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Long-range scattering for N-body Stark Hamiltonians

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§1. Introduction

In this article, we study the problem of the asymptotic completeness for N-body Stark Hamiltonians with long-range potentials. The results which are obtained in this article are due to the joint works [AT1,2] with Professor Hideo Tamura.

We consider a system of N particles moving in a given constant electric field $\mathcal{E} \in \mathbf{R}^d$, $\mathcal{E} \neq 0$. Let m_j, e_j and $r_j \in \mathbf{R}^d$, $1 \leq j \leq N$, denote the mass, charge and position vector of the j-th particle, respectively. The particles under consideration are supposed to interact with one another through the pair potentials $V_{jk}(r_j - r_k)$, $1 \leq j < k \leq N$. Then the total Hamiltonian for the system is given by

$$\tilde{H} = \sum_{1 \leq j \leq N} \left\{ -\frac{1}{2m_j} \Delta_{r_j} - e_j \mathcal{E} \cdot r_j \right\} + V,$$

where $\xi \cdot \eta = \sum_{j=1}^d \xi_j \eta_j$ for ξ, $\eta \in \mathbf{R}^d$ and

$$V = \sum_{1 \leq j < k \leq N} V_{jk}(r_j - r_k).$$

As usual, we consider the Hamiltonian $\tilde{H}$ in the center-of-mass frame. We equip $\mathbf{R}^{d \times N}$ with the metric $\langle r, \tilde{r} \rangle = \sum_{j=1}^N m_j r_j \cdot \tilde{r}_j$ for $r = (r_1, \ldots, r_N)$ and $\tilde{r} = (\tilde{r}_1, \ldots, \tilde{r}_N) \in \mathbf{R}^{d \times N}$ and denote $|r| = \langle r, r \rangle^{1/2}$. Let X and X_{cm} be the configuration spaces for the inner motion of the particles and the center of mass motion, respectively:

$$X = \left\{ r \in \mathbf{R}^{d \times N} \;\middle|\; \sum_{1 \leq j \leq N} m_j r_j = 0 \right\},$$
$$X_{\mathrm{cm}} = \left\{ r \in \mathbf{R}^{d \times N} \;\middle|\; r_j = r_k \text{ for } 1 \leq j < k \leq N \right\}.$$

X and X_{cm} are mutually orthogonal and $\mathbf{R}^{d \times N} = X \oplus X_{cm}$. We denote by $\pi : \mathbf{R}^{d \times N} \to X$ and $\pi_{\mathrm{cm}} : \mathbf{R}^{d \times N} \to X_{\mathrm{cm}}$ the orthogonal projections onto X and X_{cm}, respectively, and write $x = \pi r$ and $x_{\mathrm{cm}} = \pi_{\mathrm{cm}} r$ for $r \in \mathbf{R}^{d \times N}$, and

$$E = \pi \left(\frac{e_1}{m_1} \mathcal{E}, \ldots, \frac{e_N}{m_N} \mathcal{E} \right), \quad E_{\mathrm{cm}} = \pi_{\mathrm{cm}} \left(\frac{e_1}{m_1} \mathcal{E}, \ldots, \frac{e_N}{m_N} \mathcal{E} \right).$$

Then $\tilde{H}$ is decomposed into

$$\tilde{H} = H \otimes Id + Id \otimes T_{\rm cm} \qquad \text{on } L^2(X) \otimes L^2(X_{cm}),$$

where Id are the identity operators,

$$H = -\frac{1}{2}\Delta - \langle E, x\rangle + V \qquad \text{on } L^2(X),$$

$$T_{\rm cm} = -\frac{1}{2}\Delta_{\rm cm} - \langle E_{\rm cm}, x_{\rm cm}\rangle \qquad \text{on } L^2(X_{\rm cm}),$$

and Δ (resp. $\Delta_{\rm cm}$) is the Laplace-Beltrami operator on X (resp. $X_{\rm cm}$). When $|E| \neq 0$, which is equivalent to saying that $e_j/m_j \neq e_k/m_k$ for at least one pair (j,k), H is called an N-body Stark Hamiltonian in the center-of-mass frame.

A non-empty subset of the set $\{1, \dots, N\}$ is called a cluster. Let C_j, $1 \le j \le m$, be clusters. If $\cup_{1 \le j \le m} C_j = \{1, \dots, N\}$ and $C_j \cap C_k = \emptyset$ for $1 \le j < k \le m$, $a = \{C_1, \dots, C_m\}$ is called a cluster decomposition. We denote by $\#(a)$ the number of clusters in a. $\tilde{\mathcal{A}}$ is the set of all cluster decompositions and $\mathcal{A} = \{a \in \tilde{\mathcal{A}} \mid \#(a) \ge 2\}$. We let a, $b \in \tilde{\mathcal{A}}$. If b is obtained as a refinement of a, that is, if each cluster in b is a subset of a cluster in a, we say $b \subset a$, and its negation is denoted by $b \not\subset a$. Any a is regarded as a refinement of itself. If, in particular, b is a strict refinement of a, that is, if $b \subset a$ and $b \neq a$, we denote by $b \subsetneq a$. We identify the pair $\alpha = (j,k)$ with the $(N-1)$-cluster decomposition $\{(j,k), (1), \dots, (\hat{j}), \dots, (\hat{k}), \dots, (N)\}$.

Next we define for $a \in \tilde{\mathcal{A}}$ the two subspaces X^a and X_a of X by

$$X^a = \left\{ r \in X \;\middle|\; \sum_{j \in C} m_j r_j = 0 \text{ for each cluster } C \text{ in } a \right\},$$

$$X_a = \left\{ r \in X \;\middle|\; r_j = r_k \text{ for each pair } \alpha = (j,k) \subset a \right\}.$$

For $\alpha = (j,k)$, X^α is the configuration space for the relative position of j-th and k-th particles. Hence we can write $V_\alpha(x^\alpha) = V_{jk}(r_j - r_k)$. These spaces are mutually orthogonal and span the total space $X = X^a \oplus X_a$, so that $L^2(X)$ is decomposed as the tensor product $L^2(X) = L^2(X^a) \otimes L^2(X_a)$. We also denote by $\pi^a : X \to X^a$ and $\pi_a : X \to X_a$ the orthogonal projections onto X^a and X_a, respectively, and write $x^a = \pi^a x$ and $x_a = \pi_a x$ for a generic point $x \in X$. The intercluster potential I_a is defined by

$$I_a(x) = \sum_{\alpha \not\subset a} V_\alpha(x^\alpha),$$

and the cluster Hamiltonian

$$H_a = H - I_a = -\frac{1}{2}\Delta - \langle E, x\rangle + V^a, \quad V^a(x^a) = \sum_{\alpha \subset a} V_\alpha(x^\alpha),$$

governs the motion of the system broken into non-interacting clusters of particles. Let $E^a = \pi^a E$ and $E_a = \pi_a E$. Then the cluster Hamiltonian H_a acting on $L^2(X)$ is decomposed into

$$H_a = H^a \otimes Id + Id \otimes T_a \qquad \text{on } L^2(X^a) \otimes L^2(X_a),$$

where H^a is the subsystem Hamiltonian defined by

$$H^a = -\frac{1}{2}\Delta^a - \langle E^a, x^a \rangle + V^a \qquad \text{on } L^2(X^a),$$

T_a is the free Hamiltonian defined by

$$T_a = -\frac{1}{2}\Delta_a - \langle E_a, x_a \rangle \qquad \text{on } L^2(X_a),$$

and Δ^a (resp. Δ_a) is the Laplace-Beltrami operator on X^a (resp. X_a).

We use the following convention for smooth cut-off functions F with $0 \le F \le 1$, which is often used throughout the discussion below. For sufficiently small $\delta > 0$, we define

$$\begin{aligned} F(s \le u) &= 1 \ \text{ for } \ s \le u - \delta, \quad &= 0 \ \text{ for } \ s \ge u, \\ F(s \ge u) &= 1 \ \text{ for } \ s \ge u + \delta, \quad &= 0 \ \text{ for } \ s \le u, \\ F(s = u) &= 1 \ \text{ for } \ |s - u| \le \delta, \quad &= 0 \ \text{ for } \ |s - u| \ge 2\,\delta \end{aligned}$$

and $F(u_1 \le s \le u_2) = F(s \ge u_1)F(s \le u_2)$. To clarify the dependence on $\delta > 0$ in the definition of F, we sometimes write F_δ for F.

The problem of the asymptotic completeness is to show whether or not the ranges of the wave operators

$$W_a^\pm = s - \lim_{t \to \pm\infty} \exp(itH)U_a(t)(P^a \otimes Id), \qquad a \in \tilde{\mathcal{A}}, \tag{1.1}$$

span the continuous subspace of the full Hamiltonian H:

$$L_c^2(H) = \sum_a \oplus Ran\, W_a^\pm. \tag{1.2}$$

Here $U_a(t)$ is a propagator associated with the cluster Hamiltonian H_a, which is $\exp(-itH_a)$ in the short-range potential case, and $P^a : L^2(X^a) \to L^2(X^a)$ is the eigenprojection associated with the subsystem Hamiltonian H^a. It can be easily seen that (1.2) holds if and only if the time evolution of any scattering state $\psi \in L_c^2(H)$ is asymptotically represented as

$$\exp(-itH)\psi = \sum_{a \in \tilde{\mathcal{A}}} U_a(t)(P^a \otimes Id)\psi_a^\pm + o(1) \qquad \text{as } t \to \pm\infty \tag{1.3}$$

with some $\psi_a^\pm \in L^2(X)$. We note that each summand $U_a(t)(P^a \otimes Id)\psi_a^\pm$ describes the motion of the particles in which those in the clusters in a form a bound state and the centers of masses of the clusters move freely. Thus, the asymptotic completeness is to show that the time evolution of any scattering state $\exp(-itH)\psi$ can be asymptotically represented by a superposition of such states. This article is to prove this property for long-range N-body Stark Hamiltonians.

There are mainly two methods to prove the asymptotic completeness for N-body quantum systems, that is, the stationary method and the time-dependent method. First we consider the case when $E = 0$. When pair potentials decay like $O(|x^\alpha|^{-1-\epsilon})$ at infinity with $\epsilon > 0$, the asymptotic completeness of three-body systems was first proved by Faddeev [F] by the stationary method, and his method was extended by Ginibre-Moulin [GM], Hagedorn [Ha] and so on. But it had been difficult to use this method for arbitrary N-body systems. In the meantime, the asymptotic completeness of three-body systems was first proved by Enss [E1] by the time-dependent method. Inspired by the work, the asymptotic completeness of arbitrary N-body systems was first proved by Sigal-Soffer [SS1] for a large class of short-range pair potentials, and alternative proofs have been given by Graf [Gr1], Kitada [K1,2], Tamura [T1], Yafaev [Y] and Zielinski [Z]. When pair potentials are long-range, V.Enss [E1,2] first proved the asymptotic completeness of three-body systems with pair potentials decaying like $O(|x^\alpha|^{-\nu})$ at infinity for some $\nu > \sqrt{3} - 1$. This result has been extended by Dereziński [D2] and Zielinski [Z] to arbitrary N-body systems. The case of potentials decaying more slowly has been dealt with by Gérard [G2] and Wang [Wa1,2] for three-body systems.

In the case when $E \neq 0$ and pair potentials are short-range, the asymptotic completeness of three-body systems has been proved by Korotyaev [K] by the stationary method, and by Tamura [T2] by the time-dependent method. For arbitrary N-body systems, it has been proved by Tamura [T4] and Møller [Mø] by the time-dependent method. Thus we consider the long-range case.

The asymptotic behavior of $\exp(-itH)\psi$ depends on the values of e_j/m_j. If $e_j/m_j \neq e_k/m_k$ for any $j \neq k$, then each pair cluster has a non-zero reduced charge. Hence the particles are expected to be scattered along the direction of $\mathcal{E}$ without forming bound states and also the solution $\psi(t)$ has a single channel as the asymptotic state. On the other hand, if, for example, $e_j/m_j = e_k/m_k$ for some pair (j,k), then the pair cluster (j,k) has a zero reduced charge and these particles may escape to infinity, forming a bound state at some energy. Therefore the solution $\psi(t)$ has scattering channels associated with such bound states as the asymptotic state. Thus the asymptotic behavior of the solutions is different according to the values of e_j/m_j. We shall discuss the matter more precisely.

We state the precise assumption on the pair potentials. Let c be a maximal element of the set $\{a \in \mathcal{A} \mid E^a = 0\}$ with respect to the relation $\subset$. As is easily seen, such a cluster decomposition uniquely exists and it follows that $E^\alpha = 0$ if $\alpha \subset c$, and $E^\alpha \neq 0$ if $\alpha \not\subset c$. Thus the potential V_α with $\alpha \not\subset c$ (resp. $\alpha \subset c$) describes the pair interaction between two particles with $e_j/m_j \neq e_k/m_k$ (resp. $e_j/m_j = e_k/m_k$). If, in particular, $e_j/m_j \neq e_k/m_k$ for any $j \neq k$, then c becomes the N-cluster decomposition. We make different assumptions on V_α according as $\alpha \not\subset c$ or $\alpha \subset c$. We assume that:

$(V)_G$ $V_\alpha(x^\alpha) \in C^\infty(X^\alpha)$ is a real-valued function and has the decay property

$$\partial_{x^\alpha}^\beta V_\alpha(x^\alpha) = O(|x^\alpha|^{-(\rho+|\beta|)}), \qquad |x^\alpha| \to \infty,$$

with $0 < \rho \leq 1/2$ for $\alpha \not\subset c$ and with $\sqrt{3} - 1 < \rho \leq 1$ for $\alpha \subset c$,

or

$(V)_D$ $V_\alpha(x^\alpha) \in C^\infty(X^\alpha)$ is a real-valued function and has the decay property

$$\partial_{x^\alpha}^\beta V_\alpha(x^\alpha) = O(|x^\alpha|^{-(\rho+|\beta|)/2}) \quad \alpha \not\subset c, \qquad \partial_{x^\alpha}^\beta V_\alpha(x^\alpha) = O(|x^\alpha|^{-(\rho+|\beta|)}) \quad \alpha \subset c$$

for some $\sqrt{3} - 1 < \rho \leq 1$.

Under these assumptions, all the Hamiltonians defined above are essentially self-adjoint on C_0^∞. We denote their closures by the same notations. Throughout this article, the notations c and ρ are used with the meanings described above. If V_α satisfies these decay assumptions, then V_α is called a long-range potential. To formulate the result precisely, we define the modified wave operators. We assume that $a \subset c$. Then the subsystem operator H^a does not have a constant electric field, that is, $E^a = 0$. Hence it may have bound states in $L^2(X^a)$. We denote by $P^a : L^2(X^a) \to L^2(X^a)$ the eigenprojection associated with H^a. We also denote the direction of E by $\omega = E/|E|$ and write $z = \langle x, \omega \rangle$. We should note that $z = \langle x_a, \omega \rangle$ because $\omega^a = 0$. We set $x_\| = z\omega$ and $x_\perp = x - x_\|$, and write $x_{a,\perp} = \pi_a x_\perp$. Then we can write $x_a = (x_{a,\perp}, x_\|)$. We also write $p_a = (p_{a,\perp}, p_\|)$ for the coordinates dual to $x_a = (x_{a,\perp}, x_\|)$ and denote by $D_a = -i\nabla_a = (D_{a,\perp}, D_\|)$ the corresponding velocity operator. If we write $\partial_\| = \omega\partial_z$, we see that $D_\| = -i\partial_\|$ and $D_{a,\perp} = D_a - D_\|$. Let I_a^c be the intercluster interaction obtained from H^c:

$$I_a^c(x) = I_a^c(x^c) = \sum_{\alpha \subset c,\, \alpha \not\subset a} V_\alpha(x^\alpha).$$

First, we assume that $(V)_G$ is fulfilled. We set the time-dependent Hamiltonian

$$H_{aG}(t) = H_a + I_a^c(D_a t) + I_c\left(\frac{E}{2}t^2\right) \qquad \text{on } L^2(X). \tag{1.4}$$

The three operators on the right side of (1.4) commute with one another. This can be easily seen, if we take account of the fact that $I_a^c(p_a t) = I_a^c(\pi^c p_a t)$. Thus the propagator $U_{aG}(t)$ generated by $H_{aG}(t)$, that is, $\{U_{aG}(t)\}_{t \in \mathbf{R}}$ is a family of unitary operators such that for $\psi \in \mathcal{D}(H_{aG}(0))$, $\psi_t = U_{aG}(t)\psi$ is a strong solution of $id\psi_t/dt = H_{aG}(t)\psi_t$, $\psi_0 = \psi$, is represented by

$$U_{aG}(t) = \exp(-itH_a)\exp\left(-i\int_0^t \left\{I_a^c(D_a s) + I_c\left(\frac{E}{2}s^2\right)\right\} ds\right). \tag{1.5}$$

With these notations, the Graf-type modified wave operators are defined by

$$W_{aG}^\pm = s - \lim_{t \to \pm\infty} \exp(itH)U_{aG}(t)(P^a \otimes Id), \quad a \subset c, \tag{1.6}$$

which was introduced by Graf [Gr2] for two-body systems (see also [Zo] and [JO2]). We can easily prove that if these wave operators exist, their ranges are all closed and mutually orthogonal

$$Ran\, W_{aG}^\pm \perp Ran\, W_{bG}^\pm, \quad a \neq b,$$

and they have the intertwining property $\exp(itH)W_{aG}^\pm = W_{aG}^\pm \exp(itH_a)$ for $t \in \mathbf{R}$. If $V_\alpha(x^\alpha)$ decays like $V_\alpha(x^\alpha) = O(|x^\alpha|^{-\nu})$, $\nu > 1/2$, for $\alpha \not\subset c$ and like $V_\alpha(x^\alpha) =$

$O(|x^\alpha|^{-\nu})$, $\nu > 1$, for $\alpha \subset c$, V_α is called a short-range potential. For the class of short-range pair potentials, the ordinary wave operators

$$W_a^\pm = s - \lim_{t\to\pm\infty} \exp(itH)\exp(-itH_a)(P^a \otimes Id)$$

exist without the modifiers $I_a^c(D_a t)$ and $I_c(Et^2/2)$. The asymptotic completeness of the ordinary wave operators has been also proved by Korotyaev [Ko] and Tamura [T2] for three-body short-range systems and by Tamura [T4] and Møller [Mø] for N-body short-range systems. However it is known that the ordinary wave operators do not always exist for the long-range potentials which satisfy $(V)_G$ (see Jensen-Ozawa [JO2] and Ozawa [O]). Therefore, if $(V)_G$ is fulfilled, we need introduce the Graf-type modifiers (1.5).

One of the main results is the following theorem.

Theorem 1.1. (The Asymptotic Completeness) *Assume that $(V)_G$ is fulfilled. Let c be as above. Then the Graf-type wave operators $W_{aG}^\pm$, $a \subset c$, exist, have the intertwining property and are asymptotically complete*

$$L^2(X) = \sum_{a\subset c} \oplus Ran\, W_{aG}^\pm.$$

Next, we assume that $(V)_D$ is fulfilled. We set the time-dependent Hamiltonian

$$H_{aD}(t) = H_a + I_a^c(D_{a,\perp}t) + I_c\left(D_{a,\perp}t + \frac{E}{2}t^2\right) \qquad \text{on } L^2(X), \tag{1.7}$$

and denote by $U_{aD}(t)$ the propagator which is generated by $H_{aD}(t)$. Since $D_{a,\perp}$ commutes with H_a, the three operators on the right-hand side of (1.7) commute with one another. We note that $I_a^c(D_{a,\perp}t) = I_a^c(D_a t)$ for $I_a^c(p_{a,\perp}t) = I_a^c(\pi^c p_{a,\perp}t) = I_a^c(p_a t)$. Then $U_{aD}(t)$ is explicitly represented by

$$U_{aD}(t) = \exp(-itH_a)\exp\left(-i\int_0^t \left\{I_a^c(D_{a,\perp}s) + I_c\left(D_{a,\perp}s + \frac{E}{2}s^2\right)\right\} ds\right). \tag{1.8}$$

With these notations, the Dollard-type modified wave operators are defined by

$$W_{aD}^\pm = s - \lim_{t\to\pm\infty} \exp(itH)U_{aD}(t)(P^a \otimes Id), \quad a \subset c. \tag{1.9}$$

It can be easily proved that if these wave operators exist, their ranges are all closed and they have the intertwining property $\exp(itH)W_{aD}^\pm = W_{aD}^\pm \exp(itH_a)$ for $t \in \mathbf{R}$. We should note that it is known that both the ordinary wave operators and the Graf-type modified wave operators do not always exist for the long-range potentials which satisfy $(V)_D$ (see [JO2] for the case of two-body systems). Therefore, if $(V)_D$ is fulfilled, we need introduce the Dollard-type modifiers (1.8).

Another main result is the following theorem.

Theorem 1.2. (The Asymptotic Completeness) *Assume that $(V)_D$ is fulfilled. Let c be as above. Then the Dollard-type wave operators $W^{\pm}_{aD}$, $a \subset c$, exist, have the intertwining property and are asymptotically complete:*

$$L^2(X) = \sum_{a \subset c} \oplus Ran\, W^{\pm}_{aD}.$$

For simplification, we consider the $t \to \infty$ case only. The $t \to -\infty$ case can be treated similarly.

§2. Preliminaries

In this section, we recall the known results. First we introduce some notations. We define $S_0(X)$ by

$$S_0(X) = \{q \in C^\infty(X) \mid |\partial_x^\beta q(x)| \le C_\beta \langle x \rangle^{-|\beta|}\},$$

where we write $\langle x \rangle = (1 + |x|^2)^{1/2}$. Let $\omega = E/|E|$ be the direction of E. We denote the coordinate $z \in \mathbf{R}$ by $z = \langle x, \omega \rangle$, so that H is written as $H = -\Delta/2 - |E|\, z + V$. We should note that

$$\langle z \rangle^{-1/2} \nabla (H + i)^{-1}, \;\; \langle z \rangle^{-1} \nabla\nabla (H + i)^{-1} : L^2(X) \to L^2(X)$$

are bounded.

Now we state the recent result of Herbst-Skibsted [HS] and Herbst-Møller-Skibsted [HMS1], which is concerned with the spectral properties for N-body Stark Hamiltonians.

Theorem 2.1. *Assume that $(V)_G$ or $(V)_D$ is satisfied. Let $D = -i\nabla$ and $A = \langle \omega, D \rangle = -i\partial_z$. Then*
(1) H has no bound states.
(2) Let $R > 0$ be fixed and let $\Pi : X \to X$ be an orthogonal projection such that $\Pi E \neq 0$. Then

$$\|F_\epsilon(H = \lambda) F(|\Pi x| \le R)\| \to 0, \quad \epsilon \to 0,$$

uniformly in $\lambda \in \mathbf{R}$. In particular,

$$\|F_\epsilon(H = \lambda) F(|x^\alpha| \le R)\| \to 0, \quad \epsilon \to 0,$$

for $\alpha \not\subset c$.
(3) Let $0 < \sigma < |E|$. Then one can take $\epsilon > 0$ so small (uniformly in $\lambda \in \mathbf{R}$) that

$$F_\epsilon(H = \lambda) i[H, A] F_\epsilon(H = \lambda) \ge \sigma\, F_\epsilon(H = \lambda)^2.$$

The above theorem plays a basic role in studying the propagation properties of $\exp(-itH)$. In particular, the uniformity in high energies $\lambda \gg 1$ in the statement (2) is important. This makes it possible to take A as a conjugate operator in the form inequality in (3).

Next we recall the almost analytic extention method due to Helffer and Sjöstrand [HeSj], which is useful in analyzing operators given by functions of self-adjoint operators. For two operators B_1 and B_2, we define

$$ad^0_{B_1}(B_2) = B_2, \quad ad^n_{B_1}(B_2) = [ad^{n-1}_{B_1}(B_2), B_1], \quad n \geq 1.$$

For $m \in \mathbf{R}$, let $\mathcal{F}^m$ be the set of functions $f \in C^\infty(\mathbf{R})$ such that

$$|f^{(k)}(s)| \leq C_k \langle s \rangle^{m-k}, \quad k \geq 0.$$

If $f \in \mathcal{F}^m$ with $m \in \mathbf{R}$, then there exists $F \in C^\infty(\mathbf{C})$ such that $F(s) = f(s)$ for $s \in \mathbf{R}$, $\operatorname{supp} F(\zeta) \subset \{\zeta \in \mathbf{C} \big| |Im\,\zeta| \leq d'(1 + |Re\,\zeta|)\}$ for some $d' > 0$ and

$$|\bar{\partial}_\zeta F(\zeta)| \leq C_M \langle \zeta \rangle^{m-1-M} |Im\,\zeta|^M, \quad M \geq 0.$$

Such a function $F(\zeta)$ is called an almost analytic extension of f. Let B be a self-adjoint operator. If $f \in \mathcal{F}^{-m}$ with $m > 0$, then $f(B)$ is represented by

$$f(B) = \frac{i}{2\pi} \int_{\mathbf{C}} \bar{\partial}_\zeta F(\zeta)(B - \zeta)^{-1}\, d\zeta \wedge d\bar{\zeta}. \tag{2.1}$$

For $f \in \mathcal{F}^m$ with $m \in \mathbf{R}$, we have the following formulas of the asymptotic expansion of the commutator:

$$[B_1, f(B)] = \sum_{n=1}^{M-1} \frac{(-1)^{n-1}}{n!} ad^n_B(B_1) f^{(n)}(B) + R_M, \tag{2.2}$$

$$R_M = \frac{1}{2\pi i} \int_{\mathbf{C}} \bar{\partial}_\zeta F(\zeta)(B - \zeta)^{-1} ad^M_B(B_1)(B - \zeta)^{-M}\, d\zeta \wedge d\bar{\zeta}. \tag{2.3}$$

R_M is bounded if there exists k such that $m + k < M$ and $ad^M_B(B_1)(B + i)^{-k}$ is bounded. For the proof, see [G2].

We often use the following lemma in later sections. The lemma is more or less known (see, for example, Lemma 3.2 of [SS2]). On account of limited space, we omit its proof. For the proof, see [AT1,2].

Lemma 2.2. *Let $f_j \in C_0^\infty(\mathbf{R})$, $1 \leq j \leq 2$, and let $g \in \mathcal{F}^0$. Assume that B is a self-adjoint operator such that $ad^j_B(H)(H+i)^{-1}$, $1 \leq j \leq 2$, are bounded from $L^2(X)$ into itself. Then*
(1) $[f_1(H), g(B/t)] = [f_1(H), B/t]g'(B/t) + O(t^{-2})$.
(2) $[[f_1(H), B], f_2(B/t)] = O(t^{-1})$.
Moreover, let A be as in Theorem 2.1 and $Q = \langle x \rangle$ or $z (= \langle x, \omega \rangle)$. Then
(3) $[(H + i)^{-1}, f_1(A/t)] = O(t^{-1})$, $\quad [(H + i)^{-1}, f_1(Q/t^2)] = O(t^{-1})$.
(4) $[f_2(H), f_1(A/t)] = O(t^{-1})$, $\quad [f_2(H), f_1(Q/t^2)] = O(t^{-1})$.
(5) $[Q, f_1(A/t)] = O(t^{-1})$, $\quad [f_2(Q/t^2), f_1(A/t)] = O(t^{-3})$.
(6) $(H + i)[f_1(H), f_2(Q/t^2)] = O(t^{-1})$.

§3. Propagation estimates

In this section, we prove some propagation estimates which mean that the solution $\exp(-itH)\psi$ concentrates asymptotically on a classical Stark trajectory as $t \to \infty$. Throughout this section, we assume that $(V)_G$ or $(V)_D$ is fulfilled.

We begin by fixing some new notations. We define a conical neighborhood of $\omega = E/|E|$ by $\Gamma(\omega, \epsilon_1, r) = \{x \in X \mid \langle \omega, x/|x| \rangle \geq 1 - \epsilon_1, |x| > r\}$ for $\epsilon_1 > 0$ and $r > 0$. We denote by $\mathbf{D}_{H(t)}$ the Heisenberg derivative with respect to the time-dependent Hamiltonian $H(t)$: $\mathbf{D}_{H(t)}\Phi(t) = \partial_t \Phi(t) + i[H(t), \Phi(t)]$. Throughout the discussion below, we always denote by $f \in C_0^\infty(\mathbf{R})$ a non-negative smooth function with a compact support. We use the notations $\| \ \|$ and $(\ , \ )$ for the L^2 norm and scalar product in $L^2(X)$, respectively. We also denote by $B(t)$, $t \geq 1$, operators having the following properties: (1) $f(H)B(t)f(H) : L^2(X) \to L^2(X)$ is bounded; (2)

$$\int_1^\infty |(B(t)f(H)\exp(-itH)\psi, f(H)\exp(-itH)\psi)|dt \leq C\|\psi\|^2, \quad \psi \in L^2(X).$$

First we need the following proposition, which implies that the acceleration of the particles is finite if the total energy is finite.

Proposition 3.1. *There exists $M \gg 1$ dependent on f such that:*
(1) For $\psi \in L^2(X)$,

$$\int_1^\infty \frac{dt}{t} \left\| F\left(\frac{\langle x \rangle}{t^2} = M\right) f(H)\exp(-itH)\psi \right\|^2 \leq C\|\psi\|^2.$$

(2) For $\psi \in \mathcal{S}(X)$, $\mathcal{S}(X)$ being the Schwartz space over X,

$$\int_1^\infty \frac{dt}{t} \left\| F\left(\frac{\langle x \rangle}{t^2} \geq M\right) f(H)\exp(-itH)\psi \right\|^2 < \infty.$$

Proof. This proposition can be also proved in the same way as in the proof of Theorem 4.3 of [SS2] (see also [A]). We take the propagation observables

$$\Phi_1(t) = -F\left(\frac{\langle x \rangle}{t^2} \geq M\right) \quad \text{and} \quad \Phi_2(t) = -\left(\frac{\langle x \rangle}{t^2} - M\right) F\left(\frac{\langle x \rangle}{t^2} \geq M\right)$$

to prove (1) and (2), respectively. The detailed proof is omitted. □

Next we prove the following proposition, which means that the particles are accelerated along the direction ω with an at least positive acceleration which is less than $|E|$.

Proposition 3.2. *Let $0 < \nu < |E|$ and $L > 0$. Then for any $\psi \in L^2(X)$,*

$$\int_1^\infty \frac{dt}{t} \left\| F\left(-L \leq \frac{z}{t^2} \leq \frac{\nu}{2}\right) f(H)\exp(-itH)\psi \right\|^2 \leq C\|\psi\|^2.$$

To prove this proposition, we need the two lemmas below. On account of limited space, we omit the proofs of these lemmas. For the proofs, see [AT1,2].

Lemma 3.3. *Let ν be as above and $K > 0$. Then*

$$\int_1^\infty \frac{dt}{t} \left\| F\left(-K \leq \frac{A}{t} \leq \nu\right) f(H) \exp(-itH)\psi \right\|^2 \leq C\|\psi\|^2.$$

Lemma 3.4. *Let ν and L be as above. Then there exists $K \gg 1$ such that*

$$F\left(\frac{A}{t} \leq -K\right) F\left(-L \leq \frac{z}{t^2} \leq \frac{\nu}{2}\right) (H+i)^{-1} = O(t^{-1}).$$

Proof of Proposition 3.2. The proof is done in exactly the same way as in the proof of Theorem 4.2 of [SS2], by virtue of Lemmas 3.3 and 3.4. We omit the detail (see [AT1,2]). □

The next proposition is the most important propagation estimate, which means that the particles asymptotically concentrate in any conical neighborhood of ω.

Proposition 3.5. *Let M be as in Proposition 3.1 and ν be as in Proposition 3.2. Fix $\epsilon_1 > 0$ and $r > 0$. Assume that $q \in S_0(X)$ vanishes in $\Gamma(\omega, \epsilon_1, r)$. Then*

$$\int_1^\infty \frac{dt}{t} \left\| F\left(\frac{z}{t^2} \geq \frac{\nu}{2}\right) F\left(\frac{\langle x\rangle}{t^2} \leq M\right) q f(H) \exp(-itH)\psi \right\|^2 \leq C\|\psi\|^2.$$

To prove the above proposition, we have only to prove the following lemma. The proof is essentially the same as that of Proposition 7.3 of [T2] (see also [Mø] and [A]). We omit the detail (see [AT1,2]).

Lemma 3.6. *Let $\tilde{q}(x) = (\langle x\rangle - z)\langle x\rangle^{-1}$. Let M be as in Proposition 3.1 and ν be as in Proposition 3.2. Then*

$$\int_1^\infty \frac{dt}{t} \left\| F\left(\frac{z}{t^2} \geq \frac{\nu}{2}\right) F\left(\frac{\langle x\rangle}{t^2} \leq M\right) \tilde{q} f(H) \exp(-itH)\psi \right\|^2 \leq C\|\psi\|^2.$$

By the above propositions, we conclude the following propagation properties.

Proposition 3.7. *Let M, ν and $q \in S_0(X)$ be as above. Let $\Phi(t)$ denote one of the following three operators*

$$F\left(\frac{\langle x\rangle}{t^2} \geq M\right), \quad F\left(\frac{z}{t^2} \leq \frac{\nu}{2}\right), \quad F\left(\frac{z}{t^2} \geq \frac{\nu}{2}\right) F\left(\frac{\langle x\rangle}{t^2} \leq M\right) q.$$

Then

$$s - \lim_{t\to\infty} \Phi(t) f(H) \exp(-itH) = 0.$$

Proof. We calculate the Heisenberg derivative of $\Phi(t)$. Then the support of $\nabla\Phi(t)$ or $\partial_t\Phi(t)$ lies in the forbidden region of the propagator $\exp(-itH)$ in the sense of

Propositions 3.1, 3.2 and 3.5. Hence these propositions imply the existence of the strong limit

$$s - \lim_{t\to\infty} \exp(itH)\Phi(t)f(H)\exp(-itH). \tag{3.1}$$

In fact, taking $f_1 \in C_0^\infty(\mathbf{R})$ such that $f_1 f = f$ and noting that $\langle x\rangle^{1/2}[q, f_1(H)]$ is bounded, we have $[\Phi(t), f_1(H)] = O(t^{-1})$ by Lemma 2.2. Hence, to prove (3.1), it suffices to show the existence of the strong limit $s - \lim_{t\to\infty} \tilde{W}(t)$, where

$$\tilde{W}(t) = \exp(itH)f_1(H)\Phi(t)f(H)\exp(-itH).$$

By Propositions 3.1, 3.2 and 3.5, we have

$$|(\varphi, \tilde{W}(s_1)\psi) - (\varphi, \tilde{W}(s_2)\psi)| = o(1)\|\varphi\|, \quad s_1, s_2 \to \infty,$$

for φ, $\psi \in L^2(X)$. This implies that $\{\tilde{W}(t)\psi\}_{t\geq 1}$ is a Cauchy sequence and hence the existence of (3.1) is proved. By Propositions 3.1, 3.2 and 3.5 again, we see that for $\psi \in \mathcal{S}(X)$, there exists a subsequence $\{t_n\}_{n\in\mathbf{N}}$ with $t_n \to \infty$ such that

$$\lim_{n\to\infty} \Phi(t_n)f(H)\exp(-it_nH)\psi = 0, \tag{3.6}$$

where the choice of subsequence $\{t_n\}_{n\in\mathbf{N}}$ depends on ψ. By (3.1) and (3.2), we have for $\psi \in \mathcal{S}(X)$,

$$\lim_{t\to\infty} \Phi(t)f(H)\exp(-itH)\psi = 0.$$

Thus the proposition follows by density argument. □

§4. The asymptotic clustering

In this section, we introduce an auxiliary time-dependent Hamiltonian which approximates the full Hamiltonian H asymptotically and study the relation between $\exp(-itH)$ and the propagator generated by such a time-dependent Hamiltonian. Then we may prove the asymptotic clustering which is the key step to prove the asymptotic completeness of the modified wave operators. Throughout this section, we assume that $(V)_G$ or $(V)_D$ is fulfilled.

Let $q_c \in S_0(X)$ be such that $q_c = 1$ in $\Gamma(\omega, \epsilon_1, |E|/3)$, and $q_c = 0$ outside $\Gamma(\omega, 2\epsilon_1, |E|/4)$. Let $\tilde{q}_c \in S_0(X)$ be such that $\tilde{q}_c = 1$ in $\Gamma(\omega, 2\epsilon_1, |E|/4)$, and $\tilde{q}_c = 0$ outside $\Gamma(\omega, 3\epsilon_1, |E|/5)$. By definition, it follows that $\tilde{q}_c q_c = q_c$. We define

$$\varphi_c(t,x) = F\left(\frac{\langle x\rangle}{t^2} \leq M\right) F\left(\frac{z}{t^2} \geq \frac{|E|}{3}\right) q_c(x), \tag{4.1}$$

$$W_c(t,x) = W_c(t, x^c, x_c) = F\left(\frac{z}{t^2} \geq \frac{|E|}{4}\right) \tilde{q}_c(x) I_c(x). \tag{4.2}$$

We should note that $\varphi_c(t,x)I_c(x) = \varphi_c(t,x)W_c(t,x)$. By assumption, W_c obeys the estimate

$$|\partial_t^m \partial_x^\beta W_c(t,x)| \leq C_{m\beta}\langle t\rangle^{-m}(\langle t\rangle + \langle x\rangle^{1/2})^{-\rho-|\beta|}. \tag{4.3}$$

We consider the time-dependent Hamiltonian

$$H_c(t) = H_c + W_c(t), \qquad W_c(t) = W_c(t,x),$$

and denote by $U_c(t)$ the propagator generated by $H_c(t)$, that is, $\{U_c(t)\}_{t\geq 1}$ is a family of unitary operators such that for $\psi \in \mathcal{D}(H_c(1))$, $\psi_t = U_c(t)\psi$ is a strong solution of $id\psi_t/dt = H_c(t)\psi_t$, $\psi_1 = \psi$.

By the almost analytic extension method, we see that for $f \in C_0^\infty(\mathbf{R})$,

$$\mathbf{D}_{H_c(t)} f(H_c(t)) = \frac{d}{dt}\{f(H_c(t))\} = O(t^{-1-\rho}), \tag{4.4}$$

due to (4.3). In virtue of this estimate (4.4), we obtain the analogue of Propositions 3.1, 3.2, 3.5 and 3.7. Since the proofs are similar to those of theirs, we omit the proofs.

Proposition 4.1. *There exists $M \gg 1$ dependent on f such that:*
(1) For $\psi \in L^2(X)$,

$$\int_1^\infty \frac{dt}{t} \left\| F\left(\frac{\langle x\rangle}{t^2} = M\right) f(H_c(t))U_c(t)\psi \right\|^2 \leq C\|\psi\|^2.$$

(2) For $\psi \in \mathcal{S}(X)$, $\mathcal{S}(X)$ being the Schwartz space over X,

$$\int_1^\infty \frac{dt}{t} \left\| F\left(\frac{\langle x\rangle}{t^2} \geq M\right) f(H_c(t))U_c(t)\psi \right\|^2 < \infty.$$

Proposition 4.2. *Let $0 < \nu < |E|$ and $L > 0$. Then for any $\psi \in L^2(X)$,*

$$\int_1^\infty \frac{dt}{t} \left\| F\left(-L \leq \frac{z}{t^2} \leq \frac{\nu}{2}\right) f(H_c(t))U_c(t)\psi \right\|^2 \leq C\|\psi\|^2.$$

Proposition 4.3. *Let M be as Proposition 4.1 and ν be as Proposition 4.2. Fix $\epsilon_1 > 0$ and $r > 0$. Assume that $q \in S_0(X)$ vanishes in $\Gamma(\omega, \epsilon_1, r)$. Then*

$$\int_1^\infty \frac{dt}{t} \left\| F\left(\frac{z}{t^2} \geq \frac{\nu}{2}\right) F\left(\frac{\langle x\rangle}{t^2} \leq M\right) q f(H_c(t))U_c(t)\psi \right\|^2 \leq C\|\psi\|^2.$$

Proposition 4.4. *Let M, ν and $q \in S_0(X)$ be as above. Let $\Phi(t)$ denote one of the following three operators*

$$F\left(\frac{\langle x\rangle}{t^2} \geq M\right), \quad F\left(\frac{z}{t^2} \leq \frac{\nu}{2}\right), \quad F\left(\frac{z}{t^2} \geq \frac{\nu}{2}\right) F\left(\frac{\langle x\rangle}{t^2} \leq M\right) q.$$

Then

$$s - \lim_{t\to\infty} \Phi(t) f(H_c(t))U_c(t) = 0.$$

Next we show the existence of the following two strong limits, the first limit being often called the Deift-Simon wave operator (see [Gr1], [SS1], [Z] and [A]).

Theorem 4.5. *Let the notations be as above. Then there exist the following strong limits:*

$$s - \lim_{t\to\infty} U_c(t)^* \exp(-itH), \qquad s - \lim_{t\to\infty} \exp(itH)U_c(t).$$

Proof. By Proposition 3.7, we have only to prove the existence of the strong limit

$$s - \lim_{t\to\infty} U_c(t)^* \varphi_c(t,x) f(H) \exp(-itH). \tag{4.5}$$

Taking $f_1 \in C_0^\infty(\mathbf{R})$ such that $f_1 f = f$ and noting $\varphi_c(t,x)(W_c(t,x) - I_c(x)) = 0$, we see that

$$f_1(H_c(t))\varphi_c(t,x) - \varphi_c(t,x) f_1(H) = O(t^{-1}),$$

in virtue of the almost analytic extension method. From this fact, the existence of (4.5) can be proved by the same argument in the proof of the existence of (3.5) (see the proof of Proposition 3.7).

Next we note that for $\epsilon > 0$ small enough and $\psi \in L^2(X)$, there exists $f \in C_0^\infty(\mathbf{R})$ such that

$$\|\{Id - f(H_c(t))\}U_c(t)\psi\| = O(\epsilon)$$

uniformly in $t \geq 1$, which is proved in virtue of (4.4). Now we take M as in Proposition 4.1 for this f and define $\varphi_c(t,x)$ by (4.1) with this M. Then, in virtue of Proposition 4.4, we have only to prove the existence of the strong limit

$$s - \lim_{t\to\infty} \exp(itH)\varphi_c(t,x) f(H_c(t)) U_c(t),$$

which may be proved in the same way as that of (4.5). □

By the existence of the first strong limit in the above proposition, we obtain the following theorem immediately.

Theorem 4.6. (The Asymptotic Clustering) *Let the notation be as above. Then for $\psi \in L^2(X)$, there exists $\psi_c \in L^2(X)$ such that*

$$\exp(-itH)\psi = U_c(t)\psi_c + o(1) \qquad \text{as } t \to \infty.$$

§5. Proof of Theorem 1.1

In this section, we complete the proof of Theorem 1.1. Throughout this section, we assume that $(V)_G$ is fulfilled. By Theorem 4.6, the proof is now reduced to studying the propagation properties of the propagator $U_c(t)$ generated by the time-dependent Hamiltonian $H_c(t)$.

As in §1, we define the time-dependent Hamiltonian $H_{cG}(t)$ by

$$H_{cG}(t) = H_c + I_c\left(\frac{E}{2}t^2\right) = H_c + W_c\left(t, \frac{E}{2}t^2\right) \tag{5.1}$$

and denote by $U_{cG}(t)$ the propagator generated by $H_{cG}(t)$. We note that $U_{cG}(t)$ is represented by

$$U_{cG}(t) = \exp(-itH_c)\exp\left(-i\int_0^t I_c\left(\frac{E}{2}s^2\right)ds\right). \tag{5.2}$$

First, we replace $U_c(t)$ by $U_{cG}(t)$.

Proposition 5.1. *There exist the strong limits*

$$s - \lim_{t\to\infty} U_{cG}(t)^* U_c(t), \qquad s - \lim_{t\to\infty} U_c(t)^* U_{cG}(t).$$

To prove this proposition, we need the following lemma. By virtue of the condition $(V)_G$, we should note that we may refine the estimate (4.3) as follows:

$$|\partial_t^m \partial_x^\beta W_c(t,x)| \le C_{m\beta}\langle t\rangle^{-m}(\langle t\rangle + \langle x\rangle^{1/2})^{-2(\rho+|\beta|)}. \tag{5.3}$$

On account of limited space, we omit the proof of the lemma below. For the proof, see [AT1,2].

Lemma 5.2. *Assume that $\psi \in \mathcal{S}(X)$. Then*
(1) $\|(x_c - Et^2/2)U_c(t)\psi\| = O(t)$.
(2) $\|(x_c - Et^2/2)U_{cG}(t)\psi\| = O(t)$.
(3) $\|x^c U_c(t)\psi\| = O(t)$.
(4) $\|x^c U_{cG}(t)\psi\| = O(t)$.

Proof of Proposition 5.1. We prove the existence of the first strong limit only. We may prove that of the second strong limit similarly. Since $\mathcal{S}(X)$ is dense in $L^2(X)$, it suffices to show the limit exists for $\varphi \in \mathcal{S}(X)$. Since $I_c(Et^2/2) = W_c(t, Et^2/2)$ for $t \ge 1$, we have

$$\begin{aligned}\frac{d}{dt}\left(U_{cG}(t)^* U_c(t)\varphi\right) =& U_0^G(t)^* i\left(W_c\left(t, \frac{E}{2}t^2\right) - W_c(t,x)\right)U_c(t)\varphi \\ =& U_{cG}(t)^* i\int_0^1 \left\langle (\nabla W_c)\left(s\left(\frac{E}{2}t^2\right) + (1-s)x\right), \left(\frac{E}{2}t^2 - x\right)\right\rangle ds \\ & \times U_c(t)\varphi.\end{aligned}$$

Then by (5.3) and Lemma 5.2, we see that for $\varphi \in \mathcal{S}(X)$, the right-hand side is of $O(t^{-2\rho-1})$, which is integrable since $-2\rho - 1 < -1$. Thus the proposition follows by Cook's method. □

The following proposition is an immediate consequence of Theorem 4.5 and Proposition 5.1.

Proposition 5.3. *There exist the following strong limits:*

$$s - \lim_{t\to\infty} U_{cG}(t)^* \exp(-itH), \qquad s - \lim_{t\to\infty} \exp(itH)U_{cG}(t).$$

If we write

$$\Theta(t) = \exp\left(-i\int_0^t I_c\left(\frac{E}{2}s^2\right)ds\right),$$

we may write $U_{cG}(t) = \Theta(t)\exp(-itH_c)$. Thus it follows from the existence of the first strong limit in Proposition 5.3 that for any $\psi \in L^2(X)$,

$$\exp(-itH)\psi = \Theta(t)\{\exp(-itH^c) \otimes \exp(-itT_c)\}\psi_c + o(1), \quad t \to \infty, \tag{5.4}$$

with some $\psi_c \in L^2(X)$. Then the proof of Theorem 1.1 is reduced to analyzing the asymptotic behavior as $t \to \infty$ of $\exp(-itH^c)$.

We now use the asymptotic completeness for the subsystem Hamiltonian H^c without constant electric fields (see [D2] and [Z]). For $a \subset c$, we define

$$H^c_{a0}(t) = H^c_a + I^c_a(D_a t) = H^a \otimes Id + Id \otimes (T^c_a + I^c_a(D_a t)) \text{ on } L^2(X^c) = L^2(X^a) \otimes L^2(X^c_a),$$

and denote the propagator $U^c_{a0}(t)$ generated by $H^c_{a0}(t)$ as

$$U^c_{a0}(t) = \exp(-itH^c_a)\exp\left(-i\int_0^t I^c_a(D_a s)ds\right) \qquad \text{on } L^2(X^c),$$

where $H^c_a = H^c - I^c_a$ is the cluster Hamiltonian obtained from H^c and $T^c_a = -\Delta^c_a/2$ acts on $L^2(X^c_a)$. Then we have the following theorem by virtue of [D2].

Theorem 5.4. *Assume that $(V)_G$ is fulfilled. Then the modified wave operators*

$$\Omega^c_a = s - \lim_{t\to\infty} \exp(itH^c)U^c_{a0}(t)(P^a \otimes Id) : L^2(X^c) \to L^2(X^c)$$

exist for all $a \subset c$, and are asymptotically complete, that is,

$$L^2(X^c) = \sum_{a \subset c} \oplus Ran\, \Omega^c_a.$$

Completion of the proof of Theorem 1.1. Let $H_{aG}(t)$, $a \subset c$, be defined by (1.4). Then $H_{aG}(t)$ is decomposed into

$$H_{aG}(t) = H^c_{a0}(t) \otimes Id + Id \otimes T_c + W_c\left(t, \frac{E}{2}t^2\right)(Id \otimes Id) \text{ on } L^2(X) = L^2(X^c) \otimes L^2(X_c).$$

The three operators on the right side commute with one another. The propagator $U_{aG}(t)$ generated by $H_{aG}(t)$ is also represented by

$$U_{aG}(t) = \Theta(t)(U^c_{a0}(t) \otimes \exp(-itT_c)) \qquad \text{on } L^2(X) = L^2(X^c) \otimes L^2(X_c).$$

In virtue of the existence of the second strong limit in Proposition 5.3, the existence of W^+_{aG} defined by (1.6) can be proved by showing the existence of the strong limit

$$\begin{aligned} s - \lim_{t\to\infty} U_{cG}(t)^* U_{aG}(t)(P^a \otimes Id) &= s - \lim_{t\to\infty} \{\exp(itH^c)U^c_{a0}(t)(P^a \otimes Id)\} \otimes Id \\ &= \Omega^c_a \otimes Id \qquad \text{on } L^2(X) = L^2(X^c) \otimes L^2(X_c), \end{aligned}$$

which follows from the existence of Ω^c_a. On the other hand, by (5.4) and Theorem 5.4, we have with $\psi_c = \sum_{j:\text{finite}} \psi^c_j \otimes \psi^j_c + O(\epsilon)$, $\psi^c_j \in L^2(X^c)$ and $\psi^j_c \in L^2(X_c)$,

$$\begin{aligned} \exp(-itH)\psi &= \Theta(t) \sum_{j:\text{finite}} \exp(-itH^c)\psi^c_j \otimes \exp(-itT_c)\psi^j_c + O(\epsilon) \\ &= \Theta(t) \sum_{j:\text{finite}} \sum_{a \subset c} \exp(-itH^c)(\Omega^c_a \tilde{\psi}^c_{aj}) \otimes \exp(-itT_c)\psi^j_c + O(\epsilon) \\ &= \Theta(t) \sum_{j:\text{finite}} \sum_{a \subset c} U^c_{a0}(t)\tilde{\psi}^c_{aj} \otimes \exp(-itT_c)\psi^j_c + O(\epsilon) + o(1) \end{aligned}$$

for some $\tilde{\psi}^c_{aj} \in L^2(X^c)$, which implies

$$\left\| \psi - \sum_{j:\text{finite}} \sum_{a \subset c} W^+_{aG}(\tilde{\psi}^c_{aj} \otimes \psi^j_c) \right\| = O(\epsilon).$$

Since $\epsilon > 0$ is arbitrary and $\sum_{a \subset c} \oplus Ran\, W^+_{aG}$ is closed, we have

$$\psi \in \sum_{a \subset c} \oplus Ran\, W^+_{aG},$$

which implies the asymptotic completeness of the wave operators W^+_{aG}. The proof of Theorem 1.1 is now completed. □

§6. Proof of Theorem 1.2

In this section, we complete the proof of Theorem 1.2. Throughout this section, we assume that $(V)_D$ is fulfilled. By Theorem 4.6, the proof is now reduced to studying the propagation properties of the propagator $U_c(t)$ generated by the time-dependent Hamiltonian $H_c(t)$. For this sake, we introduce a family of the unitary operators $\{T(t)\}_{t \in \mathbf{R}}$ on $L^2(X)$ as follows: For $u(x) \in L^2(X)$, we define

$$(T(t)u)(x) = \exp\left(i|E|zt - i\frac{|E|^2}{6}t^3\right) u\left(x - \frac{E}{2}t^2\right). \tag{6.1}$$

We also introduce the time-dependent Hamiltonians

$$\begin{aligned}
\tilde{H}_{aD}(t) &= H_a + I^c_a(D_a t) + I_c\left(D_a t - \frac{E}{2}t^2\right),\\
H_{a,S1}(t) &= H_a + I^c_a(D_a t) + W_c\left(t, 0, D_a t - \frac{E}{2}t^2\right),\\
H_{c,M0}(t) &= H_{c,M} + W_c\left(t, x^c, x_c + \frac{E}{2}t^2\right),\\
H_{a,M1}(t) &= H_{a,M} + I^c_a(D_a t) + W_c\left(t, 0, D_a t + \frac{E}{2}t^2\right),
\end{aligned}$$

for $a \subset c$, where $H_{a,M} = -\Delta/2 + V^a(x^a)$ acts on $L^2(X)$ and has no electric fields. We denote by $\tilde{U}_{aD}(t)$, $U_{a,S1}(t)$, $U_{c,M0}(t)$ and $U_{a,M1}(t)$ the propagators generated by $\tilde{H}_{aD}(t)$, $H_{a,S1}(t)$, $H_{c,M0}(t)$ and $H_{a,M1}(t)$, respectively, where $\tilde{U}_{aD}(0) = Id$, $U_{a,S1}(1) = Id$, $U_{c,M0}(1) = Id$ and $U_{a,M1}(1) = Id$. Since $\exp(-itH_a)D_a\exp(itH_a) = D_a - Et$ for $a \subset c$, $\tilde{U}_{aD}(t)$ is explicitly represented by

$$\tilde{U}_{aD}(t) = \exp(-itH_a)\exp\left(-i\int_0^t \left\{I^c_a(D_a s) + I_c\left(D_a s + \frac{E}{2}s^2\right)\right\} ds\right).$$

The family of transformations $\{T(t)\}_{t\in\mathbf{R}}$ was introduced by Jensen-Yajima [JY], by which Stark Hamiltonians are transformed into Hamiltonians without electric fields (see also [AH]). In fact, we see by the argument similar to [JY] that

$$U_c(t) = T(t)U_{c,M0}(t)T(1)^{-1}, \qquad U_{a,S1}(t) = T(t)U_{a,M1}(t)T(1)^{-1}. \tag{6.2}$$

We should note that $V^a(x^a)$ in $H_{a,M}$ does not undergo a change under the transformation $T(t)$. In virtue of the relation (6.2), we have only to study the asymptotic behavior of the propagator $U_{c,M0}(t)$ generated by the time-dependent Hamiltonian $H_{c,M0}$. We now apply to $H_{c,M0}$ the result due to Dereziński [D2] on the asymptotic completeness for long-range N-body quantum systems without electric fields (see also [Z]).

Proposition 6.1. *Assume that (V) is fulfilled. Then the modified wave operators*

$$\Omega_{a,M1} = s - \lim_{t\to\infty} U_{c,M0}(t)^* U_{a,M1}(t)(P^a \otimes Id)$$

exist for all $a \subset c$, and are asymptotically complete, that is,

$$L^2(X) = \sum_{a\subset c} \oplus Ran\, \Omega_{a,M1}.$$

The condition $\rho > \sqrt{3}-1$ in the assumption (V) is essentially used to prove this proposition only. We go back to the original propagator $U_c(t)$. Since $T(t)$ commutes with $P^a \otimes Id$ for $a \subset c$, the following theorem is obtained as an immediate consequence of Proposition 6.1.

Theorem 6.2. *Assume that (V) is fulfilled. Then the modified wave operators*

$$\Omega_{a,S1} = s - \lim_{t\to\infty} U_c(t)^* U_{a,S1}(t)(P^a \otimes Id)$$

exist for all $a \subset c$, and are asymptotically complete, that is,

$$L^2(X) = \sum_{a\subset c} \oplus Ran\, \Omega_{a,S1}.$$

We also need the following lemmas to analyze the propagators $\tilde{U}_{aD}(t)$ and $U_{a,S1}(t)$.

Lemma 6.3. *Let $\psi \in \mathcal{S}(X)$. Then as $t \to \infty$,*
(1) $\|(D_a - Et)\tilde{U}_{aD}(t)\psi\| = O(1)$.
(2) $\|(D_a - Et)U_{a,S1}(t)\psi\| = O(1)$.

Proof. Since

$$\mathbf{D}_{\tilde{H}_{aD}(t)}(D_a - Et) = 0, \qquad \mathbf{D}_{H_{a,S1}(t)}(D_a - Et) = 0,$$

(1) and (2) are obtained by integration. $\square$

Lemma 6.4. *Let $\psi \in \mathcal{S}(X)$ and $q_c(x) \in S_0(X)$ be as in §4. Set $\phi_c(t,x) = q_c(x/t^2)$ for $t \geq 1$. Then as $t \to \infty$,*
(1) $\|(1-\phi_c(t,0,D_a t - Et^2/2))\tilde{U}_{aD}(t)\psi\| = O(t^{-1})$.
(2) $\|(1-\phi_c(t,0,D_a t - Et^2/2))U_{a,S1}(t)\psi\| = O(t^{-1})$.

Proof. Since $\phi_c(t, Et^2/2) \equiv 1$ for $t \geq 1$ and $|\nabla\phi_c(t,x)| \leq Ct^{-2}$, the lemma follows from Lemma 6.3. □

By these lemmas, we have the next proposition.

Proposition 6.5. *There exist the strong limits*

$$s - \lim_{t\to\infty} \tilde{U}_{aD}(t)^* U_{a,S1}(t), \qquad s - \lim_{t\to\infty} U_{a,S1}(t)^* \tilde{U}_{aD}(t).$$

Proof. We write $\tilde{\phi}_a(t) = \phi_c(t,0,D_a t - Et^2/2)$. By Lemma 6.4, we have only to prove the existence of the limits

$$\lim_{t\to\infty} \tilde{U}_{aD}(t)^* \tilde{\phi}_a(t) U_{a,S1}(t)\psi,$$
$$\lim_{t\to\infty} U_{a,S1}(t)^* \tilde{\phi}_a(t) \tilde{U}_{aD}(t)\psi$$

for $\psi \in \mathcal{S}(X)$. Since

$$\begin{aligned} &\partial_t\tilde{\phi}_a(t) + i\{\tilde{H}_{aD}(t)\tilde{\phi}_a(t) - \tilde{\phi}_a(t)H_{a,S1}(t)\} \\ &= -t^{-2}\left\langle (D_a - Et), (\nabla_a q_c)\left(\frac{D_a}{t} - \frac{E}{2}\right)\right\rangle, \end{aligned}$$

we have the proposition in virtue of Lemma 6.3. □

We replace $\tilde{U}_{aD}(t)$ by the propagator $U_{aD}(t)$ defined by (1.8). We need the following proposition.

Proposition 6.6. *There exist the strong limits*

$$s - \lim_{t\to\infty} U_{aD}(t)^* \tilde{U}_{aD}(t), \qquad s - \lim_{t\to\infty} \tilde{U}_{aD}(t)^* U_{aD}(t).$$

Proof. Noting that $I_a(D_a s) = I_a(D_{a,\perp}s)$, we have only to prove that as $t \to \infty$, $\int_0^t \{I_c(p_a s + Es^2/2) - I_c(p_{a,\perp}s + Es^2/2)\}ds$ converges locally uniformly in p_a. We write

$$I_c\left(p_a s + \frac{E}{2}s^2\right) - I_c\left(p_{a,\perp}s + \frac{E}{2}s^2\right) = \int_0^1 \left\langle (\partial_\| I_c)\left(p_\| s\tau + p_{a,\perp}s + \frac{E}{2}s^2\right), p_\| s\right\rangle d\tau.$$

We note that

$$\begin{aligned} &\partial_s\left\{I_c\left(p_\| s\tau + p_{a,\perp}s + \frac{E}{2}s^2\right)\right\} \\ &= \left\langle (\partial_\| I_c)\left(p_\| s\tau + p_{a,\perp}s + \frac{E}{2}s^2\right), p_\|\tau + Es\right\rangle \\ &\quad + \left\langle (\partial_\perp I_c)\left(p_\| s\tau + p_{a,\perp}s + \frac{E}{2}s^2\right), p_{a,\perp}\right\rangle \\ &= \left\langle (\partial_\| I_c)\left(p_\| s\tau + p_{a,\perp}s + \frac{E}{2}s^2\right), Es\right\rangle + O(s^{-(1+\rho)}) \end{aligned}$$

holds locally uniformly in p_a and uniformly in $0 \le \tau \le 1$. Here we used the fact that $(\nabla I_c)(p_\| s\tau + p_{a,\perp}s + Es^2/2) = O(s^{-(1+\rho)})$ holds locally uniformly in p_a and uniformly in $0 \le \tau \le 1$. Hence we have

$$\begin{aligned}&\left\langle (\partial_\| I_c)\left(p_\| s\tau + p_{a,\perp}s + \frac{E}{2}s^2\right), p_\| s\right\rangle \\ =&\frac{\langle p_\|, \omega\rangle}{|E|}\partial_s\left\{I_c\left(p_\| s\tau + p_{a,\perp}s + \frac{E}{2}s^2\right)\right\} + O(s^{-(1+\rho)}).\end{aligned}$$

This implies the proposition. □

Combining the above two propositions, we have the following proposition.

Proposition 6.7. *There exist the strong limits*

$$s - \lim_{t\to\infty} U_{aD}(t)^* U_{a,S1}(t), \qquad s - \lim_{t\to\infty} U_{a,S1}(t)^* U_{aD}(t).$$

We should note that these operators commute with $P^a \otimes Id$. Indeed, we decompose $H_{aD}(t)$ and $H_{a,S1}$ on $L^2(X) = L^2(X^a) \otimes L^2(X_a)$ into

$$H_{aD}(t) = H^a \otimes Id + Id \otimes T_{aD}(t), \qquad H_{a,S1}(t) = H^a \otimes Id + Id \otimes T_{a,S1}(t),$$

where

$$\begin{aligned}T_{aD}(t) &= T_a + I_a^c(D_{a,\perp}t) + I_c(0, D_{a,\perp}t + Et^2/2),\\ T_{a,S1}(t) &= T_a + I_a^c(D_a t) + W_c(t, 0, D_a t - Et^2/2).\end{aligned}$$

Thus, if we denote by $\hat{U}_{aD}(t)$ and $\hat{U}_{a,S1}(t)$ the propagators which are generated by $T_{aD}(t)$ and $T_{a,S1}(t)$, respectively, we see that

$$\begin{aligned}s - \lim_{t\to\infty} U_{aD}(t)^* U_{a,S1}(t) &= Id \otimes \left(s - \lim_{t\to\infty} \hat{U}_{aD}(t)^* \hat{U}_{a,S1}(t)\right),\\ s - \lim_{t\to\infty} U_{a,S1}(t)^* U_{aD}(t) &= Id \otimes \left(s - \lim_{t\to\infty} \hat{U}_{a,S1}(t)^* \hat{U}_{aD}(t)\right).\end{aligned}$$

Completion of the proof of Theorem 1.2. The existence of the modified wave operators W_{aD}^+ with $a \subset c$ is proved by Theorem 4.5, Theorem 6.2 and Proposition 6.7. It follows from Theorem 6.2 that $Ran\, W_{aD}^+ \perp Ran\, W_{bD}^+$ if $a \ne b$.

We shall prove the asymptotic completeness. Let $\psi \in L^2(X)$. By Theorem 4.6, Theorem 6.2 and Proposition 6.7, we have as $t \to \infty$,

$$\begin{aligned}\exp(-itH)\psi &= U_c(t)\psi_c + o(1)\\ &= \sum_{a\subset c} U_c(t)\Omega_{a,S1}\psi_c^a + o(1)\\ &= \sum_{a\subset c} U_{a,S1}(t)\psi_c^a + o(1)\\ &= \sum_{a\subset c} U_{aD}(t)\tilde{\psi}_c^a + o(1)\end{aligned}$$

for some $\psi_c^a \in Ran\,(P^a \otimes Id)$, where $\tilde{\psi}_c^a = \lim_{t\to\infty} U_{aD}(t)^* U_{a,S1}(t)\psi_c^a$. This implies

$$\psi \in \sum_{a\subset c} \oplus\, Ran\, W_{aD}^+,$$

which completes the proof of Theorem 1.2. □

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