数理解析研究所講究録
904 巻 1995 年 87-115

Moonshine Cohomology

Bong H. Lian and Gregg J. Zuckerman

ABSTRACT. We construct a new cohomology functor from a certain category of *quantum operator algebras* to the category of *Batalin-Vilkovisky algebras*. This *Moonshine cohomology* has, as a group of natural automorphisms, the Fischer-Griess Monster finite group. We prove a general vanishing theorem for this cohomology. For a certain commutative QOA attached to a rank two hyperbolic lattice, we show that the degree one cohomology is isomorphic to the so-called Lie algebra of physical states. In the case of a rank two unimodular lattice, the degree one cohomology gives a new construction of Borcherd's Monster Lie algebra. As applications, we compute the graded dimensions and signatures of this cohomology as a hermitean Lie algebra graded by a hyperbolic lattice. In the first half of this paper, we give as preparations an exposition of the theory of quantum operator algebras. Some of the results here were announced in lectures given by the first author at the Research Institute for Mathematical Sciences in Kyoto in September 94.

1 Introduction

In the 1980's, there took place the following developments, all of which are now understood to be connected with two-dimensional quantum field theory:

1) Frenkel and Kac gave a construction of the simply-laced simple Lie algebras via the vertex operator representation of the corresponding affine Kac-Moody Lie algebras. The vertex operators were already thought of as quantum fields that depend holomorphically on one complex variable. These fields had their origin in the dual resonance model, which was central to the creation of string theory. Frenkel later extended this vertex operator construction to obtain some new infinite dimensional Lie algebras that contained Kac-Moody Lie algebras of hyperbolic type. In the process, Frenkel gave a proof of the No Ghost Theorem in string theory.

2) Frenkel, Lepowsky and Meurman generalized the Frenkel-Kac vertex operator construction in order to construct an infinite dimensional graded representation of the Monster finite group. The existence of the FLM Moonshine module immediately explained the empirically observed connection between the modular function $j(\tau)$ and the dimensions of irreducible representations of the Monster. Borcherds soon afterwards discovered that the Moonshine module possessed

[0]q-alg/9501015

[0]B.H.L. is supported by grant DE-FG02-88-ER-25065. G.J.Z. is supported by NSF Grant DMS-9307086 and DOE Grant DE-FG02-92-ER-25121.

the structure of what he called a vertex algebra. The book by Frenkel, Lepowsky and Meurman gave a complete treatment of vertex *operator* algebras and the Moonshine module.

3) Belavin, Polyakov and Zamolodchikov developed the so-called operator product expansion into a powerful tool for the study of two dimensional conformal quantum fields. As a consequence, BPZ determined the possible central charges and critical dimensions of the minimal conformal field theories. This analysis required the deep work by Kac, and later Feigin and Fuchs on the structure of the highest weight representations of the Virasoro Lie algebra. It was soon recognized (see for example [29]) that there was a close relationship between the theory of the operator product expansion and the theory of vertex operator algebras.

4) Feigin invented the theory of semi-infinite cohomology for Virasoro and Kac-Moody Lie algebras. The construction of the differential in Feigin's complex was soon reinterpreted in terms of some elementary holomorphic quantum fields arising in conformal field theory and the BRST quantization construction in string theory [18][36]. Moreover, Frenkel, Garland and the second author of the current paper were able to give a new proof of the No Ghost Theorem via the analysis of a particular semi-infinite cochain complex. At the end of the eighties, the two authors of the current paper began a long series of papers building on the earlier work of FGZ.

5) Koszul discovered a new relationship between graded commutative superalgebras and graded Lie superalgebras. Specifically, Koszul found that under certain very general hypotheses, the failure of a second order differential operator to be a derivation led to the existence of a graded Lie bracket on the underlying commutative algebra. An important special case of the same relationship was found independently by the physicists Batalin and Vilkovisky. At the time, there was no perceived connection between the work of Koszul, Batalin and Vilkovisky and the rapidly developing study of conformal quantum fields.

In the early 1990's, Borcherds proved the Conway-Norton conjectures for the FLM Moonshine module. The first two developments above are fundamental for Borcherds. As a brief aside, Borcherds claims in his paper that semi-infinite cohomology theory, as discussed in 4 above, can be employed to obtain alternate constructions of the infinite dimensional Lie algebras that figure prominently in his work. However, he presents no details to support his claim.

The main purpose of the current paper, partly inspired by [28], is to forge a synthesis of Borcherds work with all five of the above developments. We construct a new functor that we call Moonshine cohomology and which fully justifies Borcherds claim. This new cohomology theory is an outgrowth of the developments sketched in 3, 4 and 5 above. In particular, we employ a new mathematical approach to the operator product expansion, and we work with a recent generalization of the notion of a VOA to the notion of a commutative quantum operator algebra. Our paper on CQOAs [24] is designed to be a companion to the current paper on Moonshine cohomology.

Degree one Moonshine cohomology provides a functor from the category of vertex operator algebras to the category of Lie algebras that carry an action of the Monster finite group by automorphisms. The degree one Moonshine cohomology of a particular VOA yields the generalized

Kac-Moody Lie algebra that is the key to Borcherds proof of the Conway-Norton conjectures.

The total Moonshine cohomology provides a functor with values in the category of Batalin-Vilkovsky algebras that carry an action of the Monster finite group by automorphisms. BV algebras are odd Poisson algebras in which the graded Lie bracket is related to the graded commutative product in the fashion first described by Koszul and Batalin-Vilkovisky (see 5 above.) The abstract notion of a BV algebra, though present in the paper of Koszul, did not become well known until the recent work of Penkava-Schwarz and Getzler.

As in any cohomology theory, the total cohomology is better behaved and more fundamental than the cohomology of any special degree. Moreover, the BV structure on the total cohomology allows us to relate a complicated Lie algebra structure to a more elementary commutative algebra structure. We hope that Moonshine cohomology will yield further insight into the structure of the Moonshine module as well as into the proof of the Conway-Norton conjectures. We also hope that our current paper unifies a number of seemingly disparate points of view in mathematics and mathematical physics.

Here is a brief summary of the contents of this paper:

In section 2, we state the definitions of our main concepts: quantum operators, matrix elements, Wick products, iterated Wick products, the infinitely many "circle" products, the operator product expansion, the notions of locality and commutativity, and finally the corresponding notions of local and commutative quantum operator algebras. Finally, we discuss some elementary known facts about commutativity.

In section 3, we introduce what we call the Wick calculus, which deals with operator products of the form $:t(z)u(z):v(w)$ as well as $t(z):u(w)v(w):$ under the assumption that the quantum operators $t(z)$, $u(z)$, and $v(z)$ are *pairwise commutative*. Here, $:t(z)u(z):$ denotes the Wick or normal ordered product of $t(z)$ with $u(z)$. The Wick calculus is essential for both computations as well as for theoretical issues, such as the explicit construction of CQOAs. Section 3 continues with the construction of the CQOA $O(b,c)$, which acts in the ghost Fock space of the BRST construction. Following this is a construction of the CQOA $O_\kappa(L)$, which arose originally in the seminal work of BPZ [3], and which acts in the state space of any conformal field theory having central charge κ. This section concludes with a construction of a CQOA from a Lie algebra equipped with an symmetric invariant form. All three examples are special, in that these algebras are spanned by Wick products of derivatives of the generating quantum operators. In fact, we exhibit explicit bases consisting of such products in the first two examples.

In section 4, we discuss the BRST construction in the language of what we call conformal QOAs. Given a conformal QOA O with central charge κ, we form the tensor product $C^*(O) = O(b,c)\otimes O$. We then construct the special quantum operator $J(z)$, which we call the BRST current. We also give a simple characterization of $J(z)$. We then recall the famous result that the coefficient $J(0) = Res_z J(z)$ is square-zero if and only if $\kappa = 26$. After that we specialize to this central charge.

The first main result of section 4 is Theorem 4.6, which states that the Wick product induces a graded commutative associative product on the cohomology of $C^*(O)$ with respect to the derivation, $[J(0), -]$. This theorem first appeared in work of E. Witten [33], who called the ghost number zero subalgebra the "ground ring of a string background". An approach to this theorem via VOA theory appears in [26]. The approach in the current paper is via CQOA theory. Continuing section 4, we develop the theory of the ghost field, $b(z)$, and its coefficient, $b(1)$. As a preparation, we remind the reader of the definition of a Batalin-Vilkovisky (BV) operator and BV algebra. The second main result of section 4 is Theorem 4.8, which states that the operator $b(1)$ induces a BV operator acting in the BRST cohomology algebra. Thus the cohomology becomes a BV algebra. This theorem was inspired by work of Witten and Zwiebach [37], and first appeared i! n [26], where it was derive At the end of section 4 we state the precise connection between BRST cohomology and semi-infinite cohomology.

In section 5, we finally present the construction of Moonshine cohomology $\mathbf{M}^*$ as a functor from conformal QOAs to BV algebras. Our main result is Theorem 5.2, which asserts that Moonshine cohomology vanishes for degrees less than zero and greater than three. In fact, we first prove Theorem 5.3, which states a vanishing theorem for semi-infinite cohomology; Theorem 5.2 follows immediately.

We specialize in section 6 to the Moonshine cohomology of the conformal QOA attached by FLM to a hyperbolic lattice of rank 2. Theorem 6.2 asserts that in this case, the degree zero and degree three Moonshine cohomology groups are both one dimensional; moreover, the degree one and degree two cohomology groups can both be canonically identified with the so-called space of physical states, whose definition dates back to the early days of string theory (see [8][9][5]). We actually prove a more general result, Theorem 6.4, which follows from a semi-infinite cohomology calculation, Theorem 6.5. As a consequence, we are able to compute the full Moonshine cohomology as a tri-graded linear space, and determine all the graded dimensions of this space. As a second application, we compute the signature of the hermitean form on degree one cohomology and show that the form is positive definite.

At the end of section 6 we discuss some open questions about Moonshine cohomology. In particular, we conjecture that the natural automorphism group of the *functor* $\mathbf{M}^*$ is isomorphic to the Monster finite group. This conjecture is suggested by the known theorem that the Monster is the full automorphism group of the Moonshine VOA [11]. We hope that future study of the functor $\mathbf{M}^*$ will cast new light on a rich amalgam of mathematics and mathematical physics.

Sections 2-4 of this paper are meant to be an exposition, and should be accessible to the readers who are new to the theory of operator product expansions. Many useful (in the authors' opinion) exercises are given. Sections 5-6 are more advanced because they draw from several different subjects. We hope that the included references will help the readers who are interested in further details.

Acknowledgments: B.H.L. would like to express special thanks to Prof. M. Miyamoto and Prof. H. Yamada for their invitation to lecture at the Research Institute for Mathematical Sciences in Kyoto in September 94. He also thanks them for their hospitality during his stay in

Kyoto, and for their patience waiting for the final draft of this paper. We thank F. Akman for carefully proofreading our manuscript.

2 Quantum Operator Algebras

Let V be a $\mathbf{Z}$ doubly graded vector space $V = \oplus V^n[m]$. The degrees of a homogeneous element v in $V^n[m]$ will be denoted by $|v| = n$, $||v|| = m$ respectively. In physical applications, $|v|$ will be the fermion or ghost number of v. In conformal field theory, $||v||$ will be the conformal dimension or weight of v. We say that V is bounded if for each n, $V^n[m] = 0$ for $m << 0$.

Let z be a formal variable with degrees $|z| = 0$, $||z|| = -1$. Then it makes sense to speak of a *homogeneous* (biinfinite) formal power series

$$u(z) = \sum_{n\in\mathbf{Z}} u(n)z^{-n-1} \tag{2.1}$$

of degrees $|u(z)|$, $||u(z)||$ where the coefficients $u(n)$ are homogeneous linear maps in V with degrees $|u(n)| = |u(z)|$, $||u(n)|| = -n-1+||u(z)||$. Note then that the terms $u(n)z^{-n-1}$ indeed have the same degrees $|u(z)|$, $||u(z)||$ for all n. *We require that for every $v \in V$, $u(n)v = 0$ for $n >> 0$.* (If V is bounded, then this requirement is superfluous.) We call a finite sum of such homogeneous series $u(z)$ a *quantum operator* on V, and we denote the linear space of quantum operators as $QO(V)$.

Notations: By the expression $(z-w)^n$, n an integer, we usually mean its formal power series expansion in the region $|z| > |w|$. Thus $(z-w)^{-2}$ and $(-w+z)^{-2}$ are different, as power series. When such expressions are to be regarded as rational functions rather than formal series, we will explicitly mention so. When $A(z) = \sum A(n)z^{-n-1}$ is a formal series with coefficients $A(n)$ in whatever linear space, we define $Res_z A(z) = A(0)$, $A(z)^+ = \sum_{n\geq 0} A(n)z^{-n-1}$, $A(z)^- = \sum_{n<0} A(n)z^{-n-1}$, $\partial A(z) = \sum -(n+1)A(n)z^{-n-2}$. If $u(z), u'(z)$ belong to QOAs O, O' respectively, we abbreviate $u(z)\otimes u'(z)$, as an element of $O\otimes O'$, simply as $u(z)u'(z)$. When no ambiguity occurs, we denote $|u(z)|, ||u(z)||$ simply as $|u|, ||u||$. The restricted dual of a graded vector space V is denoted $V^\#$. If $A_1(z)$,$A_2(z)$,... are quantum operators, an arbitrary matrix element $\langle x, A_1(z_1)A_2(z_2)\cdots y\rangle$ with $x \in V^\#$, $y \in V$, is denoted as $\langle A_1(z_1)A_2(z_2)\cdots\rangle$. In the interest of clarity, we often write signs like $(-1)^{|t||u|}$ simply as $\pm$. This convention is used only when the sign arises from permutation of elements. When in doubt, the reader can easily recover the correct sign from such a permutation. Given two homogeneous linear operators, X, Y, we write $[X, Y] = XY - (-1)^{|X||Y|}YX$. A similar notation applies to quantum operators when it makes sense.

Given two quantum operators $u(z), v(z)$, we write

$$: u(z)v(w) := u(z)^- v(w) + (-1)^{|u||v|} v(w)u(z)^+. \tag{2.2}$$

Because $v(n)t = u(n)t = 0$ for $n >> 0$, it's easy to check that if we replace w by z, the right hand side makes sense as a quantum operator and hence defines a nonassociative product on $QO(V)$. It is called the *Wick product.* Similarly given $u_1(z), \cdots, u_n(z)$, we define $: u_1(z_1) \cdots u_n(z_n) :$ inductively as $: u_1(z_1)(: u_2(z_2) \cdots u_n(z_n) :) :$.

Exercise 1 *Show that* $: u_1(z) \cdots u_n(z) :$ *makes sense as an element of* $QO(V)$.

Definition 2.1 *For each integer n we define a product on $QO(V)$:*

$$u(w) \circ_n v(w) = Res_z u(z)v(w)(z-w)^n - (-1)^{|u||v|} Res_z v(w)u(z)(-w+z)^n. \tag{2.3}$$

Explicitly we have:

$$u(z) \circ_n v(z) = \begin{cases} \frac{1}{(-n-1)!} : \partial^{-n-1} u(z)\ v(z) : & \text{if } n < 0 \\ [(\sum_{m=0}^{n} \begin{pmatrix} n \\ m \end{pmatrix} u(m)(-z)^{n-m}), v(z)] & \text{if } n \geq 0. \end{cases} \tag{2.4}$$

If A is a homogeneous linear operator on V, then it's clear that the graded commutator $[A, -]$ is a graded derivation of each of the products $\circ_n$. Since $u(z) \circ_0 v(z) = [u(0), v(z)]$, we have

Proposition 2.2 *For any $t(z), u(z), v(z)$ in $QO(V)$ and n integer, we have*

$$t(z) \circ_0 (u(z) \circ_n v(z)) = [t(z) \circ_0 u(z)] \circ_n v(z) \pm u(z) \circ_n [t(z) \circ_0 v(z)],$$

ie. $t(z) \circ_0$ is a derivation of every product in $QO(V)$.

Proposition 2.3 *For $u(z), v(z)$ in $QO(V)$, the following equality of formal power series in two variables holds:*

$$u(z)v(w) = \sum_{n \geq 0} u(w) \circ_n v(w)(z-w)^{-n-1} + : u(z)v(w) : . \tag{2.5}$$

Proof: We have $u(z)v(w) = [u(z)^+, v(w)] + : u(z)v(w) :$. On the other hand by inverting the second eqn. in (2.4), we get

$$[u(m), v(w)] = \sum_{n=0}^{m} \begin{pmatrix} m \\ n \end{pmatrix} u(w) \circ_n v(w) w^{m-n}. \tag{2.6}$$

Thus we have

$$\begin{aligned} [u(z)^+, v(w)] &= \sum_{m \geq n \geq 0} \begin{pmatrix} m \\ n \end{pmatrix} u(w) \circ_n v(w) w^{m-n} z^{-m-1} \\ &= \sum_{n \geq 0} u(w) \circ_n v(w) \frac{1}{n!} \partial_w^n (z-w)^{-1} \\ &= \sum_{n \geq 0} u(w) \circ_n v(w)(z-w)^{-n-1}. \quad \square \end{aligned} \tag{2.7}$$

In the sense of the above Proposition, $: u(z)v(w) :$ is the *nonsingular part* of the *operator product expansion* (2.5), while $u(w) \circ_n v(w)(z-w)^{-n-1}$ is the *polar part* of order $-n-1$ (see [3]). In physics literature, $u(w) \circ_n v(w)$ is often written as $\frac{1}{2\pi i}\int_C u(z)v(w)(z-w)^n dz$ where C is a small circle around w. The above proposition clearly justifies this notation. The products $\circ_n$ will become important for describing the algebraic and analytic structures of certain algebras of quantum operators. Thus we introduce the following mathematical definitions:

Definition 2.4 *A graded subspace $\mathcal{A}$ of $QO(V)$ containing the identity operator and closed with respect to all the products $\circ_n$ is called a quantum operator algebra. We say that $u(z)$ is local to $v(z)$ if $u(z) \circ_n v(z) = 0$ for all but finitely many positive n. A QOA $\mathcal{A}$ is called local if its elements are pairwise mutually local.*

We observe that for any element $a(z)$ of a QOA, we have $a(z) \circ_{-2} 1 = \partial a(z)$. Thus a QOA is closed with respect to formal differentiation.

Proposition 2.5 *Let $u(z), v(z)$ be quantum operators, and N a nonnegative integer. If $u(z) \circ_n v(z) = 0$ for $n \geq N$, then $\langle u(z)v(w)\rangle$ represents a rational function in $|z| > |w|$ with poles along $z = w$ of order at most N.*

Proof: By eqn (2.5), we have

$$\langle u(z)v(w)\rangle = \sum_{n\geq 0}\langle u(w) \circ_n v(w)\rangle (z-w)^{-n-1} + \langle : u(z)v(w) :\rangle. \tag{2.8}$$

It is trivial to check that $\langle : u(z)v(w) :\rangle$, $\langle u(w) \circ_n v(w)\rangle \in \mathbf{C}\,[z^{\pm 1}, w^{\pm 1}]$. Thus our claim follows immediately. □

Lemma 2.6 *Let $u(z)$ be local to $v(z)$, and $\langle u(z)v(w)\rangle$ represent the rational function $f(z,w)$. Then for $|w| > |z-w|$,*

$$f(z,w) = \sum_{n\in\mathbf{Z}}\langle u(w) \circ_n v(w)\rangle (z-w)^{-n-1}. \tag{2.9}$$

Proof: The Laurent polynomial $\langle : u(z)v(w) :\rangle$ in the above region is just $\sum_{i\geq 0}\frac{1}{i!}\langle : (\partial^i u(w))v(w) :\rangle (z-w)^i$. Now apply eqn. (2.4). □

We note that none of the products $\circ_n$ is associative in general. However it clearly makes sense to speak of the left, right or two sided ideals in a QOA as well as homomorphisms of QOAs and they are defined in an obvious way. For example, a linear map $f : O \to O'$ is a *homomorphism* if $f(u(z) \circ_n v(z)) = fu(z) \circ_n fv(z)$ for all $u(z), v(z) \in O$, and $f(1) = 1$. An *O-module* is a graded space M equipped with a homomorphism of QOAs $g : O \to QO(M)$.

Exercise 2 *Define the notions of a left, right, and two sided ideals for QOAs.*

Definition 2.7 *Two quantum operators $u(z), v(z)$ are said to commute if they are mutually local, and $\langle u(z)v(w)\rangle, \pm\langle v(w)u(z)\rangle$ represent the same rational function. This is equivalent (Proposition 2.5) to the following: for some $N \geq 0$, $(z-w)^N\langle u(z)v(w)\rangle = \pm(z-w)^N\langle v(w)u(z)\rangle$ as Laurent polynomials. We call a QOA O whose elements pairwise commute a commutative QOA.*

Proposition 2.8 *If $u(z), v(z)$ commute, then for all m*

$$[u(m), v(w)] = \sum_{n\geq 0} \begin{pmatrix} m \\ n \end{pmatrix} u(w) \circ_n v(w) w^{m-n}. \tag{2.10}$$

Proof: The case $m \geq 0$ is obtained by inverting the second eqn. in (2.4). Since $u(z)v(w) = [u(z)^+, v(w)]+ : u(z)v(w) :$ and $v(w)u(z) = \mp[u(z)^-, v(w)]\pm : u(z)v(w) :$, it follows from commutativity that $\langle[u(z)^-, v(w)]\rangle$ represents the same rational function as $-\langle[u(z)^+, v(w)]\rangle$ does, which is just $-\sum_{n\geq 0} \frac{u(w)\circ_n v(w)}{(z-w)^{n+1}}$. This gives

$$[u(z)^-, v(w)] = -\sum_{n\geq 0} u(w) \circ_n v(w)(-w+z)^{-n-1}. \tag{2.11}$$

Taking $Res_z[u(z)^-, v(w)]z^m$ for $m < 0$ gives the desired result. □

The notion of commutativity here is closely related to the physicists' notion of duality in conformal field theory[29]. Frenkel-Lepowsky-Meurman have reformulated the axioms of a VOA in terms of what they call rationality, associativity and commutativity. The notion of commutativity in Definition 2.7 is essentially the same as FLM's. This notion has also been reformulated in the language of formal variables in [6].

3 Wick's calculus

In this section, we derive a number of useful formulas relating various iterated products of three quantum operators. Most of these formulas are well-known to physicists who are familiar with the calculus of operator product expansions. We will also include a lemma on commutativity.

Let $t(z), u(z), v(z)$ be homogeneous quantum operators which pairwise commute.

Lemma 3.1 *(see [22]) For all n, $t(z) \circ_n u(z)$ and $v(z)$ commute.*

Proof: We include Li's proof here for completeness. For a positive integer N, $(z-w)^{2N}$ is a binomial sum of terms $(z-x)^i(x-w)^{2N-i}$, $i=1,..,2N$. So $(z-w)^{N+2N}(t(z)\circ_n u(z))v(w)$ is a binomial sum of terms

$$Res_x\left((z-w)^N(z-x)^i(x-w)^{2N-i}(t(x)u(z)(x-z)^n \mp u(z)t(x)(-z+x)^n)v(w)\right). \quad (3.12)$$

We want to show that for large enough N, and for $0\leq i\leq 2N$, term by term we have

$$\begin{aligned}&(z-w)^N(z-x)^i(x-w)^{2N-i}(t(x)u(z)(x-z)^n \mp u(z)t(x)(-z+x)^n)v(w)\\ &= \pm(z-w)^N(z-x)^i(x-w)^{2N-i}v(w)(t(x)u(z)(x-z)^n \mp u(z)t(x)(-z+x)^n). \quad (3.13)\end{aligned}$$

Consider two cases: $i\geq N$ and $i<N$. By assumption, $(z-x)^k(t(x)u(z)(x-z)^n \mp u(z)t(x)(-z+x)^n)=0$ for all large enough k. So for large enough N, (3.13) holds for $i\geq N$. Similarly for $i<N$, $(z-w)^N(x-w)^{2N-i}(t(x)u(z)(x-z)^n \mp u(z)t(x)(-z+x)^n)v(w)$ coincides with $\pm(z-w)^N(x-w)^{2N-i}v(w)(t(x)u(z)(x-z)^n \mp u(z)t(x)(-z+x)^n)$. This shows that (3.13) holds for each i. □

This lemma is useful for showing existence of commutative QOAs: it says that given a set of pairwise commuting quantum operators, the QOA generated by the set is commutative. We now develop some abstract tools for studying the structure of commutative QOAs.

Applying (2.5), we have

$$\begin{aligned}&:t(z)u(z):v(w)\\ =\ &(t(z)^-u(z)\pm u(z)t(z)^+)v(w)\\ =\ &t(z)^-u(z)v(w)\pm u(z)v(w)t(z)^+ \pm u(z)[t(z)^+,v(w)]\\ =\ &\sum_{n\geq 0}:t(z)(u(w)\circ_n v(w)):(z-w)^{-n-1}+:t(z)u(z)v(w):\\ &\pm\sum_{n,m\geq 0}u(w)\circ_m(t(w)\circ_n v(w))(z-w)^{-n-m-2}+\\ &\pm\sum_{n\geq 0}:u(z)(t(w)\circ_n v(w)):(z-w)^{-n-1}. \quad (3.14)\end{aligned}$$

Similarly,

$$\begin{aligned}&t(z):u(w)v(w):\\ =\ &\pm[u(w)^-,t(z)]v(w)\pm u(w)^-t(z)v(w)\pm t(z)v(w)u(w)^+\\ =\ &\pm\sum_{n,m\geq 0}(-1)^{n+1}u(w)\circ_n(t(w)\circ_m v(w))(z-w)^{-n-m-2}\\ &\pm\sum_{n\geq 0}(-1)^{n+1}:u(z)\circ_n t(z)\ v(w):(z-w)^{-n-1}\\ &\pm\sum_{n\geq 0}:u(w)\ t(w)\circ_n v(w):(z-w)^{-n-1}\pm:u(w)t(z)v(w): \quad (3.15)\end{aligned}$$

Lemma 3.2 *The following equalities hold in $|w| > |z-w|$:*

$$
\begin{aligned}
(i)\quad & \sum_{k\in\mathbf{Z}} \frac{\langle (: t(w)u(w) :) \circ_k v(w)\rangle}{(z-w)^{k+1}} \\
&= \sum_{n,m\geq 0} \frac{\langle : \partial^m t(w)\ u(w) \circ_n v(w) :\rangle \pm \langle : \partial^m u(w)\ t(w) \circ_n v(w) :\rangle}{m!(z-w)^{n-m+1}} \\
&\pm \sum_{n,m\geq 0} \frac{\langle u(w) \circ_n (t(w) \circ_m v(w))\rangle}{(z-w)^{n+m+2}} \\
&+ \sum_{m\geq 0} \frac{\langle : \partial^m (t(w)u(w))\ v(w) :\rangle}{m!(z-w)^{-m}} \qquad (3.16) \\
(ii)\quad & \pm \sum_{k\in\mathbf{Z}} \frac{\langle t(w)\circ_k : u(w)v(w) :\rangle}{(z-w)^{k+1}} \\
&= \sum_{n,m\geq 0} (-1)^{n+1} \frac{\langle u(w) \circ_n (t(w) \circ_m v(w))\rangle}{(z-w)^{n+m+2}} \\
&+ \sum_{n,m\geq 0} (-1)^{n+1} \frac{\langle : \partial^m (u(w) \circ_n t(w)) v(w) :\rangle}{m!(z-w)^{n-m+1}} \\
&+ \sum_{n\geq 0} \frac{\langle : u(w)\ t(w) \circ_n v(w) :\rangle}{(z-w)^{n+1}} \\
&+ \sum_{m\geq 0} \frac{\langle : u(w)(\partial^m t(w)) v(w) :\rangle}{m!(z-w)^{-m}} \qquad (3.17)
\end{aligned}
$$

Proof: To prove (i), consider matrix coefficients on both sides of eqn. (3.14). By assumption of commutativity these matrix coefficients represent rational functions. Expanding both sides using Lemma 2.6, we get the first eqn. (i). The eqn. (ii) is derived similarly from (3.15). □

By reading off coefficients of the $(z-w)^i$, we can use this lemma to simultaneously compute all products $: t(w)u(w) : \circ_k v(w)$, and $t(w)\circ_k : u(w)v(w) :$ in terms of other products among the constituents $t(w), u(w), v(w)$. Thus it is a kind of recursion relation for the products. In the examples below, we will see how it allows us to understand the structure of commutative QOAs.

Lemma 3.3 *If $t(z)^{\pm}u(w)^{\pm} = (-1)^{|t||u|}u(w)^{\pm}t(z)^{\pm}$, then* $: t(z)u(w)v(x) := (-1)^{|t||u|} : u(w)t(z)v(x) :$.

Proof: Applying the definition of the Wick product (and surpressing z, w, x):

$$
\begin{aligned}
&: tuv : -(-1)^{|t||u|} : utv : \\
&= t^-(u^-v + (-1)^{|u||v|}vu^+) + (-1)^{|t|(|u|+|v|)}(u^-v + (-1)^{|u||v|}vu^+)t^+ \\
&-(-1)^{|t||u|}\left(u^-(t^-v + (-1)^{|t||v|}vt^+) + (-1)^{|u|(|y|+|v|)}(t^-v + (-1)^{|t||v|}vt^+)u^+\right) \\
&= 0. \quad \square \qquad (3.18)
\end{aligned}
$$

3.1 Examples

Let $QO(V)^- = \{u(z)^- \mid u(z) \in QO(V)\}$. This space is obviously closed under differentiation and the Wick product. It follows that the space is also closed under all $\circ_n$, n negative. Also observe that for any $u(z), v(z) \in QO(V)$, we have $u(z)^- v(w)^- =: u(z)^- v(w)^- :$. It follows that the products $\circ_n$, $n = 0, 1, ...$, restricted to $QO(V)^-$, all vanish. Thus $QO(V)^-$ is a local QOA.

Let $LO(V)$ be the algebra spanned by homogeneous linear operators on V. We can regard each operator A as a formal series with just the constant term. This makes $LO(V)$ a subspace of $QO(V)$. It is obvious that every $\circ_n$ restricted to $LO(V)$ vanishes except for $n = -1$, in which case $\circ_{-1}$ is the usual product on $LO(V)$. Thus $LO(V)$ is a very degenerate example of a QOA. Obviously, any commutative subalgebra of $LO(V)$ is a commutative QOA.

Let $\mathcal{C}$ be the Clifford algebra with the generators $b(n), c(n)$ $(n \in \mathbf{Z})$ and the relations [13][17][1]

$$\begin{aligned} b(n)c(m) + c(m)b(n) &= \delta_{n,-m-1} \\ b(n)b(m) + b(m)b(n) &= 0 \\ c(n)c(m) + c(m)c(n) &= 0 \end{aligned} \tag{3.19}$$

Let λ be a fixed integer. The algebra $\mathcal{C}$ becomes $\mathbf{Z}$ -bigraded if we define the degrees $|b(n)| = -|c(n)| = -1$, $||b(n)|| = \lambda - n - 1$, $||c(n)|| = -\lambda - n$. Let $\bigwedge^*$ be the graded irreducible $\mathcal{C}$ *-module with generator $\mathbf{1}$ and relations

$$b(m)\mathbf{1} = c(m)\mathbf{1} = 0, \quad m \geq 0 \tag{3.20}$$

Let $b(z), c(z)$ be the quantum operators

$$\begin{aligned} b(z) &= \sum_m b(m) z^{-m-1} \\ c(z) &= \sum_m c(m) z^{-m-1} \end{aligned} \tag{3.21}$$

Let $O(b, c)$ be the smallest QOA containing $b(z), c(z)$.

Proposition 3.4 *The QOA $O(b,c)$ is commutative. It has a basis consisting of the monomials*

$$: \partial^{n_1} b(z) \cdots \partial^{n_i} b(z)\ \partial^{m_1} c(z) \cdots \partial^{m_j} c(z) : \tag{3.22}$$

with $n_1 > ... > n_i \geq 0$, $m_1 > ... > m_j \geq 0$.

Proof: Computing the OPE of $b(z), c(w)$, we have

$$\begin{aligned} b(z)c(w) &= (z-w)^{-1} + : b(z)c(w) : \\ c(w)b(z) &= (w-z)^{-1} + : c(w)b(z) : \\ : b(z)c(w) : &= - : c(w)b(z) : . \end{aligned} \tag{3.23}$$

It follows that $b(z)$ and $c(z)$ commute. Also $b(z), c(z)$ each commutes with itself, hence they form a pairwise commuting set. By Lemma 3.1, they generate a commutative QOA.

If each $u_1(z), ..., u_k(z)$ is of the form $\partial^n b(z)$ or $\partial^m c(z)$, let's call $: u_1(z) \cdots u_k(z) :$ a monomial of degree k. We claim that it's proportional to some monomial (3.22) with $n_1 > ... > n_i \geq 0$, $m_1 > ... > m_j \geq 0$. If $t(z), u(z)$ each is of the form $\partial^n b(z)$ or $\partial^m c(z)$, it is easy to check that $t(z)^{\pm} u(z)^{\pm} = -u(z)^{\pm} t(z)^{\pm}$. It follows from Lemma 3.3 that $: t(z)u(z)v(z) := -u(z)t(z)v(z) :$ for any element $v(z) \in O(b,c)$. This shows that $: u_1(z) \cdots u_k(z) :$ is equal to $(-1)^{\sigma} : u_{\sigma(1)}(z) \cdots u_{\sigma(k)}(z) :$ for any permutation σ of $1, ..., k$.

Let O' be the linear span of the monomials (3.22). We now show that $A \circ_k B \in O'$ for any k and any two monomials A, B; hence $O(b,c) = O'$. We will do a double induction on the degrees of A and B. Case 1: let $A = t(z)$, $B =: u(z)v(z) :$ with $t(z), u(z)$ each monomial of degree 1, and $v(z)$ of any degree. If $v(z) = 1$, then by (3.23) $t(w) \circ_k : u(w)v(z) :\in O'$. By induction on the degree of $v(z)$ and applying Lemma 3.2(ii), we see that $t(w) \circ_k : u(w)v(w) :\in O'$. This shows that $A \circ_k B \in O'$ for A of degree 1, B of any degree. Now case 2: suppose $A =: t(z)u(z) :$, $B = v(z)$, where $t(z)$ is of degree 1 and $u(z), v(z)$ of any degree. By induction on the degree of $u(z)$, it's clear from Lemma 3.2(i) that this case reduces to case 1.

Finally we must show that the monomials (3.22) are linearly independent. Define a map $O(b,c) \to \bigwedge$ by $u(z) \mapsto u(-1)\mathbf{1}$. We see that this map gives a 1-1 correspondence between the set of monomials (3.22) and a basis of $\bigwedge$. This completes the proof. $\Box$.

Exercise 3 *Let $j(z) =: c(z)b(z) :$. Show that $j(z)j(w) = (z-w)^{-2} + : j(z)j(w) :$ by direct computation (Hint: Eqn. (3.14) is a good guide.). Use Lemma 2.8 to conclude that $[j(m), j(n)] = n\delta_{n+m,0}$. Construct a canonical basis of $O(j)$, the QOA generated by $j(z)$ in $O(b,c)$.*

Let $M(\kappa, 0)$ be the Verma module of the Virasoro algebra with highest weight $(\kappa, 0)$ and vacuum vector v_0. Let $M(\kappa)$ be the quotient of $M(\kappa, 0)$ by the submodule generated by $L_{-1}v_0$. Let $O_\kappa(L)$ be the QOA generated by $L(z) = \sum L_n z^{-n-2}$ in $QO(M(\kappa))$.

Proposition 3.5 *(see [3][12]) The QOA $O_\kappa(L)$ is commutative. It has a basis consisting of monomials*

$$: \partial^{n_1} L(z) \cdots \partial^{n_i} L(z) : \tag{3.24}$$

with $n_1 \geq ... \geq n_i \geq 0$.

Proof: A direct computation gives

$$\begin{aligned} [L(z)^+, L(w)] &= \frac{\kappa}{2}(z-w)^{-4} + 2L(w)(z-w)^{-2} + \partial L(w)(z-w)^{-1} \\ [L(z)^-, L(w)] &= -\frac{\kappa}{2}(w-z)^{-4} - 2L(w)(w-z)^{-2} + \partial L(w)(w-z)^{-1}. \end{aligned} \tag{3.25}$$

But we also have $L(z)L(w) = [L(z)^+, L(w)] + : L(z)L(w) :$, and $L(w)L(z) = -[L(z)^-, L(w)] + : L(z)L(w) :$. Combining these with (3.25), it is obvious that $\langle L(z)L(w)\rangle$ and $\langle L(w)L(z)\rangle$ represent the same rational function. Thus $L(z)$ commutes with itself as a quantum operator. By Lemma 3.1, $O_\kappa(L)$ is commutative.

Let O' be the linear span of the monomials (3.24) with $n_1, .., n_i \geq 0$ *unrestricted.* To show O' is closed under all the products (hence $O_\kappa(L) = O'$), we apply induction and Lemma 3.2 as in the case of $O(b,c)$ above. We now show that we can restrict to those monomials (3.24) with $n_1 \geq, ... \geq n_i \geq 0$, and that the resulting monomials form a basis. First by direct computation, we see (see Lemma 4.2 of [24]) that $O_\kappa(L)$ is a Vir-module defined by the action $(L(n) = L_{n-1})$

$$L(n) \cdot u(z) = L(z) \circ_n u(z). \tag{3.26}$$

Since $O_\kappa(L)$ is spanned by the monomials (3.24), and because $L(-n-1)\cdot u(z) = \frac{1}{n!} : \partial^n L(z)u(z) :$ for $n \geq 0$, it follows that the module is cyclic. Thus we have a unique onto map of Vir-modules $M(\kappa) \to O_\kappa(L)$ sending v_0 to 1. But $M(\kappa)$ has a PBW basis consisting of $L(-n_1-1)\cdots L(-n_i-1)v_0$, $n_1 \geq, ... \geq n_i \geq 0$. This shows that the monomials (3.24) with $n_1 \geq, ... \geq n_i \geq 0$ span $O_\kappa(L)$. Now define a map $O_\kappa(L) \to M(\kappa)$ by $u(z) \mapsto u(-1)v_0$. This is the inverse to the previous map, hence it must map a basis to a basis. □

Let $(\mathbf{g}, B)$ be any Lie algebra with an invariant symmetric bilinear form B, possibly degenerate. Let $\hat{\mathbf{g}}$ be the affinization of $(\mathbf{g}, B)$, ie. $\hat{\mathbf{g}} = \mathbf{g}[t, t^{-1}] \oplus \mathbf{C}$ with bracket:

$$[Xt^n, Yt^m] = [X,Y]t^{n+m} + n\delta_{n+m,0}B(X,Y) \tag{3.27}$$

and $\mathbf{C}$ being central. Let M be any $t\mathbf{g}[t]$ locally finite $\hat{\mathbf{g}}$-module in which $1 \in \hat{\mathbf{g}}$ acts by the scalar 1. Denote by $X(n)$ the operator representing Xt^n, and define the *currents*:

$$X(z) = \sum_{n\in\mathbf{Z}} X(n)z^{-n-1} \tag{3.28}$$

for $X \in \mathbf{g}$. Let O be the QOA generated by all the currents in $QO(M)$.

Proposition 3.6 *The QOA O is commutative. It is spanned by the monomials* $: \partial^{n_1}X_1(z)\cdots\partial^{n_i}X_i(z) :$, *with* $n_1, ..., n_i \geq 0$, $X_1, ..., X_i \in \mathbf{g}$.

Proof: For $X, Y \in \mathbf{g}$, we have

$$\begin{aligned} [X(z)^+, Y(w)] &= B(X,Y)(z-w)^{-2} + [X,Y](w)(z-w)^{-1} \\ [X(z)^-, Y(w)] &= -B(X,Y)(w-z)^{-2} + [X,Y](w)(w-z)^{-1}. \end{aligned} \tag{3.29}$$

It follows that the currents are pairwise commuting quantum operators, and hence O is commutative.

To prove the second statement, it's enough to show that for any integer n and any two monomials A, A' above, $A \circ_n A'$ is a linear sum of those monomials. This follows by induction on the degrees of A, A' (ie. the number X occuring in A, A') and by applying Lemma 3.2. □

There is a vast literature closely related to the study of the algebra O above. For a small sample, see for example [20][35][32][12][23][22].

4 BRST cohomology algebras

Definition 4.1 *A conformal QOA with central charge κ is a pair (O,f), where O is a commutative QOA equipped with a homomorphism $f: O_\kappa(L) \to O$ such that for every homogeneous $u(z) \in O$,*

$$fL(z)u(w) = \cdots + ||u||u(w)(z-w)^{-2} + \partial u(w)(z-w)^{-1} + : fL(z)u(w) : \tag{4.30}$$

where "$\cdots$" denotes the higher order polar terms. In other words, $fL(z) \circ_1 u(z) = ||u||u(w)$ and $fL(z) \circ_0 u(z) = \partial u(z)$. For simplicity, we sometimes write $f: O_\kappa(L) \to O$, or simply $O_\kappa(L) \to O$, to denote a conformal QOA. A homomorphism $(O,f) \xrightarrow{h} (O',f')$ of conformal QOAs is a homomorphism of QOAs $h: O \to O'$ such that $h \circ f = f'$.

Recall that M is a *positive energy Vir-module* of central charge κ if for every $v \in M$, we have $L_n \cdot v = 0$ for $n >> 0$, and L_0 acts diagonalizably. It's easy to show that M is a positive energy Vir-module iff there's a quantum operator $X(z) \in QO(M)$ which commutes with itself and has the OPE

$$X(z)X(w) = \frac{\kappa}{2}(z-w)^{-4} + 2X(w)(z-w)^{-2} + \partial X(w)(z-w)^{-1} + : X(z)X(w) : . \tag{4.31}$$

Lemma 4.2 *Let $X(z) \in QO(M)$ define a positive energy Vir-module. Then every subalgebra of $QO(M)$ containing $X(z)$ is naturally a positive energy Vir-module defined by $L_n \cdot u(z) = X(z) \circ_{n+1} u(z)$.*

Lemma 4.3 *Let O be a commutative QOA generated by a set $S \subset QO(M)$, and let $X(z) \in O$ have OPE* (4.31)*. Suppose for all $u(z) \in S$,*

$$X(z)u(w) = \cdots + ||u||u(w)(z-w)^{-2} + \partial u(w)(z-w)^{-1} + : X(z)u(w) : . \tag{4.32}$$

Then there's a unique homomorphism $f: O_\kappa(L) \to O$ such that $fL(z) = X(z)$. Moreover (O,f) is a conformal QOA with central charge κ. In particular, every positive energy Vir-module M has a canonical $O_\kappa(L)$-module structure.

We refer the interested readers to section 4 of [24] for the complete proofs.

Consider, as an example, $O(b,c)$. For a fixed λ, let

$$X(z) = (1-\lambda) : \partial b(z)\ c(z) - \lambda : b(z)\partial c(z) : \tag{4.33}$$

and $S = \{b(z), c(z)\}$. Then we have, by direct computation [13],

$$\begin{aligned} X(z)b(w) &= \lambda b(w)(z-w)^{-2} + \partial b(w)(z-w)^{-1} + : X(z)b(w) : \\ X(z)c(w) &= (1-\lambda)c(w)(z-w)^{-2} + \partial c(w)(z-w)^{-1} + : X(z)c(w) : \\ X(z)X(w) &= \frac{\kappa}{2}(z-w)^{-4} + 2X(w)(z-w)^{-2} + \partial X(w)(z-w)^{-1} + : X(z)X(w) : \end{aligned} \tag{4.34}$$

where $\kappa = -12\lambda^2 + 12\lambda - 2$. It follows that we have a conformal QOA $f_\lambda : O_\kappa(L) \to O(b,c)$.

Similarly $id : O_\kappa(L) \to O_\kappa(L)$ is itself a conformal QOA. Thus by definition, it is the initial object in the category of conformal QOAs with central charge κ.

Exercise 4 *In the previous exercise, we define $j(z) =: c(z)b(z):$ which has $||j|| = 1$. Compute the generating function $\sum_n \dim\ O(j)[n]q^n$.*

Exercise 5 *Use Lemma 4.3 to classify the homomorphisms $f : O_\kappa(L) \to O(j)$.*

Exercise 6 *Show that $O(j)$ coincides with the subalgebra $O(b,c)^0$ of all elements of $u(z) \in O(b,c)$ with $|u| = 0$. (Hint: Compute the generating function for the graded dimensions of $O(b,c)^0$. Alternatively for the readers who know it, you can use the so-called boson-fermion correspondence.)*

Exercise 7 *Use the last two exercises to classify the (for each λ) the homomorphisms $f : O_\kappa(L) \to O(b,c)$.*

4.1 The BRST construction

It is evident that if (O,f), (O',f') are conformal QOAs on the respective spaces V, V' with central charges κ, κ', then $(O \otimes O', f \otimes f')$ is a conformal QOA on $V \otimes V'$ with central charge $\kappa + \kappa'$. From now on we fix $\lambda = 2$ which means that $(O(b,c), f_\lambda)$ now has central charge -26. Let (O,f) be any conformal QOA with central charge κ and consider

$$C^*(O) = O(b,c) \otimes O \tag{4.35}$$

where $*$ means the total first degree. For simplicity, we write $\hat{f} = f_\lambda \otimes f$.

Proposition 4.4 *For every (O,f), there is a unique homogeneous element $J_f(z) \in C^*(O)$ with the following properties:*
(i) (Cartan identity) $J_f(z)b(w) = \hat{f}L(w)(z-w)^{-1} + :J_f(z)b(w):$.
(ii) (Universality) If $(O,f) \to (O',f')$ is a homomorphism of conformal QOAs, then the induced homomorphism $C^*(O) \to C^*(O')$ sends $J_f(z)$ to $J_{f'}(z)$.

Proof: Since the category of conformal QOAs with central charge κ has $(O_\kappa(L), id)$ as the initial object, if we can show that there is a unique J_{id} satisfying property (i), then (ii) implies that the same holds for every other object in that category.

Property (i) implies $|J_{id}| = 1 = ||J_{id}||$. Let's list a basis of the degree $(1,1)$ subspace of $C(O_\kappa(L))$ given by Propositions 3.4, 3.5: $:c(z)L(z):$, $:b(z)c(z)\partial c(z):$, $\partial^2 c(z)$. Take a linear combination of these elements and compute its OPE with $b(z)$. Requiring property (i), we determine the coefficients of the linear combination and get

$$J_{id}(z) =: c(z)L(z) : + : b(z)c(z)\partial c(z) : . \tag{4.36}$$

Now given a conformal QOA (O, f), the induced map $f^* : C^*(O_\kappa(L)) \to C^*(O)$ sends $J_{id}(z)$ to $J_f(z) =: c(z)fL(z) : + : b(z)c(z)\partial c(z) :$. This completes our proof. □.

It follows from property (i) that

$$\hat{f}L(z) = J_f(z) \circ_0 b(z) = [Q_f, b(z)] \tag{4.37}$$

where $Q_f = Res_z J_f(z)$.

Lemma 4.5 *[18][7][9] Given $f : O_\kappa(L) \to O$, we have*

$$J_f(w) \circ_0 J_f(w) = \frac{3}{2}\partial(\partial^2 c(w)\ c(w)) + \frac{\kappa - 26}{12}\partial^3 c(w)\ c(w) \tag{4.38}$$

Thus $Q_f^2 = 0$ iff $\kappa = 26$.

Proof: We'll drop the subscripts for J_f, Q_f and write $fL(z)$ as $L(z)$. Since $J(w) \circ_0 J(w)$ is the coefficient of $(z-w)^{-1}$ in the OPE $J(z)J(w)$, we can extract this term from the OPE. Now $J(z)J(w)$ is the sum of 4 terms:

$$\begin{array}{ll} (i) & c(z)L(z)c(w)L(w) \\ (ii) & c(z)L(z) : b(w)c(w)\partial c(w) : \\ (iii) & : b(z)c(z)\partial c(z) : c(w)L(w) \\ (iv) & : b(z)c(z)\partial c(z) :: b(w)c(w)\partial c(w) : . \end{array} \tag{4.39}$$

Extracting the coefficient of $(z-w)^{-1}$ (which is done by applying Lemma 3.2 repeatedly) in each of these 4 OPEs, we get respectively (surpressing w):

$$\begin{array}{ll} (i) & 2\partial c\, cL + \frac{\kappa}{12}\partial^3 c\ c \\ (ii) & c\partial c\ L \\ (iii) & c\partial c\ L \\ (iv) & \frac{3}{2}\partial(\partial^2 c\ c) - \frac{13}{6}\partial^3 c\ c \end{array} \tag{4.40}$$

Thus (4.38) follows. Now $2Q^2 = [Q, Q] = Res_w[Q, J(w)] = Res_w J(w) \circ_0 J(w)$, which is zero iff $\kappa = 26$. □

Given $f : O_{26}(L) \to O$, $[Q_f, -] = J_f(z)\circ_0$ is a derivation of the QOA $C^*(O)$ (Lemma 2.2). For $\kappa = 26$, which we assume from now on, $[Q_f, -]$ becomes a differential on $C^*(O)$ and we have a cochain complex

$$[Q_f, -] : C^*(O) \longrightarrow C^{*+1}(O). \tag{4.41}$$

It is called *the BRST complex* associated to $f : O_{26}(L) \to O$. Its cohomology will be denoted as $H^*(O)$. All the operations $\circ_n$ on $C^*(O)$ descend to the cohomology. However, all but one is trivial.

Theorem 4.6 *[33][37][26] The Wick product $\circ_{-1}$ induces a graded commutative associative product on $H^*(O)$ with unit element represented by the identity operator. Moreover, every cohomology class is represented by a quantum operator $u(z)$ with $||u|| = 0$.*

Exercise 8 *Check that* 1 *represents the unit of the commutative algebra $H^*(O)$. Show that for all $n \neq -1$, $\circ_n$ is homologically trivial, ie. if $u(z), v(z)$ represent two cohomology classes, then $u(z) \circ_n v(z) = [Q_f, t(z)]$ for some $t(z)$. (Hint: Recall that $\partial A(w) = [Q_f, b(w)] \circ_0 A(w)$ and $||A||A(w) = [Q_f, b(w)] \circ_1 A(z)$.)*

4.2 Batalin-Vilkovisky Algebras

Let A^* be a $\mathbf{Z}$ graded commutative associative algebra. For every $a \in A$, let l_a denote the linear map on A given by the left multiplication by a. Recall that a (graded) derivation d on A is a homogeneous linear operator such that $[d, l_a] - l_{da} = 0$ for all a. A BV operator [34][31][15] Δ on A^* is a linear operator of degree -1 such that:
(i) $\Delta^2 = 0$;
(ii) $[\Delta, l_a] - l_{\Delta a}$ is a derivation on A for all a, ie. Δ is a *second* order derivation.

A BV algebra is a pair (A, Δ) where A is a graded commutative algebra and Δ is a BV operator on A. The following is an elementary but fundamental lemma:

Lemma 4.7 *[21][15][30] Given a BV algebra (A, Δ), define the BV bracket $\{,\}$ on A by:*

$$(-1)^{|u|}\{u, v\} = [\Delta, l_u]v - l_{\Delta u}v.$$

Then $\{,\}$ is a graded Lie bracket on A of degree -1, ie.

$$\begin{aligned}
&\{u, v\} + (-1)^{(|u|-1)(|v|-1)}\{v, u\} = 0 \\
&(-1)^{(|u|-1)(|t|-1)}\{u, \{v, t\}\} + (-1)^{(|t|-1)(|v|-1)}\{t, \{u, v\}\} + (-1)^{(|v|-1)(|u|-1)}\{v, \{t, u\}\} = 0
\end{aligned}$$

By property (ii) above, it follows immediately that for every $u \in A$, $\{u, -\}$ is a derivation on A. Thus a BV algebra is a special kind of an odd Poisson algebra which, in mathematics, is also known as a Gerstenhaber algebra [14]. It's important to note that A^1 *is canonically a Lie algebra and that each A^p is an A^1-module.*

Exercise 9 *Let* $\mathbf{g}$ *be an Lie algebra, $\wedge^*\mathbf{g}$ its exterior algebra and δ the Chevalley-Eilenberg differential on $\wedge^*\mathbf{g}$. Check that $(\wedge^*\mathbf{g}, \delta)$ is a BV algebra. Show that $\wedge^1\mathbf{g}$ is canonically isomorphic to* $\mathbf{g}$ *as a Lie algebra.*

Exercise 10 *More generally, let B be any commutative algebra and $f : \mathbf{g} \to Der\ B$ be a Lie algebra homomorphism (making B a $\mathbf{g}$-module). Consider the Lie algebra homology complex $\wedge^*\mathbf{g} \otimes B$. Show that the Chevalley-Eilenberg differential is a BV operator on this complex.*

Given $f : O_{26}(L) \to O$, consider the linear operator $\Delta_f : C^*(O) \longrightarrow C^{*-1}(O)$, $u(z) \mapsto b(z) \circ_1 u(z)$.

Theorem 4.8 *[26] The operator Δ_f descends to the cohomology $H^*(O)$. Morever, it is a BV operator on the commutative algebra $H^*(O)$. Thus $H^*(O)$ is naturally a BV algebra.*

For complete proofs of the two theorems above, see section 4 of [24]. The theorems were originally proved in [26] in the context of vertex operator algebras. (For related versions of Theorem 4.8, see [15][31][19][16].)

Program 4.9 *Study $H^*(O \otimes O')$ as a bifunctor from pairs of conformal QOAs to BV algebras. In particular, fix O and study $H(O \otimes -)$ as a functor from conformal QOAs to BV algebras. An automorphism group of O acts by natural automorphisms on the functor $H(O \otimes -)$.*

4.3 Modules

Consider $f : O_{26}(L) \to O$, and an O-module M equipped with the structure homomorphism $g : O \to QO(M)$. The homomorphism g induces $g^* : C^*(O) \to QO(\wedge \otimes M)$. We write $g^* J_f(z)$ as $J_{f,g}(z)$, its residue as $Q_{f,g}$, and the $C^*(O)$-module $\wedge \otimes M$ as $C^*(O, M)$. By Lemma 4.5, $Q_{f,g}$ turns $C^*(O, M)$ into a complex whose cohomology is denoted as $H^*(O, M)$. It turns out that $H^*(O, M)$ is a module over the BV algebra $H^*(O)$ in a suitable sense. This will be the topic of a future paper.

Let N be a positive energy Vir-module of central charge 26. By Lemma 4.3, we have a canonical homomorphism $g : O_{26}(L) \to QO(N)$. This makes $\wedge \otimes N$ into a $C^*(O_{26}(L))$-module with BRST differential $Q_{id,g}$. On the other hand, $\wedge \otimes N$ is by definition the space of semi-infinite cochains $C^{\frac{\infty}{2}+*}(Vir, N)$ of Vir with coefficients in N (see [7][9]). The differential of this cochain complex is denoted by d_N, and its cohomology as $H^{\frac{\infty}{2}+*}(Vir, N)$. It's easily seen that we have $d_N = Q_{id,g}$ [9]. It follows that we have

$$H^*(O_{26}(L), N) \cong H^{\frac{\infty}{2}+*}(Vir, N). \tag{4.42}$$

Now given a conformal QOA (O, f) and an O-module M equipped with the homomorphism $g : O \to QO(M)$, we can regard M as an $O_{26}(L)$-module via $g \circ f : O_{26}(L) \to QO(M)$. It follows that $(C^*(O, M), Q_{f,g}) = (C^*(O_{26}(L), M), Q_{id, g\circ f}) = (C^{\frac{\infty}{2}+*}(Vir, M), d_M)$ as complexes.

Recall the linear isomorphism $O(b, c) \xrightarrow{\sim} \wedge$, $u(z) \mapsto u(-1)\mathbf{1}$ (see proof of Lemma 3.4). Given a conformal QOA (O, f), we have $C^*(O) = O(b, c) \otimes O \xrightarrow{\sim} \wedge \otimes O$. Call this map k. We claim that the differential induced on $\wedge \otimes O$ via k coincides with the semi-infinite differential d_O. We must check that $J_f(z) \circ_0 (u(z) \otimes v(z)) \stackrel{k}{\mapsto} d_O(u(-1)\mathbf{1} \otimes v(z))$.

Let $a(z) =: b(z)c(z)\partial c(z) :$. It acts only on $\wedge$, and hence $a(z) \circ_0 (u(z) \otimes v(z)) = [a(0), u(z)] \otimes v(z) \stackrel{k}{\mapsto} a(0)u(-1)\mathbf{1} \otimes v(z)$. Use Lemma 2.6 to compute the OPE $c(z)fL(z)(u(w) \otimes v(w))$ and get $c(z)fL(z) \circ_0 (u(z) \otimes v(z)) = \sum(c(z) \circ_n u(z)) \otimes (fL(z) \circ_{-n-1} v(z))$. It's also easy to check, using eqn. (2.4), that under $O(b,c) \to \wedge$, we have $c(z) \circ_n u(z) \mapsto c(n)u(-1)\mathbf{1}$. It follows that

$$\begin{aligned} J_f(z) \circ_0 (u(z) \otimes v(z)) \stackrel{k}{\mapsto} \quad & \sum c(n)u(-1)\mathbf{1} \otimes L(-n-1) \cdot v(z) + a(0)u(-1)\mathbf{1} \otimes v(z) \\ & = d_O(u(-1)\mathbf{1} \otimes v(z)). \end{aligned} \tag{4.43}$$

The last equality follows from the definition of d_O. Thus we've shown that

Lemma 4.10 *Given a conformal QOA (O, f) and an O-module M equipped with homomorphism $g : O \to QO(M)$, we have*
(i) $(C^*(O), Q_f) \cong (C^{\frac{\infty}{2}+*}(Vir, O), d_O)$.
(ii)$(C^*(O, M), Q_{f,g}) = (C^{\frac{\infty}{2}+*}(Vir, M), d_M)$.

5 Moonshine Cohomology

We'll now construct a functor $\mathbf{M}$ using the Moonshine VOA of rank 24 and the BRST construction above. This will be a functor from the category of conformal QOAs of central charge 2 to the category of BV algebras (Program 4.9). In particular, it assigns a Lie algebra $\mathbf{M}^1(O)$ to every such conformal QOA O. As a special case, if O is the conformal QOA corresponding to the unimodular rank 2 hyperbolic lattice $II_{1,1}$ (see below), then $\mathbf{M}^1(O)$ is Borcherds' Monster Lie algebra.

Let $(V, \mathbf{1}, \omega, Y(-, z))$ be the Moonshine VOA as studied by Frenkel-Lepowsky-Meurman and Borcherds [10][4][11] (see Definitions 8.10.1-8.10.18 of [11]). It's now known that
(i) The Fischer-Griess Monster finite group F_1 is the automorphism group of the VOA V.
(ii) $\sum dim\ V[n]q^{n-1} = j(q) - 744$ where $j(q)$ is the Dedekind-Klein j-function.
(iii) $Y(\omega, z)$ defines a unitarizable Vir-module structure on V.

Proposition 5.1 *(see [25]) Let $(U, \mathbf{1}, \omega, Y(-, z))$ be a VOA of rank κ. Let $O(U)$ be the linear space of vertex operators, ie. $O(U) = \{Y(a, z) | a \in U\}$. Then $O(U) \subset QO(U)$ is a conformal QOA with $O_\kappa(L) \to O(U)$, $L(z) \mapsto Y(\omega, z)$. Moreover, $O(U)$ has an $O(U)$-module structure $O(U) \to QO(O(U))$ defined by $u(z) \mapsto \hat{u}(z)$, $\hat{u}(n) \cdot v(z) \stackrel{def}{=} u(z) \circ_n v(z)$.*

Proof: By Lemma 2.6 above and Proposition 8.10.5 of [11], we have
$\sum\langle Y(u, w) \circ_n Y(v, w)\rangle(z-w)^{-n-1} = \langle Y(Y(u, z-w)v, w)\rangle$. Thus $O(U)$ is closed under all the

products $\circ_n$. We also see that $O(U)$ has an $O(U)$-module structure as claimed. By Proposition 8.10.3 of [11], $O(U)$ is commutative. By definition, $Y(\omega, z) = \sum \omega(n) z^{-n-1}$ satisfies, for all u, $Y(\omega(0)u, z) = \partial Y(u, z)$, $\omega(1)u = ||u||u$, $\omega(2)\omega = \frac{\kappa}{2}\omega$. By Lemma 4.3, this means that $O(U)$ is a conformal QOA as claimed. □

It follows immediately that the linear bijection $U \to O(U)$, $a \mapsto Y(a, z)$ is an isomorphism of $O(U)$-modules. It's also clear that any automorphism of the VOA U yields an automorphism of the conformal QOA $O(U)$. In the case of the Moonshine VOA V, it follows that
(iv) $O(V)$ is a conformal QOA of central charge 24, in which F_1 acts by automorphisms.

Thus for any conformal QOA O of central charge 2, we can consider the BRST QOA $C^*(O(V) \otimes O)$. We denote its cohomology as $\mathbf{M}^*(O)$, which by Theorem 4.8 is a BV algebra. We call $\mathbf{M}^*(O)$ *the Moonshine cohomology of* O. If M is an O-module, then $V \otimes M$ is naturally an $O(V) \otimes O$-module. It follows from the preceeding section that $C^*(O(V) \otimes O, V \otimes M)$ is a cochain complex. We denote its cohomology as $\mathbf{M}^*(O, M)$, which we call *the Moonshine cohomology of* (O, M). It follows from (iv) above that F_1 is a group of automorphisms of both $\mathbf{M}^*(O)$ and $\mathbf{M}^*(O, M)$ (Program 4.9).

5.1 Vanishing Theorem

A Vir-module is *tame* if it's graded dimensions are finite. A Vir-module is *hermitean* if it's a direct sum of a tame positive energy modules equipped with an invariant nondegenerate hermitean form. A hermitean Vir-module is *unitarizable* if its hermitean form is positive definite. *Unless specified otherwise, Vir-modules and conformal QOAs from now on are assumed to have the first degree* $|\cdot| \equiv 0$.

Theorem 5.2 *For any conformal QOA O of central charge 2, and any O-module M, $\mathbf{M}^p(O)$, $\mathbf{M}^p(O, M)$ vanish for all $p \neq 0, 1, 2,$ or 3.*

Theorem 5.3 *Let r be a real number with $1 < r < 25$. Let P and N be positive energy Vir-modules of central charges $26 - r, r$ respectively. Assume P is unitarizable. Then $H^{\frac{\infty}{2}+p}(Vir, P \otimes N)$ vanishes for all $p \neq 0, 1, 2,$ or 3.*

Proof: By the unitarizability of P, it's a direct sum of irreducible modules $L(26 - r, h)$, $h \geq 0$, with suitable multiplicities. Thus it's enough to do the case $P = L(26 - r, h)$.

Recall that N is a Vir-module of central charge r, in which L_0 acts diagonalizably. Since every cohomology class in $H^{\frac{\infty}{2}+*}(Vir, L(26 - r, h) \otimes N) = 0$ is represented by an element of zero weight, we may assume, without loss of generality, that L_0 only has real eigenvalues in N. Thus any irreducible module $L(r, k)$ occuring in the composition series of N must have real k. ¿From the structure of the Verma modules, $L(r, k) = M(r, k)$ unless $k = 0$, and $L(26 - r, h) = M(26 - r, h)$ for $h > 0$.

By our reduction theorem on semi-infinite cohomology [27], for $k \neq 0$ or $h > 0$, we have $H^{\frac{\infty}{2}+p}(Vir, L(26-r,h) \otimes L(r,k)) = 0$ for $p \neq 1,2$. It's easy to verify that $H^p(O_{26}(L), L(26-r,0) \otimes L(r,0))$ is zero if $p \neq 0,3$, and one dimensional if $p = 0,3$. Thus if N is a module of finite length, then $H^{\frac{\infty}{2}+p}(Vir, L(24,0) \otimes N) = 0$ for $p \neq 0,..,3$. Every finitely generated positive energy Vir-module of central charge r has finite length. Now any module is a direct limit of finitely generated modules, and direct limit is exact with respect to cohomology. If follows that $H^{\frac{\infty}{2}+p}(Vir, L(24,0) \otimes N) = 0$ for $p \neq 0,..,3$. This completes the proof. □

Proof of Theorem 5.2: Specialize Theorem 5.3 to the case $r = 2$, $P = O(V)$ (which is unitarizable), $N = O$, and applying Lemma 4.10, we see that $\mathbf{M}^p(O)$ vanishes for all $p \neq 0, 1, 2,$ *or* 3. For $N = M$, we have a similar statement for $\mathbf{M}^p(O, M)$. □

6 Moonshine cohomology and the Monster Lie algebra

Let M be a positive energy Vir-module. Let $Vir_\pm$ be respectively the subalgebras spanned by the L_n, $\pm n > 0$. Define the *physical space associated to* M:

$$\mathbf{P}(M) = M[1]^{Vir_+}/N(M) \tag{6.44}$$

where $N(M) = (Vir_- \cdot M) \cap M[1]^{Vir_+}$. If M is a hermitean module of central charge 26, then there are two natural linear maps $\nu_i : \mathbf{P}(M) \to H^{\frac{\infty}{2}+i}(Vir, M)$, $i = 1, 2$, given respectively by $v \mapsto c(-1)v$, $v \mapsto c(-2)c(-1)v$ (see [26] section 2.4 for details). To emphasize their dependence on M, we'll refer to these maps as ν_1, ν_2 *for the module* M. Let O be a conformal QOA. Suppose the Vir-module structure $f : O_\kappa(L) \to QO(O)$ on O, given by Lemma 4.2, is hermitean. Then it makes sense to consider the maps ν_1, ν_2 for O.

If $u(z) \in O^{Vir_+}$ then it's easy to show using commutativity that $u(z) \circ_0 fL(z) = fL(z) \circ_0 u(z) - \partial u(z) = 0$. This implies that

$$u(z) \circ_0 (fL(z) \circ_n v(z)) = fL(z) \circ_n (u(z) \circ_0 v(z)). \tag{6.45}$$

For $v(z) \in O^{Vir_+}$, this shows that $u(z) \circ_0 v(z) \in O^{Vir_+}$. For $v(z) \in N(O)$, it shows that $u(z) \circ_0 v(z) \in N(O)$. Using commutativity, we show that $u(z) \circ_0 v(z) + v(z) \circ_0 u(z) = \partial A(z)$ for some $A(z)$. Thus $\circ_0$ is a skew symmetric product on O^{Vir_+} modulo $N(O)$, and it also factors through $N(O)$. The fact that $u(z)\circ_0$ is a derivation of the product $v(z) \circ_0 t(z)$ says exactly that the skew symmetric operation $\circ_0$ on $\mathbf{P}(O)$ satisfies that Lie algebra Jacobi identity. Thus $\mathbf{P}(O)$ is a Lie algebra with bracket $\circ_0$. We'll use the convention that $-u(z) \circ_0 v(z)$ is the Lie bracket of $u(z)$ with $v(z)$.

If O has central charge 26, then the maps ν_i together with Lemma 4.10 yield two new maps (which we also call ν_i), $\nu_i : \mathbf{P}(O) \to H^i(O)$. The bracket in $H^1(O)$ can be written as $\{A(z), B(z)\} = (-1)^{|A|}(b(z) \circ_0 A(z)) \circ_0 B(z)$. (see [24] section 5 for details). Thus,

$$\{\nu_1 u(z), \nu_1 v(z)\} \quad = \quad \{c(z)u(z), c(z)v(z)\}$$

$$
\begin{aligned}
&= -u(z) \circ_0 (c(z)v(z)) \\
&= \nu_1(-u(z) \circ_0 v(z)).
\end{aligned} \tag{6.46}
$$

Thus ν_1 is a Lie algebra homomorphism. Since $H^2(O)$ is canonically a $H^1(O)$-module, it becomes a $\mathbf{P}(O)$-module via ν_1. But we also have

$$\{\nu_1 u(z), \nu_2 v(z)\} = \nu_2(-u(z) \circ_0 v(z)). \tag{6.47}$$

Thus ν_2 is a $\mathbf{P}(O)$-module homomorphism from the adjoint module $\mathbf{P}(O)$ to $H^2(O)$. To emphasize their dependence on the QOA O, we'll refer to those homomorphisms as ν_1, ν_2 *for the QOA O*. To summarize,

Lemma 6.1 *Given a hermitean Vir-module M of central charge 26, we've two linear maps ν_i : $\mathbf{P}(M) \to H^{\frac{\infty}{2}+i}(Vir, M) = H^i(O_{26}(L), M)$, $i = 1, 2$, given by $u \mapsto c(-1)u$, $u \mapsto c(-2)c(-1)v$ respectively. Given a hermitean conformal QOA O of central charge 26, we've a Lie algebra homomorphism and a module homomorphism $\nu_i : \mathbf{P}(O) \to H^i(O)$, $i = 1, 2$, given by $u(z) \mapsto c(z)u(z)$, $u(z) \mapsto \partial c(z)\ c(z)u(z)$ respectively.*

Let Λ be any rank 2 hyperbolic even integral lattice, and $(V_\Lambda, \mathbf{1}_\Lambda, \omega_\Lambda, Y_\Lambda(-, z))$ be the canonical rank 2 VOA associated to Λ [11]. The Vir-module $O(V_\Lambda) \cong V_\Lambda$ is a direct sum of the so-called Fock modules, which are hermitean. By the lemma above, we have $\nu_i : \mathbf{P}(O(V) \otimes O(V_\Lambda)) \to H^i(O(V) \otimes O(V_\Lambda)) = \mathbf{M}^i(O(V_\Lambda))$, $i = 1, 2$.

Theorem 6.2 *The homomorphisms ν_1, ν_2 for the QOA $O(V) \otimes O(V_\Lambda)$ are isomorphisms, and we've*

$$\mathbf{M}^p(O(V_\Lambda)) = \begin{cases} \mathbf{C}\ 1 & \text{if } p = 0 \\ \nu_1 \mathbf{P}(O(V) \otimes O(V_\Lambda)) & \text{if } p = 1. \\ \nu_2 \mathbf{P}(O(V) \otimes O(V_\Lambda)) & \text{if } p = 2. \\ \mathbf{C}\ \partial^2 c(z) \partial c(z) c(z) & \text{if } p = 3 \\ 0 & \text{otherwise.} \end{cases} \tag{6.48}$$

Corollary 6.3 *Let Λ be the unimodular lattice $II_{1,1}$. Then $\mathbf{M}^1(O(V_\Lambda))$ is canonically isomorphic to the Monster Lie algebra, and $\mathbf{M}^2(O(V_\Lambda))$ to the adjoint module.*

Proof: By definition [5], the Monster Lie algebra has as its underlying space $\mathbf{P}(V \otimes V_\Lambda)$, and its bracket $[u, v] = -Res_z Y(u, z)v$. Now $\mathbf{P}(O(V) \otimes O(V_\Lambda)) \cong \mathbf{P}(V \otimes V_\Lambda)$ follows from Proposition 5.1. □

The rest of this paper is devoted to proving and generalizing the theorem above.

6.1 Hyperbolic lattices

Let Λ be a rank $r \leq 26$ even integral Lorentzian lattice, and $(V_\Lambda, \mathbf{1}_\Lambda, \omega_\Lambda, Y_\Lambda(-,z))$ the canonical rank r VOA associated to Λ [11]. This VOA has rank r. Let O be any conformal QOA of central charge $26 - r$ such that it's unitarizable as a Vir-module.

Theorem 6.4 *Under the above assumptions the homomorphisms ν_1, ν_2 for the QOA $O \otimes O(V_\Lambda)$ are isomorphisms, and*

$$H^p(O \otimes O(V_\Lambda)) = \begin{cases} Hom_{Vir}(L(26-r,0), O) & \text{if } p = 0 \\ \nu_1 \mathbf{P}(O \otimes O(V_\Lambda)) & \text{if } p = 1. \\ \nu_2 \mathbf{P}(O \otimes O(V_\Lambda)) & \text{if } p = 2. \\ \partial^2 c(z) \partial c(z) c(z) Hom_{Vir}(L(26-r,0), O) & \text{if } p = 3 \\ 0 & \text{otherwise.} \end{cases} \tag{6.49}$$

Theorem 6.2 is clearly an immediate consequence when $r = 2$ and $O = O(V)$. By Lemma 4.10, Theorem 6.4 is equivalent to

Theorem 6.5 *Under the above assumptions the linear maps ν_1, ν_2 for the module $O \otimes V_\Lambda$ are bijections, and*

$$H^{\frac{\infty}{2}+p}(Vir, O \otimes V_\Lambda) = \begin{cases} Hom_{Vir}(L(26-r,0), O) & \text{if } p = 0 \\ \nu_1 \mathbf{P}(O \otimes V_\Lambda) & \text{if } p = 1. \\ \nu_2 \mathbf{P}(O \otimes V_\Lambda) & \text{if } p = 2. \\ c(-3)c(-2)c(-1) Hom_{Vir}(L(26-r,0), O) & \text{if } p = 3 \\ 0 & \text{otherwise.} \end{cases} \tag{6.50}$$

We now proceed to prove this. Let $\mathbf{R}^{k,l}$ be the standard pseudo-euclidean space of signature (k,l). The inner product is written as $\alpha \cdot \alpha$. Given $\alpha \in \mathbf{R}^{k,l}$, let $F_{k,l}(\alpha)$ be the standard representation of the Heisenberg algebra with generators $j^a(n)$ and relations $[j^a(n), j^b(m)] = n\delta_{n+m,0}\eta^{ab} id$ $(a, b = 1, .., k+l,\ n, m \in \mathbf{Z}$, $\eta = diag(+, .., +, -, ..-)$ with k + and l $-$). Here the $j^a(0)$ acts by the scalar α^a. The canonical generator of the module is denoted by $|\alpha\rangle$. We now regard each $F_{k,l}(\alpha)$ as a Vir-module in which Vir acts by $L(z) = \frac{1}{2} : j^a(z) j^b(z) : \eta^{ab}$. This module has the standard hermitean form [9].

Theorem 6.6 *[9] For any $\alpha \in \mathbf{R}^{25,1}$, the linear maps ν_1, ν_2 for the module $F_{25,1}(\alpha)$ are bijections, and*

$$H^{\frac{\infty}{2}+p}(Vir, F_{25,1}(\alpha)) = \begin{cases} \delta_{\alpha,0} \mathbf{C}\ \mathbf{1} \otimes |0\rangle & \text{if } p = 0 \\ \nu_1 \mathbf{P}(F_{25,1}(\alpha)) & \text{if } p = 1. \\ \nu_2 \mathbf{P}(F_{25,1}(\alpha)) & \text{if } p = 2. \\ \delta_{\alpha,0} \mathbf{C}\ c(-3)c(-2)c(-1)\mathbf{1} \otimes |0\rangle & \text{if } p = 3 \\ 0 & \text{otherwise.} \end{cases} \tag{6.51}$$

For a proof, see the original reference.

Denote the highest vector of the Vir-modules $L(\kappa,h)$ or $M(\kappa,h)$ by $|\kappa,h\rangle$.

Lemma 6.7 *For any $h \geq 0$ and $\beta \in \mathbf{R}^{r-1,1}$, the linear maps ν_1, ν_2 for the module $L(26-r,h)\otimes F_{r-1,1}(\beta)$ are bijections, and*

$$H^{\frac{\infty}{2}+p}(Vir, L(26-r,h)\otimes F_{r-1,1}(\beta)) = \begin{cases} \delta_{h,0}\delta_{\beta,0}\mathbf{C}\ \mathbf{1}\otimes|26-r,0\rangle\otimes|\beta\rangle & \text{if } p=0 \\ \nu_1\mathbf{P}(L(26-r,h)\otimes F_{r-1,1}(\beta)) & \text{if } p=1. \\ \nu_2\mathbf{P}(L(26-r,h)\otimes F_{r-1,1}(\beta)) & \text{if } p=2. \\ \delta_{h,0}\delta_{\beta,0}\mathbf{C}\ c(-3)c(-2)c(-1)\mathbf{1}\otimes|26-r,0\rangle\otimes|\beta\rangle & \text{if } p=3 \\ 0 & \text{otherwise} \end{cases} \tag{6.52}$$

Proof: The case $r = 26$ is just Theorem 6.6. So let's assume $r < 26$. The case $h = 0, \beta = 0$ can be easily checked by hand. So let's assume that either h or β is nonzero. We claim that any irreducible module $L(26-r,h)$ is direct summand in some $F_{26-r,0}(\alpha)$. Choose γ so that $\frac{\gamma\cdot\gamma}{2} = h$, and we've a homomorphism $M(26-r,h) \to F_{26-r,0}(\gamma)$ with $|26-r,h\rangle \mapsto |\gamma\rangle$. Since $F_{26-r,0}(\gamma)$ is unitarizable, the image must also be unitarizable. Thus it must also be an irreducible direct summand.

Now observe that both $H^{\frac{\infty}{2}+p}(Vir,-)$ and $\mathbf{P}(-)$ are exact with respect to direct sum. Since $F_{26-r,0}(\gamma)\otimes F_{r-1,1}(\beta)$ is isomorphic to $F_{25,1}(\alpha)$ for some $\alpha \neq 0$, we see that Theorem 6.6 implies (6.52). □

Proof of Theorem 6.5: By assumption, O is unitarizable and hence is a direct sum of $L(26-r,h)$, $h \geq 0$. On the other hand $V_\Lambda = \oplus_{\beta\in\Lambda}F_{r-1,1}(\beta)$ as Vir-modules, if we choose an identification $\mathbf{R}^{r-1,1} = \Lambda\otimes_{\mathbf{Z}}\mathbf{R}$. Now the theorem follows from Lemma 6.7 and the fact that both $H^{\frac{\infty}{2}+p}(Vir,-)$ and $\mathbf{P}(-)$ are exact with respect to direct sum. □

6.2 Some applications

The BV algebra $H^*(O\otimes O(V_\Lambda))$ in Theorem 6.4 is graded by Λ because as a Vir-module, $O(V_\Lambda) \cong V_\Lambda \cong \oplus_{\alpha\in\Lambda}F_{1,1}(\alpha)$ is graded by Λ. In particular, we've a decomposition of the Moonshine cohomology $\mathbf{M}^*(O(V_\Lambda)) = \oplus_{\alpha\in\Lambda}\mathbf{M}^*(O(V_\Lambda))_\alpha$. Since each $F_{1,1}(\alpha)$ is tame as Vir-module, and since $O(V)$ is also tame, the graded dimensions $dim\ \mathbf{M}^*(O(V_\Lambda))_\alpha$ (see Theorem 6.2) are in fact finite. Thus the $\mathbf{M}^*(O(V_\Lambda))_\alpha$ are finite dimensional representations of the group F_1. We'll compute the dimensions using our results above together with well-known techniques in semi-infinite cohomology theory (see [9][27]).

Since each $O(V)\otimes F_{1,1}(\alpha)$ is hermitean, there is an induced nondegenerate hermitean form on the cohomology $\mathbf{M}^1(O(V_\Lambda))_\alpha$ (see below). We will compute the signature of this hermitean form and show that it's positive definite for nonzero α ("no-ghost theorem"). We will restrict ourselves to the case when Λ is a rank 2 hyperbolic lattice. How to generalize our computations to other Lorentzian lattices will become clear, and is left as an exercise to the readers.

Since we're only interested in dimensions and signatures of cohomology, it's enough to work with the additive version of our results Theorem 6.5 and Lemma 6.7. We begin by introducing one other tool: the notion of relative semi-infinite cohomology. We refer to readers to original references for details.

Let $C_{\Delta}^{\frac{\infty}{2}+*}(Vir, M)$ be the subspace of the Vir-module $C^{\frac{\infty}{2}+*}(Vir, M)$ annihilated by $b(1)$ and L_0. Because $[d_M, b(1)] = L_0$, this subspace is a complex with differential d_M. Call this subspace the relative complex, and its cohomology $H_{\Delta}^{\frac{\infty}{2}+*}(Vir, M)$ the relative cohomology. Note that if M is tame and its weight $||\cdot||$ is bounded from below, then $C_{\Delta}^{\frac{\infty}{2}+*}(Vir, M)$ is finite dimensional. Relative cohomology is an important tool for studying (absolute) semi-infinite cohomology. For example, technically to prove Theorem 6.6, we'd have to *first* prove a similar vanishing theorem for relative cohomology. In this paper, we manage to prove all our results so far without using it. However for computing dimensions and signatures, relative cohomology is indispensable.

In this section, we'll be interested in $H_{\Delta}^{\frac{\infty}{2}+*}(Vir, V \otimes F_{1,1}(\alpha))$, $\alpha \in \Lambda$. For simplicity, we'll abbreviate the absolute complex $C^{\frac{\infty}{2}+*}(Vir, V \otimes F_{1,1}(\alpha))$ simply as $C^{\frac{\infty}{2}+*}(\alpha)$. Similar notations apply to the absolute cohomology, the relative complex, and the relative cohomology.

Lemma 6.8

$$H_{\Delta}^{\frac{\infty}{2}+p}(\alpha) \cong \begin{cases} \delta_{\alpha,0}\mathbf{C} & \textit{if } p = 0,2 \\ H^{\frac{\infty}{2}+1}(\alpha) & \textit{if } p = 1. \\ 0 & \textit{otherwise.} \end{cases} \tag{6.53}$$

Proof: The case $\alpha = 0$ is a trivial exercise. There's a long exact sequence (see [27] for details):

$$\cdots \to H_{\Delta}^{\frac{\infty}{2}+p}(\alpha) \to H^{\frac{\infty}{2}+p}(\alpha) \to H_{\Delta}^{\frac{\infty}{2}+p-1}(\alpha) \to H_{\Delta}^{\frac{\infty}{2}+p+1}(\alpha) \to \cdots. \tag{6.54}$$

Assume α nonzero. By decomposing V in terms of its irreducible submodules $L(24, h)$ and applying Lemma 6.7 (for $r = 2$), we see that the long exact sequence above degenerates and yields the desired result. □

Corollary 6.9 $\mathbf{M}^1(O(V_\Lambda))_\alpha \cong H_{\Delta}^{\frac{\infty}{2}+1}(\alpha)$.

Thus to compute the graded dimensions of the Moonshine cohomology in degree 1 (which is a Λ-graded Lie algebra) for the conformal QOA $O(V_\Lambda)$, it's enough to compute the corresponding degree 1 relative cohomology. For a tame hermitian $||\cdot||$-graded vector space M, let $ch_q M = \sum dim\ M[n]q^n$, $sign_q M = \sum sign\ M[n]q^n$.

By the Euler-Poincaré Principle, we have

$$\sum_i (-1)^i dim\ H_{\Delta}^{\frac{\infty}{2}+i}(\alpha) = \sum_i (-1)^i dim\ C_{\Delta}^{\frac{\infty}{2}+i}(\alpha). \tag{6.55}$$

Note that $Ker\ b(1) = Im\ b(1)$. Thus the RHS of (6.55) is just the constant term of the following q-series:

$$\begin{aligned} ch_q V \cdot ch_q F_{1,1}(\alpha) \cdot \sum_i (-1)^i ch_q b(1)\wedge^{i+1} &= q(j(q)-744)\cdot q^{\frac{\alpha\cdot\alpha}{2}}\varphi(q)^{-2}\cdot(-1)q^{-1}\varphi(q)^2 \\ &= -q^{\frac{\alpha\cdot\alpha}{2}}(j(q)-744) \qquad (6.56) \end{aligned}$$

where $\varphi(q) = \Pi_{n>0}(1-q^n)$. For $\alpha = 0$, the constant term on this RHS is zero – consistent with the fact that $dim\ H_\Delta^{\frac{\infty}{2}+i}(0)$, $i = 0,1,2$, are respectively 1,2,1. For α nonzero, combining Lemma 6.8, Corollary 6.9, and eqn. (6.55), we get

$$dim\ \mathbf{M}^1(O(V_\Lambda))_\alpha = Res_q q^{\frac{\alpha\cdot\alpha}{2}-1}(j(q)-744). \qquad (6.57)$$

The space $b(1)\wedge^*$ has a unique hermitean form $\langle\cdot,\cdot\rangle$ such that $\langle \mathbf{1}, c(-3)c(-1)\mathbf{1}\rangle = 1$, $b(n)^\dagger = b(-n+2)$, and $c(n)^\dagger = c(-n-4)$ [9]. This makes each complex $C_\Delta^{\frac{\infty}{2}+*}(\alpha)$ a hermitian space such that the semi-infinite differential d (dropping the subscript) is self-adjoint. By the Euler-Poincaré Principle for signature, we have

$$\sum_i sign\ H_\Delta^{\frac{\infty}{2}+i}(\alpha) = \sum_i sign\ C_\Delta^{\frac{\infty}{2}+i}(\alpha). \qquad (6.58)$$

As before, the RHS here is the constant term of the following q-series:

$$\begin{aligned} sign_q V \cdot sign_q F_{1,1}(\alpha) \cdot \sum_i sign_q b(1)\wedge^{i+1} &= q(j(q)-744)\cdot q^{\frac{\alpha\cdot\alpha}{2}}\lambda(q)^{-1}\cdot q^{-1}\lambda(q) \\ &= q^{\frac{\alpha\cdot\alpha}{2}}(j(q)-744) \qquad (6.59) \end{aligned}$$

where $\lambda(q) = \Pi_{n>0}(1-q^n)(1+q^n)$. Thus we conclude that for α nonzero:

$$sign\ H_\Delta^{\frac{\infty}{2}+1}(\alpha) = dim\ H_\Delta^{\frac{\infty}{2}+1}(\alpha) \qquad (6.60)$$

which is the statement of the no-ghost theorem. The interested readers should compare our results with the results of Borcherds [5] on graded dimensions and positive definiteness for the Monster Lie algebra.

6.3 Remarks

In the light of our results, a number of interesting questions come to mind. We discuss two of them.

1. Recall that the Virasoro element ω of the Moonshine VOA V is an F_1-invariant element of V. This means that $Y(\omega, z)$ is an F_1-invariant quantum operator, and hence generates an F_1-invariant subalgebra in $O(V)$. It's easy to show that this subalgebra is isomorphic to $O_{24}(L)$. By unitarizability, $O_{24}(L)$ is a direct summand in $O(V)$ as Vir-module. Thus for any conformal

QOA O of central charge 2, the Moonshine cohomology $\mathbf{M}^*(O)$, as a BV algebra, has a canonical subalgebra $H^*(O_{24}(L)\otimes O)$. In particular, the vanishing theorem 5.2 holds for this algebra. Also, $H^2(O_{24}(L)\otimes O)$ is the adjoint module over the Lie algebra $H^1(O_{24}(L)\otimes O)$.

Problem: For any rank two hyperbolic lattice Λ, study the Λ-graded Lie subalgebra $H^1(O_{24}(L)\otimes O(V_\Lambda))$ of $\mathbf{M}^1(O(V_\Lambda))$.

2. Recall that the Monster finite group F_1 acts naturally on $\mathbf{M}^*(O)$; thus, F_1 acts as a group of invertible natural transformations of $\mathbf{M}^*$ as a functor. Since the full automorphism group of the Moonshine VOA V, hence of the conformal QOA $O(V)$, is isomorphic to F_1, the following seems quite plausible:

Conjecture: The group of natural automorphisms of the Moonshine cohomology functor is isomorphic to F_1.

References

[1] F. Akman, "The semi-infinite Weil complex of a graded Lie algebra", Yale Thesis 1993.

[2] I.A. Batalin and G.A. Vilkovisky, "Quantization of gauge theories with linearly dependent generators", Phys. Rev. D28 (1983) 2567.

[3] A. Belavin, A.M. Polyakov and A.A. Zamolodchikov, "Infinite conformal symmetry in two dimensional quantum field theory", Nucl. Phys. B241 (1984) 333.

[4] R.E. Borcherds, "Vertex operator algebras, Kac-Moody algebras and the Monster", Proc. Natl. Acad. Sci. USA. 83 (1986) 3068.

[5] R.E. Borcherds, "Monstrous moonshine and monstrous Lie superalgebras", Invent. math. 109 (1992) 405-444.

[6] C.-Y. Dong and J. Lepowsky, "Generalized vertex algebras and relative vertex operators", Prog. in Math. vol 112, Birkhauser, Boston, 1993.

[7] B. Feigin, "Semi-infinite homology for Virasoro and Kac-Moody algebras", Usp. Mat. Nauk. 39 (1984) 195-196.

[8] I.B. Frenkel, "Representations of Kac-Moody algebras and dual resonance models", in Lect. Appl. Math., vol 21, ed. Flato et al, AMS, Providence (1985) 325-353.

[9] I.B. Frenkel, H. Garland and G.J. Zuckerman, "Semi-infinite cohomology and string theory ", Proc. Nat. Acad. Sci. U.S.A. 83 (1986) 8442.

[10] I.B. Frenkel, J. Lepowsky and A. Meurman, "A natural representation of the Fischer-Griess Monster with the modular function J as character", Proc. Nat. Acad. Sci. U.S.A. 81 (1984) 3256.

[11] I.B. Frenkel, J. Lepowsky and A. Meurman, "Vertex Operator Algebras and the Monster", Academic Press, New York, 1988.

[12] I.B. Frenkel and Y. Zhu, "Vertex Operator Algebras associated to affine Lie algebras and the Virasoro algebra", Duke Math J. 66 (1992) 123.

[13] D. Friedan, E. Martinec and S. Shenker, "Conformal invariance, supersymmetry and string theory", Nucl. Phys. B271 (1986) 93.

[14] M. Gerstenhaber, "The cohomology structure of an associative ring", Ann. of Math. 78, No.2 (1962) 267.

[15] E. Getzler, "Batalin-Vilkovisky algebras and two dimensional topological field theory", preprint hepth/9212043.

[16] Y-Z. Huang, "Operadic formulation of topological vertex algebras and Gerstenhaber or Batalin-Vilkovisky algebras", Univ of Penn preprint (1993).

[17] V. Kac and A. Raina, *Highest weight representations of infinite dimensional Lie algebras*, Advanced Series in Math Physics, Vol. 2, World Scientific (1987).

[18] M. Kato and K. Ogawa, "Covariant quantization of string based on BRS invariance", Nucl. Phys. B212 (1983) 443.

[19] T. Kimura, J. Stasheff and A. Voronov, "On operad structures of moduli spaces and string theory", hep-th/9307114.

[20] V.G. Knizhnik and A.B. Zamolodchikov, "Current algebra and Wess-Zumino model in two dimensions", Nucl. Phys. B247 (1984) 83-103.

[21] J.-L. Koszul, "Crochet de Schouten-Nijenhuis et cohomologie", Asterique, hors serie, (1985) 257-271.

[22] H.-S. Li, "Local systems of vertex operators, vertex superalgebras and modules", hep-th/9406185.

[23] B.H. Lian, "On the classification of simple vertex operator algebras", Commun. Math. Phys. 163 (1994) 307.

[24] B.H. Lian and G.J. Zuckerman, "Commutative quantum operator algebras", q-alg/9501014, to appear in Journ. Pure Appl. Alg. (1995).

[25] B.H. Lian and G.J. Zuckerman, "Some classical and quantum algebras", hep-th/9404010.

[26] B.H. Lian and G.J. Zuckerman, "New perspectives on the BRST-algebraic structure in string theory", hep-th/9211072, Commun. Math. Phys. 154 (1993) 613.

[27] B.H. Lian and G.J. Zuckerman, "BRST cohomology and highest weight vectors I", Commun. Math. Phys. 135 (1991) 547-580.

[28] G. Moore, "Finite in All Directions", hepth/9305139.

[29] G. Moore and N. Seiberg, "Classical and quantum conformal field theory", Comm. Math. Phys. 123 (1989) 177-254.

[30] M. Penkava, "A note on BV algebras", UC Davis preprint November 92.

[31] M. Penkava and A. Schwarz, "On some algebraic structures arising in string theory", UC Davis preprint hepth/9212072.

[32] A. Tsuchiya, K. Ueno and Y. Yamada, in Advanced Studies on Pure Math. vol 19, 1989.

[33] E. Witten, "Ground ring of the two dimensional string theory", Nucl. Phys. B373 (1992) 187.

[34] E. Witten, "The anti-bracket formalism", preprint IASSNS-HEP-90/9.

[35] E. Witten, "Nonabelian bosonization in two-dimensions", Commun. Math. Phys. 92 (1984) 455-472.

[36] E. Witten, "String field theory and non-commutative geometry", Nucl. Phys. B268 (1986) 253.

[37] E. Witten and B. Zwiebach, "Algebraic structures and differential geometry in 2d string theory", Nucl. Phys. B377 (1992) 55.

DEPARTMENT OF MATHEMATICS, HARVARD UNIVERSITY CAMBRIDGE, MA 02138.
lian@math.harvard.edu
DEPARTMENT OF MATHEMATICS, YALE UNIVERSITY NEW HAVEN, CT 06520.
gregg@math.yale.edu