

On a curve in E^2 and S^2 which is close to a circle or a straight line

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Let γ be a closed C^2 curve of length L in the 2-dimensional Euclidean space E^2 . Let s be an arclength parameter of γ and k be the signed curvature of γ . If n denotes the rotational number of γ , we have

$$\int_{\gamma} k ds = 2\pi n. \quad (1)$$

By Cauchy-Schwarz inequality, we have

$$\begin{aligned} \int_{\gamma} k^2 ds &\geq \frac{1}{L} \left(\int_{\gamma} k ds \right)^2 \\ &= \frac{(2\pi n)^2}{L}. \end{aligned}$$

When $n \neq 0$, the equality holds if and only if γ is the n -time covering of a circle of radius $L/2\pi n$. Looking at this, we would like to consider the following question: How close is γ to a circle when $\int_{\gamma} k^2 ds - (2\pi n)^2/L$ is close to zero? Our first theorem gives an answer to this question.

Theorem 1. Let ε be any positive constant. Let γ be a closed curve of length L in E^2 with rotational number $n \neq 0$. If

$$\int_{\gamma} k^2 ds < \frac{(2\pi n)^2}{L} + \frac{2\pi^2 n^2 \varepsilon^2}{L^3},$$

then γ lies between two concentric circles of radii r and R with $|R - r| < \varepsilon$.

A similar problem can be considered for a simple closed curve in the 2-dimensional unit sphere S^2 . If γ is an oriented simple closed C^2 curve in S^2 which has length $L \leq 2\pi$ and bounds the region of area A , we have

$$\int_{\gamma} k ds = 2\pi - A. \quad (2)$$

By Cauchy–Schwarz inequality and the isoperimetric inequality

$$L^2 - 4\pi A + A^2 \geq 0, \quad (3)$$

we have

$$\begin{aligned} \int_{\gamma} k^2 ds &\geq \frac{1}{L} \left(\int_{\gamma} k ds \right)^2 \\ &= \frac{(2\pi - A)^2}{L} \\ &\geq \frac{4\pi^2 - L^2}{L}. \end{aligned}$$

When $L \leq 2\pi$, the equality holds if and only if γ is a small circle. When $\int_{\gamma} k^2 ds$ is “almost” equal to $(4\pi^2 - L^2)/L$, we have the following theorem.

Theorem 2. Let ε be any positive constant less than $\pi/2$. Let γ be a simple closed curve of length $L \leq 2\pi$ in S^2 . If

$$\int_{\gamma} k^2 ds < \frac{4\pi^2 - L^2}{L} + \frac{1}{8\pi} \varepsilon^2,$$

then γ lies between two concentric small circles of radii r and R with $|R - r| < \varepsilon$.

Now we would like to look at a curve of infinite length in E^2 . Let $\gamma : x(s)$ be a curve in E^2 which is parameterized by arclength s for $-\infty < s < \infty$. If $k(s) = 0$ for all s , γ is, of course, a straight line. If $k(s)$ decays to zero in a certain order as $s \rightarrow \pm\infty$, γ has an asymptotic line on each end. In [1] it was shown that if γ is properly immersed and there exist positive constants C_0 and ε such that $|k(s)||x(s)|^{2+\varepsilon} \leq C_0$ holds for all s , then each end of C has an asymptotic line. Here an asymptotic line means a straight line $\ell : y(t)$ which has a property that the function $h(s) := \inf_t |x(s) - y(t)|$ tends to zero as $s \rightarrow +\infty$ or $s \rightarrow -\infty$. If $|k(s)||x(s)|^{2+\varepsilon}$ is bounded, we have $\int_{-\infty}^{\infty} |k(s)||x(s)| ds < \infty$. In our third theorem, we will show that this condition is sufficient for γ to have an asymptotic line on each end.

Theorem 3. Let $\gamma : x(s)$ be a curve in E^2 which is parameterized by arclength s for $-\infty < s < \infty$. If $\int_{-\infty}^{\infty} |k(s)||x(s)| ds < \infty$, then γ is properly immersed and has an asymptotic line on each end.

§1. Proof of Theorem 1

We have

$$\begin{aligned}
\int_{\gamma} k^2 ds - \frac{(2\pi n)^2}{L} &= \int_{\gamma} k^2 ds - \frac{1}{L} \left(\int_{\gamma} k ds \right)^2 \\
&= \frac{1}{2L} \int_0^L \int_0^L (k(s) - k(t))^2 ds dt \\
&\geq \frac{1}{2L^2} \int_0^L \left(\int_0^L (k(s) - k(t)) dt \right)^2 ds \\
&= \frac{1}{2L^2} \int_0^L \left(Lk(s) - \int_0^L k(t) dt \right)^2 ds \\
&= \frac{1}{2} \int_0^L \left(k(s) - \frac{2n\pi}{L} \right)^2 ds. \tag{4}
\end{aligned}$$

We set $k_0 = 2n\pi/L$. Since $n \neq 0$, $k_0 \neq 0$. Suppose that

$$\int_{\gamma} k^2 ds - \frac{(2\pi n)^2}{L} < \delta. \tag{5}$$

Then (4) and (5) give

$$\int_0^L (k(s) - k_0)^2 ds < 2\delta. \tag{6}$$

(6) means that $k(s)$ is close to k_0 except for a set of a small measure. But, in general, this does not imply that γ is close to a circle.

Let $e(s)$ be the unit normal vector of γ with $\frac{d^2\gamma}{ds^2} = -ke$. We may assume that

$$\gamma(0) = \left(\frac{1}{k_0}, 0 \right), \quad \frac{d\gamma}{ds}(0) = (0, 1), \quad e(0) = \left(\frac{k_0}{|k_0|}, 0 \right).$$

For all $s \in [0, L]$ we have

$$\begin{aligned}
\left| \gamma(s) - \frac{1}{k_0} e(s) \right| &= \left| \int_0^s \left(\frac{d\gamma}{ds} - \frac{1}{k_0} \frac{de}{ds} \right) ds \right| \\
&= \left| \int_0^s \left(1 - \frac{k(s)}{k_0} \right) \frac{d\gamma}{ds} ds \right| \\
&\leq \frac{1}{|k_0|} \int_0^s |k(s) - k_0| ds \\
&\leq \frac{1}{|k_0|} \int_0^L |k(s) - k_0| ds
\end{aligned}$$

$$\begin{aligned}
&\leq \frac{1}{|k_0|} L^{1/2} \left(\int_0^L (k(s) - k_0)^2 ds \right)^{1/2} \\
&\leq \frac{\sqrt{2\delta L}}{|k_0|} \\
&= \frac{\sqrt{2\delta L^3}}{2\pi|n|}.
\end{aligned}$$

(7) implies that

$$\begin{aligned}
|\gamma(s)| &\leq \left| \frac{1}{k_0} e(s) \right| + \left| \gamma(s) - \frac{1}{k_0} e(s) \right| \\
&\leq \frac{L}{2\pi|n|} + \frac{\sqrt{2\delta L^3}}{2\pi|n|}
\end{aligned} \tag{8}$$

and

$$\begin{aligned}
|\gamma(s)| &\geq \left| \frac{1}{k_0} e(s) \right| - \left| \frac{1}{k_0} e(s) - \gamma(s) \right| \\
&\geq \frac{L}{2\pi|n|} - \frac{\sqrt{2\delta L^3}}{2\pi|n|}.
\end{aligned} \tag{9}$$

Hence γ lies between two concentric circles of radii $\frac{L}{2\pi|n|} - \frac{\sqrt{2\delta L^3}}{2\pi|n|}$ and $\frac{L}{2\pi|n|} + \frac{\sqrt{2\delta L^3}}{2\pi|n|}$. This proves Theorem 1 by setting

$$\delta = \frac{2\pi^2 n^2 \varepsilon^2}{L^3}.$$

Remark. When γ is a simple closed convex plane curve with bounding area A , Theorem 1 is obtained from Gage's inequality ([2])

$$\pi \frac{L}{A} \leq \int_{\gamma} k^2 ds$$

and the isoperimetric inequality of Bonnesen type

$$L^2 - 4\pi A \geq \pi^2 (R - r)^2.$$

Note that Gage's inequality does not hold for nonconvex curves.

§2. Proof of Theorem 2

Suppose that

$$\int_{\gamma} k^2 ds < \frac{4\pi^2 - L^2}{L} + \delta. \quad (10)$$

Since

$$\begin{aligned} \int_{\gamma} k^2 ds &\geq \frac{1}{L} \left(\int_{\gamma} k ds \right)^2 \\ &= \frac{(2\pi - A)^2}{L}, \end{aligned}$$

we have

$$\frac{(2\pi - A)^2}{L} < \frac{4\pi^2 - L^2}{L} + \delta,$$

or equivalently,

$$L^2 - 4\pi A + A^2 < L\delta. \quad (11)$$

The isoperimetric inequality given by Osserman ([3]) states

$$L^2 - 4\pi A + A^2 \geq \left(L - \cot \frac{r}{2} A \right)^2, \quad (12)$$

where r is the radius of the inscribed circle of γ . It follows from (11) and (12) that

$$\left(L - \cot \frac{r}{2} A \right)^2 < L\delta.$$

Hence

$$\frac{L - \sqrt{L\delta}}{A} < \cot \frac{r}{2} < \frac{L + \sqrt{L\delta}}{A}. \quad (13)$$

Another inequality obtained from (11) and (12) is

$$0 \leq L^2 - 4\pi A + A^2 < L\delta. \quad (14)$$

(14) implies that either

$$2\pi - \sqrt{4\pi^2 - L^2 + L\delta} < A \leq 2\pi - \sqrt{4\pi^2 - L^2} \quad (15)$$

or

$$2\pi + \sqrt{4\pi^2 - L^2} \leq A < 2\pi + \sqrt{4\pi^2 - L^2 + L\delta} \quad (16)$$

holds. By reversing the orientation of γ if necessary, we may assume that (15) holds.

Now we assume that $\delta < L$. Then, combining (13) and (15), we obtain

$$\frac{L - \sqrt{L\delta}}{2\pi - \sqrt{4\pi^2 - L^2}} < \cot \frac{r}{2} < \frac{L + \sqrt{L\delta}}{2\pi - \sqrt{4\pi^2 - L^2 + L\delta}},$$

or equivalently,

$$2 \arctan \left(\frac{2\pi - \sqrt{4\pi^2 - L^2 + L\delta}}{L + \sqrt{L\delta}} \right) < r < 2 \arctan \left(\frac{2\pi - \sqrt{4\pi^2 - L^2}}{L - \sqrt{L\delta}} \right). \quad (17)$$

Let γ_- be the curve which is identical to γ except for the orientation. Since (16) holds for γ_- , the radius r_- of the inscribed circle of γ_- satisfies

$$2 \arctan \left(\frac{2\pi + \sqrt{4\pi^2 - L^2}}{L + \sqrt{L\delta}} \right) < r_- < 2 \arctan \left(\frac{2\pi + \sqrt{4\pi^2 - L^2 + L\delta}}{L - \sqrt{L\delta}} \right). \quad (18)$$

Let R be the radius of the circumscribed circle of γ . Since $R = \pi - r_-$, we have

$$\begin{aligned} \pi - 2 \arctan \left(\frac{2\pi + \sqrt{4\pi^2 - L^2 + L\delta}}{L - \sqrt{L\delta}} \right) &< R \\ &< \pi - 2 \arctan \left(\frac{2\pi + \sqrt{4\pi^2 - L^2}}{L + \sqrt{L\delta}} \right). \end{aligned} \quad (19)$$

It follows from (17) and (19) that

$$\begin{aligned} R - r &< \pi - 2 \arctan \left(\frac{2\pi + \sqrt{4\pi^2 - L^2}}{L + \sqrt{L\delta}} \right) \\ &\quad - 2 \arctan \left(\frac{2\pi - \sqrt{4\pi^2 - L^2 + L\delta}}{L + \sqrt{L\delta}} \right). \end{aligned} \quad (20)$$

Set

$$\alpha = \frac{2\pi + \sqrt{4\pi^2 - L^2}}{L + \sqrt{L\delta}}, \quad \beta = \frac{2\pi - \sqrt{4\pi^2 - L^2 + L\delta}}{L + \sqrt{L\delta}}.$$

Then we see that

$$\tan(\pi - 2\alpha - 2\beta) = 2 \left(\frac{\alpha + \beta}{1 - \alpha\beta} \right) \left(\left(\frac{\alpha + \beta}{1 - \alpha\beta} \right)^2 - 1 \right)^{-1}. \quad (21)$$

Now we assume that $\delta < \frac{L}{16}$. Since

$$\beta \geq \frac{2\pi - \sqrt{4\pi^2 - L^2} - \sqrt{L\delta}}{L + \sqrt{L\delta}}, \quad (22)$$

we have

$$\begin{aligned}
\frac{\alpha + \beta}{1 - \alpha\beta} &\geq \frac{(4\pi - \sqrt{L\delta})(L + \sqrt{L\delta})}{(2L + 2\pi + \sqrt{4\pi^2 - L^2} + \sqrt{L\delta})\sqrt{L\delta}} \\
&\geq \frac{(4\pi - \frac{L}{4})L}{(2L + 2\pi + \sqrt{4\pi^2 - L^2} + \frac{L}{4})\frac{L}{4}} \\
&= \frac{64\pi - 4L}{9L + 8\pi + 4\sqrt{4\pi^2 - L^2}} \\
&> 1.
\end{aligned} \tag{23}$$

Since $f(y) = \frac{2y}{y^2 - 1}$ is decreasing for $y > 1$, it follows from (23) that

$$\begin{aligned}
&2 \left(\frac{\alpha + \beta}{1 - \alpha\beta} \right) \left(\left(\frac{\alpha + \beta}{1 - \alpha\beta} \right)^2 - 1 \right)^{-1} \\
&< \frac{2(4\pi - \sqrt{L\delta})(L + \sqrt{L\delta})}{(2L + 2\pi + \sqrt{4\pi^2 - L^2} + \sqrt{L\delta})\sqrt{L\delta}} \\
&\quad \times \left(\left(\frac{(4\pi - \sqrt{L\delta})(L + \sqrt{L\delta})}{(2L + 2\pi + \sqrt{4\pi^2 - L^2} + \sqrt{L\delta})\sqrt{L\delta}} \right)^2 - 1 \right)^{-1} \\
&= \frac{2(4\pi - \sqrt{L\delta})(L + \sqrt{L\delta})(2L + 2\pi + \sqrt{4\pi^2 - L^2} + \sqrt{L\delta})\sqrt{L\delta}}{(4\pi - \sqrt{L\delta})^2(L + \sqrt{L\delta})^2 - (2L + 2\pi + \sqrt{4\pi^2 - L^2} + \sqrt{L\delta})^2 L\delta} \\
&< \frac{2 \cdot 4\pi(L + \frac{L}{4})(2L + 2\pi + \sqrt{4\pi^2 - L^2} + \frac{L}{4})\sqrt{L\delta}}{(4\pi - \frac{L}{4})^2 L^2 - (2L + 2\pi + \sqrt{4\pi^2 - L^2} + \frac{L}{4})^2 \frac{L^2}{16}} \\
&= \frac{10\pi(\frac{9}{4}L + 2\pi + \sqrt{4\pi^2 - L^2})\sqrt{\delta}}{(4\pi - \frac{L}{4})^2 - \frac{1}{16}(\frac{9}{4}L + 2\pi + \sqrt{4\pi^2 - L^2})^2}\sqrt{L} \\
&< C_1 \frac{\sqrt{\delta}}{\sqrt{L}},
\end{aligned}$$

where

$$\begin{aligned}
C_1 &= \frac{10\pi(\frac{\sqrt{97}}{2}\pi + 2\pi)}{(4\pi - \frac{2\pi}{4})^2 - \frac{1}{16}(\frac{\sqrt{97}}{2}\pi + 2\pi)^2} \\
&\approx 7.48 \dots
\end{aligned}$$

Thus, for any positive constant $\varepsilon < \pi/2$, if

$$\delta \leq \frac{L\varepsilon^2}{C_1^2}, \tag{24}$$

then $R - r \leq \tan(R - r) < \varepsilon$. Note that the requirement that $\delta < \frac{L}{16}$ is automatically satisfied since $\varepsilon < \frac{\pi}{2}$. This proves Theorem 2 except for curves of length

close to zero. For curves with $0 < L < \pi$, we use

$$\beta \geq \frac{2\pi - \sqrt{4\pi^2 - L^2} - \frac{L\delta}{2\sqrt{4\pi^2 - L^2}}}{L + \sqrt{L\delta}}, \quad (25)$$

instead of (22). Again we assume that $\delta < \frac{L}{16}$. Then we have

$$\begin{aligned} \frac{\alpha + \beta}{1 - \alpha\beta} &\geq \frac{(4\pi - \frac{L\delta}{2\sqrt{4\pi^2 - L^2}})(L + \sqrt{L\delta})}{(2L^{3/2} + \frac{3}{2}L\sqrt{\delta} + \frac{\pi L}{\sqrt{4\pi^2 - L^2}}\sqrt{\delta})\sqrt{\delta}} \\ &> \frac{(4\pi - \frac{L^2}{32\sqrt{4\pi^2 - L^2}})L}{(2L^{3/2} + \frac{3}{8}L^{3/2} + \frac{\pi L}{\sqrt{4\pi^2 - L^2}}\frac{\sqrt{L}}{4})\sqrt{\delta}} \\ &> \frac{4\pi - \frac{\pi^2}{32\sqrt{4\pi^2 - \pi^2}}}{(2\sqrt{\pi} + \frac{3}{8}\sqrt{\pi} + \frac{\pi}{\sqrt{4\pi^2 - \pi^2}}\frac{\sqrt{\pi}}{4})\sqrt{\delta}} \\ &= \frac{(128\sqrt{3} - 1)\sqrt{\pi}}{(76\sqrt{3} + 8)\sqrt{\delta}} \\ &> \frac{(128\sqrt{3} - 1)\sqrt{\pi}}{(76\sqrt{3} + 8)\frac{\sqrt{\pi}}{4}} \\ &> 1. \end{aligned} \quad (26)$$

Hence

$$\begin{aligned} &2 \left(\frac{\alpha + \beta}{1 - \alpha\beta} \right) \left(\left(\frac{\alpha + \beta}{1 - \alpha\beta} \right)^2 - 1 \right)^{-1} \\ &< \frac{2(128\sqrt{3} - 1)\sqrt{\pi}}{(76\sqrt{3} + 8)\sqrt{\delta}} \left(\left(\frac{(128\sqrt{3} - 1)\sqrt{\pi}}{(76\sqrt{3} + 8)\sqrt{\delta}} \right)^2 - 1 \right)^{-1} \\ &= \frac{2(128\sqrt{3} - 1)\sqrt{\pi}}{76\sqrt{3} + 8} \left(\left(\frac{(128\sqrt{3} - 1)\sqrt{\pi}}{76\sqrt{3} + 8} \right)^2 - \delta \right)^{-1} \sqrt{\delta} \\ &< C_2\sqrt{\delta}, \end{aligned}$$

where

$$\begin{aligned} C_2 &= \frac{2(128\sqrt{3} - 1)\sqrt{\pi}}{76\sqrt{3} + 8} \left(\left(\frac{(128\sqrt{3} - 1)\sqrt{\pi}}{76\sqrt{3} + 8} \right)^2 - \frac{\pi}{16} \right)^{-1} \\ &\approx 0.73 \dots \end{aligned}$$

Thus, for any positive constant $\varepsilon < \pi/2$, if we have

$$0 < L < \pi, \quad \delta < \frac{\varepsilon^2}{C_2^2}, \quad \delta < \frac{L}{16}, \quad (27)$$

then $R - r \leq \tan(R - r) < \varepsilon$.

The remaining case is when $0 < L < \pi$ and $\delta \geq \frac{L}{16}$. In this case, we take

$$\delta = \frac{\varepsilon^2}{8\pi}. \quad (28)$$

Then we have

$$L \leq 16\delta < \frac{2\varepsilon^2}{\pi} < \varepsilon.$$

Hence $R - r < \varepsilon$.

Now the proof of Theorem 2 is complete from (24), (27), (28) and

$$\min \left\{ \frac{\pi}{C_1^2}, \frac{1}{C_2^2}, \frac{1}{8\pi} \right\} = \frac{1}{8\pi}.$$

§3. Proof of Theorem 3

First we show that γ is properly immersed. Since $|k(s)||x(s)| \geq 0$, $\int_0^\infty |k(s)||x(s)| ds$ converges to some positive constant A . Set $r(s) = |x(s)|$. We have

$$\begin{aligned} \frac{dr^2}{ds}(s) - \frac{dr^2}{ds}(0) &= \int_0^s \frac{d^2r^2}{ds^2} ds \\ &= \int_0^s \left(2 \left\langle \frac{dx}{ds}, \frac{dx}{ds} \right\rangle + 2 \left\langle x, \frac{d^2x}{ds^2} \right\rangle \right) ds \\ &\geq \int_0^s (2 - 2|x(s)| \left| \frac{d^2x}{ds^2} \right|) ds \\ &\geq 2s - 2A. \end{aligned}$$

This shows that $\frac{dr^2}{ds} \rightarrow \infty$ as $s \rightarrow \infty$. Hence $|x(s)| \rightarrow \infty$ as $s \rightarrow \infty$. A similar argument shows that $|x(s)| \rightarrow \infty$ as $s \rightarrow -\infty$. Thus γ is properly immersed.

As the second step, we show that $\lim_{s \rightarrow \infty} x^\perp(s)$ and $\lim_{s \rightarrow -\infty} x^\perp(s)$ exist, where $x^\perp(s) = x(s) - \langle x(s), \frac{dx}{ds} \rangle \frac{dx}{ds}$. It follows from $\int_0^\infty |k(s)||x(s)| ds = A$ that, for any $\varepsilon > 0$, there exists $s_0 > 0$ such that $\int_{s_1}^{s_2} |k(s)||x(s)| ds < \varepsilon$ for any $s_1, s_2 \geq s_0$.

Since

$$\begin{aligned} \left| \frac{dx^\perp}{ds} \right|^2 &= \left| \frac{dx}{ds} - \left\langle \frac{dx}{ds}, \frac{dx}{ds} \right\rangle \frac{dx}{ds} - \left\langle x(s), \frac{d^2x}{ds^2} \right\rangle \frac{dx}{ds} - \left\langle x(s), \frac{dx}{ds} \right\rangle \frac{d^2x}{ds^2} \right|^2 \\ &= \left\langle x(s), \frac{d^2x}{ds^2} \right\rangle^2 + \left\langle x(s), \frac{dx}{ds} \right\rangle^2 |k(s)|^2 \\ &= |k(s)|^2 |x(s)|^2, \end{aligned}$$

we have

$$\begin{aligned}
 |x^\perp(s_2) - x^\perp(s_1)| &= \left| \int_{s_1}^{s_2} \frac{dx^\perp}{ds} ds \right| \\
 &\leq \int_{s_1}^{s_2} \left| \frac{dx^\perp}{ds} \right| ds \\
 &= \int_{s_1}^{s_2} |k(s)| |x(s)| ds \\
 &< \varepsilon.
 \end{aligned}$$

This shows that $\lim_{s \rightarrow \infty} x^\perp(s)$ exists. Similarly, $\lim_{s \rightarrow -\infty} x^\perp(s)$ exists.

Set $x_\infty^\perp = \lim_{s \rightarrow \infty} x^\perp(s)$. Now we show that $\lim_{s \rightarrow \infty} \langle x^\top(s), x_\infty^\perp \rangle = 0$, where $x^\top(s) = \langle x(s), \frac{dx}{ds} \rangle \frac{dx}{ds}$. Let $\theta(s)$ be the angle between $x^\perp(s)$ and x_∞^\perp . Then we have

$$\left| \frac{d\theta}{ds} \right| = |k(s)|$$

and

$$\begin{aligned}
 |\langle x^\top(s), x_\infty^\perp \rangle| &\leq |x(s)| |x_\infty^\perp| |\sin \theta(s)| \\
 &\leq |x(s)| |x_\infty^\perp| |\theta(s)|.
 \end{aligned}$$

As above, let s_0 be a positive constant such that $\int_{s_0}^{\infty} |k(s)| |x(s)| ds < \varepsilon$. For any $s \geq s_0$ we have

$$\begin{aligned}
 |x(s)| |\theta(s)| &\leq |x(s)| \int_s^{\infty} \left| \frac{d\theta}{dt} \right| dt \\
 &= |x(s)| \int_s^{\infty} |k(t)| dt \\
 &\leq \int_s^{\infty} |k(t)| |x(t)| dt \\
 &< \varepsilon.
 \end{aligned}$$

This gives

$$\lim_{s \rightarrow \infty} |x(s)| |\theta(s)| = 0.$$

Hence we have

$$\lim_{s \rightarrow \infty} \langle x^\top(s), x_\infty^\perp \rangle = 0,$$

which implies that

$$\lim_{s \rightarrow \infty} \langle x(s) - x_\infty^\perp, x_\infty^\perp \rangle = 0.$$

This shows that the straight line $\ell := \{y : \langle y - x_\infty^\perp, x_\infty^\perp \rangle = 0\}$ is an asymptotic line of γ .

References

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