2)c. \tag{2.3}$$

Proof. Similar to the case for commutativity, in order to obtain the associativity we force the second term on the left hand side to be zero. Taking Res_{z_1} of the Jacobi identity, we obtain the following *iterate formula:*

$$\begin{aligned} & Y(Y(a, z_0)b, z_2) \\ = \; & \mathrm{Res}_{z_1} \left(z_0^{-1}\delta\left(\frac{z_1 - z_2}{z_0}\right) Y(a, z_1)Y(b, z_2) - z_0^{-1}\delta\left(\frac{-z_2 + z_1}{z_0}\right) Y(b, z_2)Y(a, z_1) \right) \\ = \; & Y(a, z_0 + z_2)Y(b, z_2) - Y(b, z_2)(Y(a, z_0 + z_2) - Y(a, z_2 + z_0)). \end{aligned} \tag{2.4}$$

For any $c \in V$, let m be a positive integer such that $z^m Y(a, z)c$ involves only positive powers of z, so that

$$(z_0 + z_2)^m (Y(a, z_0 + z_2) - Y(a, z_2 + z_0))c = 0. \tag{2.5}$$

Then we obtain the associativity (2.3). □

The proof of the following theorem was taken from [L2].

Theorem 2.3. *The Jacobi identity is equivalent to the commutativity together with associativity.*

Proof. We only need to prove that the Jacobi identity follows from commutativity and associativity. Choosing a nonnegative integer k such that $a_m c = 0$ for all $m \geq k$, we get

$$\begin{aligned} & z_0^k z_1^k \left(z_0^{-1}\delta\left(\frac{z_1 - z_2}{z_0}\right) Y(a, z_1)Y(b, z_2)c - z_0^{-1}\delta\left(\frac{-z_2 + z_1}{z_0}\right) Y(b, z_2)Y(a, z_1)c \right) \\ = \; & z_0^{-1}\delta\left(\frac{z_1 - z_2}{z_0}\right) z_1^k (z_1 - z_2)^k Y(a, z_1)Y(b, z_2)c \end{aligned}$$

$$\begin{aligned}
&-z_0^{-1}\delta\left(\frac{-z_2+z_1}{z_0}\right) z_1^k(z_1-z_2)^k Y(b,z_2)Y(a,z_1)c \\
&= z_2^{-1}\delta\left(\frac{z_1-z_0}{z_2}\right)\left(z_1^k(z_1-z_2)^k Y(b,z_2)Y(a,z_1)c\right) \\
&= z_2^{-1}\delta\left(\frac{z_1-z_0}{z_2}\right)\left(z_0^k(z_0+z_2)^k Y(b,z_2)Y(a,z_2+z_0)c\right). \qquad (2.6)
\end{aligned}$$

Since $a_m c = 0$ for all $m \geq k$, $(z_0+z_2)^k Y(a, z_2+z_0)c$ involves only nonnegative powers of (z_2+z_0), so that

$$z_0^k(z_0+z_2)^k Y(b,z_2)Y(a,z_2+z_0)c = z_0^k(z_0+z_2)^k Y(b,z_2)Y(a,z_0+z_2)c. \qquad (2.7)$$

Therefore

$$\begin{aligned}
&z_0^k z_1^k\left(z_0^{-1}\delta\left(\frac{z_1-z_2}{z_0}\right)Y(a,z_1)Y(b,z_2)c - z_0^{-1}\delta\left(\frac{z_2-z_1}{-z_0}\right)Y(b,z_2)Y(a,z_1)c\right) \\
&= z_2^{-1}\delta\left(\frac{z_1-z_0}{z_2}\right)\left(z_0^k(z_0+z_2)^k Y(b,z_2)Y(a,z_0+z_2)c\right) \\
&= z_2^{-1}\delta\left(\frac{z_1-z_0}{z_2}\right)\left(z_0^k(z_0+z_2)^k Y(a,z_0+z_2)Y(b,z_2)c\right) \\
&= z_2^{-1}\delta\left(\frac{z_1-z_0}{z_2}\right)\left(z_0^k(z_0+z_2)^k Y(Y(a,z_0)b,z_2)c\right) \\
&= z_2^{-1}\delta\left(\frac{z_1-z_0}{z_2}\right)\left(z_0^k z_1^k Y(Y(a,z_0)b,z_2)c\right) \\
&= z_0^k z_1^k z_2^{-1}\delta\left(\frac{z_1-z_0}{z_2}\right)Y(Y(a,z_0)b,z_2)c. \qquad (2.8)
\end{aligned}$$

Then the Jacobi identity follows. □

Remark 2.4. Notice that the commutativity (2.1) is really a commutativity for "left multiplications." The exact analogue of the classical commutativity of product is the *skew-symmetry* [FHL]: $Y(a,z)b = e^{zL(-1)}Y(b,-z)a$ for any $a,b \in V$. Let A be a classical algebra with a right identity 1 and denote by ℓ_a the left multiplication by an element a. Suppose that $\ell_a\ell_b = \ell_b\ell_a$ for any $a,b \in A$. Then $a(bc) = b(ac)$ for any $a,b,c \in A$. Setting $c = 1$, we obtain the commutativity $ab = ba$. Furthermore, we obtain the associativity:

$$a(cb) = a(bc) = b(ac) = (ac)b \quad \text{for any } a,b,c \in A. \qquad (2.9)$$

Therefore, A is a commutative associative algebra. This classical fact suggests that the commutativity (2.1) together with the vacuum property implies the associativity (2.3).

Therefore, the commutativity implies the Jacobi identity. This has been proved at different levels in many references such as [FLM], [FHL], [G], [DL] and [L2]. The proof of the following Theorem was taken from [L2].

Theorem 2.5. *In the definition of vertex operator algebra, the Jacobi identity can be equivalently substituted by commutativity.*

Proof. Our proof, which consists of three steps, is exactly an analogue of the argument given in Remark 2.4.

(1) The skew-symmetry holds. Let k be a positive integer such that $b_m a = 0$ for all $m \geq k$ and that the commutativity (2.1) holds. Then

$$\begin{aligned} & (z_1 - z_2)^k Y(a, z_1) Y(b, z_2)\mathbf{1} \\ = \; & (z_1 - z_2)^k Y(b, z_2) Y(a, z_1)\mathbf{1} \\ = \; & (z_1 - z_2)^k Y(b, z_2) e^{z_1 L(-1)} a \\ = \; & (z_1 - z_2)^k e^{z_1 L(-1)} Y(b, z_2 - z_1) a. \end{aligned} \tag{2.10}$$

Since $(z_1 - z_2)^k Y(b, z_2 - z_1)a$ involves only nonnegative powers of $(z_2 - z_1)$, we may set $z_2 = 0$. Thus

$$z_1^k Y(a, z_1) b = z_1^k e^{z_1 L(-1)} Y(b, -z_1) a. \tag{2.11}$$

Multiplying both sides of (2.11) by z_1^{-k} we obtain $Y(a, z_1)b = e^{z_1 L(-1)} Y(b, -z_1)a$.

(2) The associativity (2.3) holds. For any $a, c \in V$, let k be a positive integer such that the commutativity (2.1) for (a, c) holds. Then for any $b \in V$, we have:

$$\begin{aligned} & (z_0 + z_2)^k Y(a, z_0 + z_2) Y(b, z_2) c \\ = \; & (z_0 + z_2)^k Y(a, z_0 + z_2) e^{z_2 L(-1)} Y(c, -z_2) b \\ = \; & e^{z_2 L(-1)} (z_0 + z_2)^k Y(a, z_0) Y(c, -z_2) b \\ = \; & e^{z_2 L(-1)} (z_0 + z_2)^k Y(c, -z_2) Y(a, z_0) b \\ = \; & (z_0 + z_2)^k Y(Y(a, z_0) b, z_2) c. \end{aligned} \tag{2.12}$$

It follows from Theorem 2.43that the Jacobi identity holds. □

The following Proposition 2.6, Corollary 2.7 and Proposition 2.8 are taken from [L2].

Proposition 2.6. *The Jacobi identity in the definition of vertex operator algebra V can be equivalently replaced by the skew symmetry and the associativity (2.3).*

Proof. For any $a, b, c \in V$, let k be a positive integer such that $z^k Y(b,z)c$ involves only positive powers of z and that the following associativities hold:

$$\begin{aligned}(z_0+z_2)^k Y(a, z_0+z_2)Y(b,z_2)c &= (z_0+z_2)^k Y(Y(a,z_0)b, z_2)c,\\ (-z_0+z_1)^k Y(b,-z_0+z_1)Y(a,z_1)c &= (-z_0+z_1)^k Y(Y(b,-z_0)a, z_1)c. \end{aligned} \tag{2.13}$$

Then

$$\begin{aligned}&z_1^k z_2^k \left(z_0^{-1}\delta\left(\frac{z_1-z_2}{z_0}\right) Y(a,z_1)Y(b,z_2)c - z_0^{-1}\delta\left(\frac{-z_2+z_1}{z_0}\right) Y(b,z_2)Y(a,z_1)c\right)\\ =\ & z_0^{-1}\delta\left(\frac{z_1-z_2}{z_0}\right)\left((z_0+z_2)^k z_2^k Y(Y(a,z_0)b,z_2)c\right)\\ &-z_0^{-1}\delta\left(\frac{-z_2+z_1}{z_0}\right)\left(z_1^k(-z_0+z_1)^k Y(Y(b,-z_0)a,z_1)c\right)\\ =\ & z_0^{-1}\delta\left(\frac{z_1-z_2}{z_0}\right)\left((z_0+z_2)^k z_2^k Y(Y(a,z_0)b,z_2)c\right)\\ &-z_0^{-1}\delta\left(\frac{-z_2+z_1}{z_0}\right)\left(z_1^k(-z_0+z_1)^k Y(e^{-z_0L(-1)}Y(a,z_0)b,z_1)c\right)\\ =\ & z_0^{-1}\delta\left(\frac{z_1-z_2}{z_0}\right)\left((z_0+z_2)^k z_2^k Y(Y(a,z_0)b,z_2)c\right)\\ &-z_0^{-1}\delta\left(\frac{-z_2+z_1}{z_0}\right)\left(z_1^k(-z_0+z_1)^k Y(Y(a,z_0)b,z_1-z_0)c\right). \end{aligned} \tag{2.14}$$

Since $z_2^k(z_0+z_2)^k Y(Y(a,z_0)b,z_2)c = (z_0+z_2)^k Y(a,z_0+z_2)(z_2^k Y(b,z_2)c)$ involves only positive powers of z_2, by (2.13) we have:

$$\begin{aligned}&z_0^{-1}\delta\left(\frac{z_1-z_2}{z_0}\right)\left((z_0+z_2)^k z_2^k Y(Y(a,z_0)b,z_2)c\right)\\ =\ & z_0^{-1}\delta\left(\frac{z_1-z_2}{z_0}\right)\left(z_1^k(z_1-z_0)^k Y(Y(a,z_0)b,z_1-z_0)c\right). \end{aligned} \tag{2.15}$$

Thus

$$z_1^k z_2^k \left(z_0^{-1}\delta\left(\frac{z_1-z_2}{z_0}\right) Y(a,z_1)Y(b,z_2)c - z_0^{-1}\delta\left(\frac{-z_2+z_1}{z_0}\right) Y(b,z_2)Y(a,z_1)c\right)$$

$$\begin{aligned}
&= z_0^{-1}\delta\left(\frac{z_1-z_2}{z_0}\right)\left(z_1^k(z_1-z_0)^k Y(Y(a,z_0)b,z_1-z_0)c\right)\\
&\quad -z_0^{-1}\delta\left(\frac{-z_2+z_1}{z_0}\right)\left(z_1^k(-z_0+z_1)^k Y(Y(a,z_0)b,z_1-z_0)c\right)\\
&= z_2^{-1}\delta\left(\frac{z_1-z_0}{z_2}\right)\left(z_1^k(z_1-z_0)^k Y(Y(a,z_0)b,z_1-z_0)c\right)\\
&= z_1^k z_2^k z_2^{-1}\delta\left(\frac{z_1-z_0}{z_2}\right) Y(Y(a,z_0)b,z_2)c. \qquad (2.16)
\end{aligned}$$

Multiplying both sides by $z_1^{-k}z_2^{-k}$, we obtain the Jacobi identity. □

In [B], Borcherds first defined the notion of vertex algebra with a set of axioms consisting of the vacuum property, the skew-symmetry and the iterate formula (2.4). Without assuming the existence of a Virasoro element in the notion of vertex algebra, one can define the operator D is defined by $Da = a_{-2}\mathbf{1}$ for $a \in V$. Therefore, we have:

Corollary 2.7. *Borcherds' definition [B] and FLM's definition for a vertex algebra are equivalent.*

Proposition 2.8. *Let V be a vertex algebra. Then the Jacobi identity of a V-module can be equivalently substituted by the associativity (2.3).*

2.2 Commutativity and associativity in terms of analytic functions

In the last subsection, we consider a vertex operator $Y(a,z)$ as a generating function of a sequence of operators, where z is a formal variable. In this subsection we shall consider a vertex operator $Y(a,z)$ as an operator valued functional, where z will be considered as a nonzero complex number.

Let V be a vertex operator algebra and define $V' = \oplus_{n\in\mathbf{Z}} V_{(n)}^*$ to be the restricted dual of V [FHL]. For any $a,b \in V$, by the definition of a vertex operator algebra, $Y(a,z)b$ involves only finitely many negative powers of z. Then it follows from the definition of V' that for any $f \in V'$, $\langle f, Y(a,z)b\rangle$ is a Laurent polynomial in z. Therefore, we may consider z as a nonzero complex number so that $\langle f, Y(a,z)b\rangle$ is a rational function of z.

Next, we consider the following *n-point functions*:

$$\langle f, Y(a_1, z_1)Y(a_2, z_2)\cdots Y(a_n, z_2)c\rangle, \tag{2.17}$$

The following Proposition 2.9 was taken from [FLM] and [FHL] with a slightly different proof.

Proposition 2.9 *Let V be a vertex operator algebra and let $a, b, c \in V, f \in V'$. Then (I) (Rationality)*

$$\langle f, Y(a, z_1)Y(b, z_2)c\rangle, \tag{2.18}$$

as a formal series converges in the domain of $|z_1| > |z_2| > 0$ to a rational function $g(z_1, z_2) = h(z_1, z_2)z_1^m z_2^n(z_1 - z_2)^k$, where $h(z_1, z_2)$ is a polynomial in z_1 and z_2, and m, n, k are integers and k only depends on a and b.

(II) (Commutativity)

$$\langle f, Y(b, z_2)Y(a, z_1)c\rangle, \tag{2.19}$$

as a formal series converges in the domain of $|z_2| > |z_1| > 0$ to the same rational function $g(z_1, z_2)$ as that in (I).

(III) (Associativity)

$$\langle f, Y(Y(a, z_0)b, z_2)c\rangle, \tag{2.20}$$

as a formal series converges in the domain of $|z_2 + z_0| > |z_2| > |z_0| > 0$ to the rational function $g(z_2 + z_0, z_2)$, where $g(z_1, z_2)$ is the same as that in (I) and (II).

Proof. For any $a, b \in V$, it follows from the first version of commutativity that there is a nonnegative integer k such that the commutativity (2.1) holds. It follows from the definition of V' and the meromorphic condition on vertex operators that the matrix-coefficient $\langle f, (z_1 - z_2)^k Y(a, z_1)Y(b, z_2)c\rangle$ involves only finitely many negative powers of z_2 and finitely many positive powers of z_1. Similarly, $\langle f, Y(b, z_2)Y(a, z_1)c\rangle$ involves only finitely many negative powers of z_1 and finitely many positive powers of z_2. Therefore the common formal series

$$\langle f, (z_1 - z_2)^k Y(a, z_1) Y(b, z_2) c \rangle = \langle f, (z_1 - z_2)^k Y(b, z_2) Y(a, z_1) c \rangle \tag{2.21}$$

involves only finitely many negative and positive powers of both z_1 and z_2. Consequently, it gives a rational function in the form of $z_1^m z_2^n h(z_1, z_2)$, where $h(z_1, z_2)$ is a polynomial in z_1 and z_2. Then both (I) and (II) have been proved. Similarly one can prove (III). □

Proposition [DL] *The two versions of commutativity and associativity are equivalent.*

3 An analogue of the endomorphism ring for VOA

In this section, we shall introduce what we call "local systems of vertex operators" for any vector space M and we prove that any local system has a natural vertex algebra structure with M as a module. Furthermore, we prove that for a fixed vertex algebra V, giving a V-module M is equivalent to giving a vertex algebra homomorphism from V to some local system of vertex operators on M. This whole section was taken from [L2]. An analogue of the homomorphism module for vertex operator algebras has been also developed in [L2] and the notion of local systems of vertex operators and applications have been generalized to the notion of local systems of twisted vertex operators in [L3].

Definition 3.1. Let M be any vector space. A *weak vertex operator* on M is a formal series $a(z) = \sum_{n \in \mathbf{Z}} a_n z^{-n-1} \in (\mathrm{End}\, M)[[z, z^{-1}]]$ such that

$$a(z)u \in M((z)) \quad \text{for any } u \in M. \tag{3.1}$$

That is, $a_n u = 0$ for n sufficiently large. Let (M, d) be a pair consisting of a vector space M and an endomorphism d of M. A *weak vertex operator on* (M, d) is a weak vertex operator $a(z)$ on M such that

$$[d, a(z)] = a'(z) \left(= \frac{d}{dz} a(z) \right). \tag{3.2}$$

Denote by $F(M)$ (resp. $F(M,d)$) the space of all weak vertex operators on M (resp. (M,d)).

By definition, it is clear that if $a(z)$ is a weak vertex operator on M (resp. (M,d)), the formal derivative $a'(z)$ is also a weak vertex operator on M (resp. (M,d)). Then we have an endomorphism $D = \frac{d}{dz}$ for both $F(M)$ and $F(M,d)$.

Definition 3.2. Let M be a restricted Vir-module of central charge ℓ. A weak vertex operator $a(z)$ on $(M, L(-1))$ is said to be of *weight* $h \in \mathbf{C}$ if it satisfies the following condition:

$$[L(0), a(z)] = ha(z) + za'(z). \tag{3.3}$$

Denote by $F(M, L(-1))_{(h)}$ the space of weak vertex operators on $(M, L(-1))$ of weight h and set

$$F^{o}(M, L(-1)) = \oplus_{h\in\mathbf{C}} F(M, L(-1))_{(h)}. \tag{3.4}$$

Remark 3.3. For any vector space M, the identity operator $I(z) = \mathrm{id}_M$ is a weak vertex operator on M. Let M be a restricted Vir-module. Then $I(z) = \mathrm{id}_M$ is a weak vertex operator on $(M, L(-1))$ of weight zero and $L(z) = \sum_{n\in\mathbf{Z}} L(n)z^{-n-2}$ is a weak vertex operator on $(M, L(-1))$ of weight two. If $a(z)$ is a weak vertex operator on $(M, L(-1))$ of weight h, then $a'(z) = \frac{d}{dz}a(z)$ is a weak vertex operator of weight $h+1$.

Lemma 3.4. *Let M be a vector space and let $a(z)$ and $b(z)$ be weak vertex operators on M. For any integer n, set*

$$a(z)_n b(z) = \mathrm{Res}_{z_1}\left((z_1 - z)^n a(z_1)b(z) - (-z + z_1)^n b(z)a(z_1)\right). \tag{3.5}$$

Then $a(z)_n b(z)$ is a weak vertex operator.

Proof. For any $u \in M$, by definition we have

$$(a(z)_n b(z))u$$

$$= \mathrm{Res}_{z_1}\left((z_1-z)^n a(z_1)b(z)u - (-z+z_1)^n b(z)a(z_1)u\right)$$
$$= \sum_{k=0}^{\infty}\binom{n}{k}\left((-z)^k a_{n-k}b(z)u - (-z)^{n-k}b(z)a_k u\right). \tag{3.6}$$

It is easy to see that $(a(z)_n b(z))u \in M((z))$. Therefore, $a(z)_n b(z)$ is a weak vertex operator on M. $\square$

Definition 3.5. Let M be a vector space and let $a(z)$ and $b(z)$ be weak vertex operators on M. Then we define

$$Y(a(z),z_0)b(z)$$
$$= \ :\sum_{n\in\mathbf{Z}} a(z)_n b(z) z_0^{-n-1}$$
$$= \mathrm{Res}_{z_1}\left(z_0^{-1}\delta\left(\frac{z_1-z}{z_0}\right)a(z_1)b(z) - z_0^{-1}\delta\left(\frac{z-z_1}{-z_0}\right)b(z)a(z_1)\right). \tag{3.7}$$

Extending the definition bilinearly, we obtain a linear map

$$Y(\cdot,z_0): \quad F(M) \to (\mathrm{End}F(M))[[z_0,z_0^{-1}]];$$
$$a(z) \mapsto Y(a(z),z_0). \tag{3.8}$$

Lemma 3.6. *For any $a(z)\in F(M)$, we have*

$$Y(I(z),z_0)a(z) = a(z); \tag{3.9}$$
$$Y(a(z),z_0)I(z) = e^{z_0\frac{\partial}{\partial z}}a(z)\ (= a(z+z_0)). \tag{3.10}$$

Proof. By definition, we have:

$$Y(I(z),z_0)a(z)$$
$$= \mathrm{Res}_{z_1}\left(z_0^{-1}\delta\left(\frac{z_1-z}{z_0}\right)a(z) - z_0^{-1}\delta\left(\frac{-z+z_1}{z_0}\right)a(z)\right)$$
$$= \mathrm{Res}_{z_1} z^{-1}\delta\left(\frac{z_1-z_0}{z}\right)a(z)$$
$$= a(z)$$

and

$$Y(a(z),z_0)I(z)$$

$$
\begin{aligned}
&= \mathrm{Res}_{z_1}\left(z_0^{-1}\delta\left(\frac{z_1-z}{z_0}\right)a(z_1)I(z)-z_0^{-1}\delta\left(\frac{-z+z_1}{z_0}\right)I(z)a(z_1)\right)\\
&= \mathrm{Res}_{z_1}z^{-1}\delta\left(\frac{z_1-z_0}{z}\right)a(z_1)\\
&= \mathrm{Res}_{z_1}z_1^{-1}\delta\left(\frac{z+z_0}{z_1}\right)a(z_1)\\
&= \mathrm{Res}_{z_1}z_1^{-1}\delta\left(\frac{z+z_0}{z_1}\right)a(z+z_0)\\
&= a(z+z_0)\\
&= e^{z_0\frac{\partial}{\partial z}}a(z). \quad \square
\end{aligned}
\tag{3.11}
$$

Lemma 3.7. *Let $M \in obC$ and $a(z), b(z) \in F(M)$. Then we have*

$$
\frac{\partial}{\partial z_0}Y(a(z),z_0)b(z) = Y(D(a(z)),z_0)b(z) = [D, Y(a(z),z_0)]b(z). \tag{3.12}
$$

Proof. By definition, we have

$$
\begin{aligned}
&\frac{\partial}{\partial z_0}Y(a(z),z_0)b(z)\\
&= \mathrm{Res}_{z_1}\left(\frac{\partial}{\partial z_0}\left(z_0^{-1}\delta\left(\frac{z_1-z}{z_0}\right)\right)a(z_1)b(z)-\frac{\partial}{\partial z_0}\left(z_0^{-1}\delta\left(\frac{-z+z_1}{z_0}\right)\right)b(z)a(z_1)\right)\\
&= -\mathrm{Res}_{z_1}\left(\frac{\partial}{\partial z_1}z_0^{-1}\delta\left(\frac{z_1-z}{z_0}\right)\right)a(z_1)b(z)\\
&\quad +\mathrm{Res}_{z_1}\left(\frac{\partial}{\partial z_1}z_0^{-1}\delta\left(\frac{-z+z_1}{z_0}\right)\right)b(z)a(z_1) \quad \text{(by Lemma 2.1)}\\
&= \mathrm{Res}_{z_1}\left(z_0^{-1}\delta\left(\frac{z_1-z}{z_0}\right)a'(z_1)b(z)-z_0^{-1}\delta\left(\frac{-z+z_1}{z_0}\right)b(z)a'(z_1)\right)\\
&= Y(a'(z),z_0)b(z)
\end{aligned}
\tag{3.13}
$$

and

$$
\begin{aligned}
&[D, Y(a(z),z_0)]b(z)\\
&= D(Y(a(z),z_0)b(z)) - Y(a(z),z_0)Db(z)\\
&= \frac{\partial}{\partial z}(Y(a(z),z_0)b(z)) - Y(a(z),z_0)b'(z)\\
&= \mathrm{Res}_{z_1}\left(\left(\frac{\partial}{\partial z}z_0^{-1}\delta\left(\frac{z_1-z}{z_0}\right)\right)a(z_1)b(z)-\left(\frac{\partial}{\partial z}z_0^{-1}\delta\left(\frac{z-z_1}{-z_0}\right)\right)b(z)a(z_1)\right)
\end{aligned}
$$

$$
\begin{aligned}
&= \operatorname{Res}_{z_1}\left(\frac{\partial}{\partial z_0}\left(z_0^{-1}\delta\left(\frac{z_1-z}{z_0}\right)\right)a(z_1)b(z) - \frac{\partial}{\partial z_0}\left(z_0^{-1}\delta\left(\frac{z-z_1}{-z_0}\right)\right)b(z)a(z_1)\right)\\
&= \frac{\partial}{\partial z_0}Y(a(z),z_0)b(z)\\
&= Y(a'(z),z_0)b(z)\\
&= Y(D\cdot a(z),z_0)b(z). \quad \square \qquad (3.14)
\end{aligned}
$$

Lemma 3.8. *Let (M,d) be an object of C^o and let $a(z),b(z) \in F(M,d)$. Then $a(z)_n b(z) \in F(M,d)$. Furthermore, if M is a restricted Vir-module with central charge ℓ and $a(z),b(z)$ are weak vertex operators on $(M,L(-1))$ of weights α,β, respectively, then for any integer n, $a(z)_n b(z)$ is a weak vertex operator of weight $(\alpha+\beta-n-1)$ on $(M,L(-1))$.*

Proof. It is equivalent to prove the following:

$$
[L(-1),Y(a(z),z_0)b(z)] = \frac{\partial}{\partial z}(Y(a(z),z_0)b(z)); \qquad (3.15)
$$

$$
\begin{aligned}
&[L(0),Y(a(z),z_0)b(z)]\\
&= (\alpha+\beta)Y(a(z),z_0)b(z) + z_0\frac{\partial}{\partial z_0}(Y(a(z),z_0)b(z)) + z\frac{\partial}{\partial z}(Y(a(z),z_0)b(z)). \qquad (3.16)
\end{aligned}
$$

By definition, we have:

$$
\begin{aligned}
&\frac{\partial}{\partial z}(Y(a(z),z_0)b(z))\\
&= \frac{\partial}{\partial z}\operatorname{Res}_{z_1}\left(z_0^{-1}\delta\left(\frac{z_1-z}{z_0}\right)a(z_1)b(z) - z_0^{-1}\delta\left(\frac{z-z_1}{-z_0}\right)b(z)a(z_1)\right)\\
&= \operatorname{Res}_{z_1}\left(\left(\frac{\partial}{\partial z}z_0^{-1}\delta\left(\frac{z_1-z}{z_0}\right)\right)a(z_1)b(z) - \left(\frac{\partial}{\partial z}z_0^{-1}\delta\left(\frac{z-z_1}{-z_0}\right)\right)b(z)a(z_1)\right)\\
&\quad +\operatorname{Res}_{z_1}\left(\delta\left(\frac{z_1-z}{z_0}\right)a(z_1)b'(z) - z_0^{-1}\delta\left(\frac{z-z_1}{-z_0}\right)b'(z)a(z_1)\right)\\
&= -\operatorname{Res}_{z_1}\left(\left(\frac{\partial}{\partial z_1}z_0^{-1}\delta\left(\frac{z_1-z}{z_0}\right)\right)a(z_1)b(z) - \left(\frac{\partial}{\partial z_1}z_0^{-1}\delta\left(\frac{z_1-z}{z_0}\right)\right)b(z)a(z_1)\right)\\
&\quad +\operatorname{Res}_{z_1}\left(z_0^{-1}\delta\left(\frac{z_1-z}{z_0}\right)a(z_1)b'(z) - z_0^{-1}\delta\left(\frac{z_1-z}{z_0}\right)b'(z)a(z_1)\right)\\
&= \operatorname{Res}_{z_1}\left(z_0^{-1}\delta\left(\frac{z_1-z}{z_0}\right)a(z_1)b'(z) - z_0^{-1}\delta\left(\frac{z-z_1}{-z_0}\right)b'(z)a(z_1)\right)\\
&\quad -\operatorname{Res}_{z_1}\left(z_0^{-1}\delta\left(\frac{z_1-z}{z_0}\right)a'(z_1)b(z) - z_0^{-1}\delta\left(\frac{z-z_1}{-z_0}\right)b(z)a'(z_1)\right)
\end{aligned}
$$

$$= [L(-1), Y(a(z), z_0)b(z)]$$

and

$$\begin{aligned}
&[L(0), Y(a(z), z_0)b(z)] \\
= \;& \mathrm{Res}_{z_1}\left(z_0^{-1}\delta\left(\frac{z_1 - z}{z_0}\right)[L(0), a(z_1)b(z)] - z_0^{-1}\delta\left(\frac{z - z_1}{-z_0}\right)[L(0), b(z)a(z_1)]\right) \\
= \;& \mathrm{Res}_{z_1} z_0^{-1}\delta\left(\frac{z_1 - z}{z_0}\right)(a(z_1)[L(0), b(z)] + [L(0), a(z_1)]b(z)) \\
& -\mathrm{Res}_{z_1} z_0^{-1}\delta\left(\frac{z - z_1}{-z_0}\right)(b(z)[L(0), a(z_1)] + [L(0), b(z)]a(z_1)) \\
= \;& \mathrm{Res}_{z_1} z_0^{-1}\delta\left(\frac{z_1 - z}{z_0}\right)(\beta a(z_1)b(z) + z a(z_1)b'(z) + \alpha a(z_1)b(z) + z_1 a'(z_1)b(z)) \\
& -\mathrm{Res}_{z_1} z_0^{-1}\delta\left(\frac{z - z_1}{-z_0}\right)(\alpha b(z)a(z_1) + z_1 b(z)a'(z_1) + \beta b(z)a(z_1) + z b'(z)a(z_1)) \\
= \;& (\alpha + \beta)Y(a(z), z_0)b(z) \\
& +\mathrm{Res}_{z_1}\left(z_0^{-1}\delta\left(\frac{z_1 - z}{z_0}\right) z a(z_1)b'(z) - z_0^{-1}\delta\left(\frac{z - z_1}{-z_0}\right) z b'(z)a(z_1)\right) \\
& -\mathrm{Res}_{z_1}\left(\left(\frac{\partial}{\partial z_1} z_1 z_0^{-1}\delta\left(\frac{z_1 - z}{z_0}\right)\right) a(z_1)b(z) - \left(\frac{\partial}{\partial z_1} z_1 z_0^{-1}\delta\left(\frac{z - z_1}{-z_0}\right)\right) b(z)a(z_1)\right) \\
= \;& (\alpha + \beta)Y(a(z), z_0)b(z) \\
& +\mathrm{Res}_{z_1}\left(z_0^{-1}\delta\left(\frac{z_1 - z}{z_0}\right) z a(z_1)b'(z) - z_0^{-1}\delta\left(\frac{z - z_1}{-z_0}\right) z b'(z)a(z_1)\right) \\
& -\mathrm{Res}_{z_1}(z_0 + z)\left(\frac{\partial}{\partial z_1} z_0^{-1}\delta\left(\frac{z_1 - z}{z_0}\right)\right) a(z_1)b(z) \\
& +\mathrm{Res}_{z_1}(z_0 + z)\left(\frac{\partial}{\partial z_1} z_0^{-1}\delta\left(\frac{z - z_1}{-z_0}\right)\right) b(z)a(z_1) \\
= \;& (\alpha + \beta)Y(a(z), z_0)b(z) \\
& +\mathrm{Res}_{z_1}\left(z_0^{-1}\delta\left(\frac{z_1 - z}{z_0}\right) z a(z_1)b'(z) - z_0^{-1}\delta\left(\frac{z - z_1}{-z_0}\right) z b'(z)a(z_1)\right) \\
& +\mathrm{Res}_{z_1} z_0\left(\left(\frac{\partial}{\partial z_0} z_0^{-1}\delta\left(\frac{z_1 - z}{z_0}\right)\right) a(z_1)b(z) - \left(\frac{\partial}{\partial z_1} z_0^{-1}\delta\left(\frac{z - z_1}{-z_0}\right)\right) b(z)a(z_1)\right) \\
& +\mathrm{Res}_{z_1} z\left(\left(\frac{\partial}{\partial z} z_0^{-1}\delta\left(\frac{z_1 - z}{z_0}\right)\right) a(z_1)b(z) - \left(\frac{\partial}{\partial z_1} z_0^{-1}\delta\left(\frac{z - z_1}{-z_0}\right)\right) b(z)a(z_1)\right) \\
= \;& (\alpha + \beta)Y(a(z), z_0)b(z) + z\frac{\partial}{\partial z}(Y(a(z), z_0)b(z)) + z_0\frac{\partial}{\partial z_0}(Y(a(z), z_0)b(z)). \quad \square
\end{aligned}$$

The following definition is motivated by physicists' work for example [Go].

Definition 3.9. Two weak vertex operators $a(z_1)$ and $b(z_2)$ are said to be *mutually local* if there a positive integer n such that

$$(z_1 - z_2)^n a(z_1)b(z_2) = (z_1 - z_2)^n b(z_2)a(z_1). \tag{3.17}$$

A weak vertex operator is called a *vertex operator* if it is local with itself, a subspace A of $F(M)$ is said to be *local* if any two weak vertex operators in A are mutually local, and *a local system* of vertex operators on M is a maximal local subspace of $F(M)$.

Remark 3.10. Let V be a vertex algebra and let (M, Y_M) be a V-module. Then the image of V under the linear map $Y_M(\cdot, z)$ is a local subspace of $F(M)$.

Remark 3.11. Let M be a vector space and let $a(z)$ and $b(z)$ be homogeneous mutually local weak vertex operators on M. Let k be a positive integer satisfying (3.1). Then $a(z)_n b(z) = 0$ whenever $n \geq k$. Thus $Y(a(z), z_0)b(z)$ involves only finitely many negative powers of z_0. (This corresponds to the truncation condition (V1).)

Lemma 3.12. *If $a(z_1)$ is local with $b(z_2)$, then $a(z_1)$ is local with $b'(z_2)$.*

Proof. Let n be a positive integer such that (3.17) holds. Then

$$(z_1 - z_2)^{n+1} a(z_1)b(z_2) = (z_1 - z_2)^{n+1} b(z_2)a(z_1). \tag{3.18}$$

Differentiating (3.18) with respect to z_2, then using (3.17) we obtain

$$(z_1 - z_2)^{n+1} a(z_1)b'(z_2) = (z_1 - z_2)^{n+1} b'(z_2)a(z_1). \quad \square \tag{3.19}$$

Remark 3.13. For any vector space M, it follows from Zorn's lemma that there always exist local systems of vertex operators on M. Since the identity operator $I(z) = \mathrm{id}_M$ is mutually local with any weak vertex operator on M, any local system contains $I(z)$. ¿From Remark 3.11 and Lemma 3.12, any local system is closed under the derivative operator $D = \dfrac{d}{dz}$.

Lemma 3.14. *Let M be a restricted Vir-module with central charge ℓ. Then $L(z) = \sum_{n\in\mathbf{Z}} L(n)z^{-n-2}$ is a (local) vertex operator on $(M, L(-1))$ of weight two.*

Proof. It follows from Remark 2.11 and Lemma 2.12 that

$$(z_1 - z_2)^k [L(z_1), L(z_2)] = 0 \quad \text{for } k \geq 4. \tag{3.20}$$

Then $L(z)$ is a local vertex operator on M. □

The proof of the following proposition was given by Professor Chongying Dong.

Proposition 3.15. *Let $a(z)$, $b(z)$ and $c(z)$ be weak vertex operators on M. Suppose both $a(z)$ and $b(z)$ are local with $c(z)$. Then $a(z)_n b(z)$ is local with $c(z)$ for all $n \in \mathbf{Z}$.*

Proof. Let r be a positive integer greater than $-n$ such that the following identities hold:

$$(z_1 - z_2)^r a(z_1) b(z_2) = (z_1 - z_2)^r b(z_2) a(z_1),$$

$$(z_1 - z_2)^r a(z_1) c(z_2) = (z_1 - z_2)^r c(z_2) a(z_1),$$

$$(z_1 - z_2)^r b(z_1) c(z_2) = (z_1 - z_2)^r c(z_2) b(z_1).$$

By definition, we have

$$a(z)_n b(z) = \mathrm{Res}_{z_1} \left((z_1 - z)^n a(z_1) b(z) - (-z + z_1)^n b(z) a(z_1) \right). \tag{3.21}$$

Since

$$\begin{aligned}
& (z - z_3)^{4r} \left((z_1 - z)^n a(z_1) b(z) c(z_3) - (-z + z_1)^n b(z) a(z_1) c(z_3) \right) \\
= \; & \sum_{s=0}^{3r} \binom{3r}{s} (z - z_1)^{3r-s} (z_1 - z_3)^s (z - z_3)^r \cdot \\
& \cdot \left((z_1 - z)^n a(z_1) b(z) c(z_3) - (-z + z_1)^n b(z) a(z_1) c(z_3) \right) \\
= \; & \sum_{s=r+1}^{3r} \binom{3r}{s} (z - z_1)^{3r-s} (z_1 - z_3)^s (z - z_3)^r \cdot \\
& \cdot \left((z_1 - z)^n a(z_1) b(z) c(z_3) - (-z + z_1)^n b(z) a(z_1) c(z_3) \right) \\
= \; & \sum_{s=r+1}^{3r} \binom{3r}{s} (z - z_1)^{3r-s} (z_1 - z_3)^s (z - z_3)^r \cdot \\
& \cdot \left((z_1 - z)^n c(z_3) a(z_1) b(z) - (-z + z_1)^n c(z_3) b(z) a(z_1) \right) \\
= \; & (z - z_3)^{4r} \left((z_1 - z)^n c(z_3) a(z_1) b(z) - (-z + z_1)^n c(z_3) b(z) a(z_1) \right),
\end{aligned} \tag{3.22}$$

we have

$$(z-z_3)^{4r}(a(z)_n b(z))c(z_3) = (z-z_3)^{4r}c(z_3)(a(z)_n b(z)). \quad \square \tag{3.23}$$

Remark 3.16. Let M be any super vector space and let V be any local system of vertex operators on M. Then it follows from Proposition 3.15, Remarks 3.11 and 3.12 and Lemmas 3.13 and 3.14 that the quadruple $(V, I(z), D, Y)$ satisfies (V1)-(V4) of Definition 2.1.

Proposition 3.17. *Let V be any local system of vertex operators on M. Then for any vertex operators $a(z)$ and $b(z)$ in V, $Y(a(z), z_1)$ and $Y(b(z), z_2)$ are mutually local on (V, D).*

Proof. Let $c(z)$ be any weak vertex operator on M. Then we have

$$\begin{aligned}
& Y(a(z), z_3)Y(b(z), z_0)c(z_2) \\
= \;& \mathrm{Res}_{z_1} z_3^{-1}\delta\left(\frac{z_1-z_2}{z_3}\right) a(z_1)(Y(b(z), z_0)c(z_2)) \\
& -z_3^{-1}\delta\left(\frac{-z_2+z_1}{z_3}\right)(Y(b(z), z_0)c(z_2))a(z_1) \\
= \;& \mathrm{Res}_{z_1}\mathrm{Res}_{z_4} A
\end{aligned}$$

where

$$\begin{aligned}
A \;=\; & z_3^{-1}\delta\left(\frac{z_1-z_2}{z_3}\right) z_0^{-1}\delta\left(\frac{z_4-z_2}{z_0}\right) a(z_1)b(z_4)c(z_2) \\
& -z_3^{-1}\delta\left(\frac{z_1-z_2}{z_3}\right) z_0^{-1}\delta\left(\frac{-z_2+z_4}{z_0}\right) a(z_1)c(z_2)b(z_4) \\
& -z_3^{-1}\delta\left(\frac{-z_2+z_1}{z_3}\right) z_0^{-1}\delta\left(\frac{z_4-z_2}{z_0}\right) b(z_4)c(z_2)a(z_1) \\
& +z_3^{-1}\delta\left(\frac{-z_2+z_1}{z_3}\right) z_0^{-1}\delta\left(\frac{-z_2+z_4}{z_0}\right) c(z_2)b(z_4)a(z_1).
\end{aligned}$$

Similarly, we have

$$Y(b(z), z_0)Y(a(z), z_3)c(z_2) = \mathrm{Res}_{z_1}\mathrm{Res}_{z_4} B \tag{3.24}$$

where

$$\begin{aligned} B &= z_3^{-1}\delta\left(\frac{z_1-z_2}{z_3}\right)z_0^{-1}\delta\left(\frac{z_4-z_2}{z_0}\right)b(z_4)a(z_1)c(z_2) \\ &\quad -z_3^{-1}\delta\left(\frac{-z_2+z_1}{z_3}\right)z_0^{-1}\delta\left(\frac{z_4-z_2}{z_0}\right)b(z_4)c(z_2)a(z_1) \\ &\quad -z_3^{-1}\delta\left(\frac{z_1-z_2}{z_3}\right)z_0^{-1}\delta\left(\frac{-z_2+z_4}{z_0}\right)a(z_1)c(z_2)b(z_4) \\ &\quad +z_3^{-1}\delta\left(\frac{-z_2+z_1}{z_3}\right)z_0^{-1}\delta\left(\frac{-z_2+z_4}{z_0}\right)c(z_2)a(z_1)b(z_4). \end{aligned}$$

Let k be any positive integer such that

$$(z_1-z_4)^k a(z_1)b(z_4) = (z_1-z_4)^k b(z_4)a(z_1).$$

Since

$$(z_3-z_0)^k z_3^{-1}\delta\left(\frac{z_1-z_2}{z_3}\right)z_0^{-1}\delta\left(\frac{z_4-z_2}{z_0}\right) = (z_1-z_4)^k z_3^{-1}\delta\left(\frac{z_1-z_2}{z_3}\right)z_0^{-1}\delta\left(\frac{z_4-z_2}{z_0}\right),$$

it is clear that locality of $a(z)$ with $b(z)$ implies the locality of $Y(a(z),z_1)$ with $Y(b(z),z_2)$. □

Now, we are ready to present our main theorem:

Theorem 3.18. *Let M be any vector space and let V be any local system of vertex operators on M. Then V is a vertex algebra and M satisfies all the conditions for module except the existence of d in (M2). If V is a local system on (M,d), then (M,d) is a V-module.*

Proof. It follows from Proposition 2.4, Remark 3.16 and Proposition 3.17 that V is a vertex algebra. It follows from Proposition 2.3 and Remark 2.3 that M is a V-module through the linear map $Y_M(a(z),z_0) = a(z_0)$ for $a(z)\in V$. □

Corollary 3.19. *Let M be any vector space and let S be any set of mutually local vertex operators on M. Let $< S >$ be the subspace of $F(M)$ generated by $S\cup\{I(z)\}$ under the vertex operator multiplication (3.7) (or (3.5) for components). Then $(< S >, I(z), D, Y)$ is a vertex algebra with M as a module.*

Proof. It follows from Proposition 3.15 that $< S >$ is a local subspace of $F(M)$. Let A be a local system containing $< S >$ as a subspace. Then by Theorem 3.18, A is a vertex superalgebra with M as a module. Since $< S >$ is closed under (3.7), $< S >$ is a vertex subalgebra. Since the "multiplication" (3.7) does not depend on the choice of the local system A, $\langle S\rangle$ is canonical. $\square$

Proposition 3.20. *Let M be a restricted Vir-module with central charge ℓ and let V be a local system of vertex operators on $(M, L(-1))$, containing $L(z)$. Then the vertex operator $L(z)$ is a Virasoro element of the vertex algebra V.*

Proof. First, by Theorem 3.18 V is a vertex algebra with M as a V-module. Set $\omega = L(z) \in V$. By Lemma 2.7, the components of vertex operator $Y(\omega, z_0)$ give rise to a representation on V of central charge ℓ for the Virasoro algebra Vir. For any $a(z) \in V_{(h)}$, by definition we have

$$L(z)_0 a(z) = [L(-1), a(z)] = a'(z); \tag{3.25}$$

$$L(z)_1 a(z) = [L(0), a(z)] - z^{-1}[L(-1), a(z)] = ha(z). \tag{3.26}$$

Therefore V satisfies all conditions for a vertex operator superalgebra except the requirements on the homogeneous subspaces. $\square$

Let V be a vertex (operator) algebra and let (M, d) be a V-module. Then the image $\bar{V}$ of V inside $F(M, d)$ is a local subspace. By Zorn's lemma, there exists a local system A containing $\bar{V}$ as a subspace. From the vacuum property (M2) we have:

$$Y_M(\cdot, z)(\mathbf{1}) = Y_M(\mathbf{1}, z) = \mathrm{id}_M = I(z). \tag{3.27}$$

For any elements $a, b \in V$, we have:

$$\begin{aligned}
& Y_M(\cdot, z)(Y(a, z_0)b) \\
= \;& Y_M(Y(a, z_0)b, z) \\
= \;& \mathrm{Res}_{z_1}\left(z_0^{-1}\delta\left(\frac{z_1 - z}{z_0}\right) Y_M(a, z_1)Y_M(b, z) - z_0^{-1}\delta\left(\frac{z - z_1}{-z_0}\right) Y_M(b, z)Y_M(a, z_1)\right) \\
= \;& Y_A(Y_M(a, z), z_0)Y_M(b, z).
\end{aligned} \tag{3.28}$$

Thus $Y_M(\cdot, z)$ is a vertex algebra homomorphism from V to A. Conversely, let ϕ be a vertex algebra homomorphism from V to some local system A of vertex operators on (M, d). Since $\phi(u) \in A \in (\mathrm{End}M)[[z, z^{-1}]]$ for any $u \in V$, we use ϕ_z for ϕ to indicate the dependence of $\phi(u)$ on z. For any formal variable z_1, set

$\phi_{z_1}(a) = \phi_z(a)|_{z=z_1}$ for any $a \in V$. We define $Y_M(a, z)u = \phi_z(a)$ for $a \in V$. By definition we have:

$$Y_M(\mathbf{1}, z) = \phi_z(\mathbf{1}) = I(z) = \mathrm{id}_M. \tag{3.29}$$

For any elements $a, b \in V$, we have:

$$\begin{aligned}
& Y_M(Y(a, z_0)b, z_2) \\
= \; & \phi_z(Y(a, z_0)b)|_{z=z_2} \\
= \; & (Y_A(\phi_z(a), z_0)\phi_z(b))\,|_{z=z_2} \\
= \; & \mathrm{Res}_{z_1}\left(z_0^{-1}\delta\left(\frac{z_1 - z_2}{z_0}\right)\phi_{z_1}(a)\phi_{z_2}(b) - z_0^{-1}\delta\left(\frac{z_2 - z_1}{-z_0}\right)\phi_{z_2}(b)\phi_{z_1}(a)\right) \\
= \; & \mathrm{Res}_{z_1}\left(z_0^{-1}\delta\left(\frac{z_1 - z_2}{z_0}\right)Y_M(a, z_1)Y_M(b, z_2) - z_0^{-1}\delta\left(\frac{z_2 - z_1}{-z_0}\right)Y_M(b, z_2)Y_M(a, z_1)\right).
\end{aligned} \tag{3.30}$$

It follows from Remark 2.4 that (M, d, Y_M) is a V-module. Therefore, we have proved:

Proposition 3.21. *Let V be a vertex (operator) algebra. Then giving a V-module (M, d) is equivalent to giving a vertex algebra homomorphism from V to some local system of vertex operators on (M, d).*

4 Vertex operator algebras and modules associated to some infinite-dimensional Lie algebras

In this section, we shall use the machinery we built in Section 3 to study vertex operator algebras and modules associated to the representations for some well-known infinite-dimensional Lie algebras such as the Virasoro algebra and affine Lie algebras. Most of the material presented in this section was taken from [L2]. (See [FZ] for a different approach.)

Let us start with an abstract result which will be used in this section.

Proposition 4.1. *Let $(V, D, \mathbf{1}, Y)$ be a vertex algebra and let (M, d, Y_M) be a V-module. Let $u \in M$ such that $du = 0$. Then the linear map*

$$f : V \to M; a \mapsto a_{-1}u \quad \text{for } a \in V, \tag{4.1}$$

is a V-homomorphism.

Proof. It follows from the proof of Proposition 3.4 [L1]. $\square$

For any complex numbers c and h, let $M(c,h)$ be the Verma module for the Virasoro algebra Vir with central charge c and with lowest weight h. Let $\mathbf{1}$ be a lowest weight vector of $M(c,0)$. Then $L(-1)\mathbf{1}$ is a singular vector, i.e., $L(n)\mathbf{1} = 0$ for $n \geq 1$. Set $\bar{M}(c,0) = M(c,0)/ < L(-1)\mathbf{1} >$, where $< L(-1)\mathbf{1} >$ denotes the submodule of $M(c,0)$ generated by $L(-1)\mathbf{1}$. Denote by $L(c,h)$ the (unique) irreducible quotient module of $M(c,h)$. By slightly abusing notations, we still use $\mathbf{1}$ for the image of $\mathbf{1}$ for both $\bar{M}(c,0)$ and $L(c,0)$.

Proposition 4.2. *For any complex number c, $\bar{M}(c,0)$ has a natural vertex operator algebra structure and any restricted Vir-module M of central charge c is a weak $\bar{M}(c,0)$-module. In particular, for any complex number h, $M(c,h)$ is a $\bar{M}(c,0)$-module.*

Proof. Let M be any restricted Vir-module with central charge c. Then $\bar{M}(c,0) \oplus M$ is a restricted Vir-module. By Lemma 3.14, $L(z)$ is a local vertex operator on $(\bar{M}(c,0) \oplus M, L(-1))$. Then by Corollary 3.19, $V =< L(z) >$ is a vertex algebra with $\bar{M}(c,0) \oplus M$ as a module. Consequently, both $\bar{M}(c,0)$ and M are V-modules. By Lemma 2.5, the components of $Y(L(z), z_0)$ on V satisfy the Virasoro relation. Since $L(z)_n I(z) = 0$ for $n \geq 0$, V is a lowest weight Vir-module with lowest weight 0, so that V is a quotient module of $\bar{M}(c,0)$. Let $\mathbf{1}$ be a lowest weight vector of $\bar{M}(c,0)$. Since $L(-1)\mathbf{1} = 0$, by Proposition 4.1, we have a V-homomorphism from V to $\bar{M}(c,0)$ mapping $I(z)$ to $\mathbf{1}$. Then it follows that V is isomorphic to $\bar{M}(c,0)$. Therefore $\bar{M}(c,0)$ is a vertex operator algebra and any restricted Vir-module M is a weak module. $\square$

Remark 4.3. It follows that $L(c,0)$ is a quotient vertex operator algebra of $\bar{M}(c,0)$.

Let $(\mathbf{g}, B)$ be a pair consisting of a finite-dimensional Lie algebra $\mathbf{g}$ and a nondegenerate symmetric invariant bilinear form B on $\mathbf{g}$. Set

$$\tilde{\mathbf{g}} = \mathbf{C}[t,t^{-1}] \otimes \mathbf{g} \oplus \mathbf{C}c. \tag{4.2}$$

Then we define

$$[a_m, b_n] = [a,b]_{m+n} + m\delta_{m+n,0}\langle a,b\rangle c, \;\; [c, x_m] = 0 \tag{4.3}$$

for any $a, b, x \in \mathbf{g}$ and $m, n \in \mathbf{Z}$, where x_m stands for $t^m \otimes x$. For any $x \in \mathbf{g}$, we set

$$x(z) = \sum_{n \in \mathbf{Z}} x_n z^{-n-1}. \tag{4.4}$$

Then the defining relations (4.3) of $\tilde{\mathbf{g}}$ are equivalent to the following equations:

$$[a(z_1), b(z_2)] = z_1^{-1}\delta\left(\frac{z_2}{z_1}\right)[a,b](z_2) + z_1^{-2}\delta'\left(\frac{z_2}{z_1}\right)\langle a,b\rangle c, \tag{4.5}$$

$$[x(z), c] = 0 \quad \text{for any } a, b, x \in \mathbf{g}, m, n \in \mathbf{Z}. \tag{4.6}$$

Set

$$N_+ = t\mathbf{C}[t] \otimes \mathbf{g}, \; N_- = t^{-1}\mathbf{C}[t^{-1}] \otimes \mathbf{g}, \; N_0 = \mathbf{g} \oplus \mathbf{C}c. \tag{4.7}$$

Then we obtain a triangular decomposition $\tilde{\mathbf{g}} = N_+ \oplus N_0 \oplus N_-$. Let $P = N_+ + N_0$ be the parabolic subalgebra. For any $\mathbf{g}$-module U and any complex number ℓ, denote by $M_{(\mathbf{g},B)}(\ell, U)$ the generalized Verma module [Lep] or Weyl module with c acting as scalar ℓ. Namely, $M_{(\mathbf{g},B)}(\ell, U) = U(\tilde{\mathbf{g}}) \otimes_{U(P)} U$. For any $\tilde{\mathbf{g}}$-module M, we may consider $x(z)$ for $x \in \mathbf{g}$ as an element of $(\mathrm{End} M)[[z, z^{-1}]]$. Recall that a $\tilde{\mathbf{g}}$-module M is said to be *restricted* if for any $u \in M$, $(t^k\mathbf{C}[t] \otimes \mathbf{g})u = 0$ for k sufficiently large. Then a $\tilde{\mathbf{g}}$-module M is restricted if and only if $x(z)$ for all $x \in \mathbf{g}$ are weak vertex operators on M.

Theorem 4.4. *For any complex number ℓ, $M_{(\mathbf{g},B)}(\ell, \mathbf{C})$ has a natural vertex algebra structure and any restricted $\tilde{\mathbf{g}}$-module M of level ℓ is a $M_{(\mathbf{g},B)}(\ell, \mathbf{C})$-module.*

Proof. Let M be any restricted $\tilde{\mathbf{g}}$-module of level ℓ. Then $W = M_{(\mathbf{g},B)}(\ell, \mathbf{C}) \oplus M$ is also a restricted $\tilde{\mathbf{g}}$-module of level ℓ. It follows from Lemma 2.3 and (4.5)-(4.6) that $\bar{\mathbf{g}} = \{a(z) | a \in \mathbf{g}\}$ is a local subspace of $F(W)$. Let V be the subspace of $F(W)$ generated by all $\bar{\mathbf{g}}$. Then by Corollary 3.19, V is a vertex superalgebra and W is a V-module. Consequently, both M and $M_{(\mathbf{g},B)}(\ell, \mathbf{C})$ are V-modules. It follows from Lemma 2.5 and (4.5)-(4.6) that V is a $\tilde{\mathbf{g}}$-module (of level ℓ) with a vector $I(z)$ satisfying $P \cdot I(z) = 0$, so that V is a quotient $\tilde{\mathbf{g}}$-module of $M_{(\mathbf{g},B)}(\ell, \mathbf{C})$.

To finish the proof, we only need to prove that V is isomorphic to $M_{(\mathbf{g},B)}(\ell, \mathbf{C})$ as a V-module. Let d be the endomorphism of $M_{(\mathbf{g},B)}(\ell, \mathbf{C})$ such that

$$d \cdot \mathbf{1} = 0, \ [d, a_m] = -m a_{m-1} \quad \text{for } a \in \mathbf{g}. \tag{4.8}$$

Then $[d, a(z)] = a'(z)$ for any $a \in \mathbf{g}$. Then $(M_{(\mathbf{g},B)}(\ell, \mathbf{C}), d)$ is a V-module. It follows from Proposition 4.1 and the universal property of $M_{(\mathbf{g},B)}(\ell, \mathbf{C})$ that V and $M_{(\mathbf{g},B)}(\ell, \mathbf{C})$ are isomorphic V-modules. $\square$

Remark 4.6. It is clear that $M_{(\mathbf{g},B)}(\ell, \mathbf{C}) = M_{(\mathbf{g},\alpha B)}(\alpha^{-1}\ell, \mathbf{C})$ for any nonzero complex number α (see for example [Lian]).

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