数理解析研究所講究録
909巻 1995年 51-53

TOWARD WAVE MODELS OF REPRESENTATIONS OF REAL SEMISIMPLE LIE GROUPS

by Takayuki Oda 織田孝幸(東大・数理)

This is an introduction for a number of talks given in this conference.

A typical situation to construct automorphic L-funcions from automorphic forms, is the following.

We are given a semisimple algebraic group G over a field k of Kronecker dimension 1. Sometime G is assumed to be quasi-split. This is particularly the case, when one use Whittaker models. Also a closed algebraic subgroup R of G is given which is large enough so that it is *spherical.*

Assume that G is quasi-split for simplicity. Fix a minimal parabolic subgroup P of G over k. Then the subgroup R of G is called spherical, when the orbit of the extension of scalars $\bar{R} = R \otimes_k \bar{k}$ of R on the flag manifold $G \otimes \bar{k}/P \otimes \bar{k}$ has open orbit. As shown by M Brion ([B1]), this condition is equivalent to the follwing.

(*) The restriction to $\bar{R}$ of any finite dimensional irreducible representation of $\bar{G}$ is multiplicity-free.

Also as shown in [B1], this condition implies and is equivalent to the condition that the double coset $\bar{R}\backslash\bar{G}/\bar{P}$ is finite.

Remark. The notion of "spherical" seems to be discussed only for the case of groups over algebraically closed fields, in the literature. Of course one can generalize this concept for any k, and can use the terminology "k-spherical groups", if one can prove any non-trival statement about that.....

Now, let us replace the global field k by its completion at a finite place, which we also denote by the same k. The k-valued points of algebraic groups G, P, R are denoted by the same symbol by an abuse of symbol.

An optimist might hope the following.

Folklore. *Given an admissible irreducible representation η of R. Let $C_\eta^\infty(R\backslash G) = Ind_R^G(\eta)$ be the smooth induction of η to G. Then for any irreducible admissible representation π of G, the dimension of intertwinig space*

$$Hom_G(\pi, C_\eta^\infty(R\backslash G))$$

is at most 1.

In view of Frobenius reciprocity with respect G and R, this is no other than a kind of multiplicity free statement for the restriction of π to the subgroup R.

When k is a p-adic fielf, this problem is considered by Murase and Sugano in their work (see their articles in this proceedings) on the construction of automorphic L-functions, under the name "Shintani function". Among others they (and S. Kato) now have explicit formula of K-invariant vector in the image of the unique element $\Phi \in Hom_G(\Pi, C_\eta^\infty(R\backslash G))$, when π is of class 1 (i.e. as spherical principal series representation), in rather general situation.

Naive analogue at archemean places does not seem to hold in general. The typical counter-example is the case when R is a maximal unipotent subgroup and η is a non-degenerate character. Then the above space is the space of Whittaker vectors (or functionals if you prefer this terminology). As shown by Kostant ([K]), the dimension of the above intertwing space for irreducible principal series representation of G is the order of the (little) Weyl group. In this case, to have multiplicity one statement, we have impose an increasing condition for the functions in target space $C_\eta^\infty(R\backslash G)$, to replace it by much smaller space. As shown by Wallach ([W]), we have multiplicity-free statement after this modification.

At the real place, Yamashita [Y] proved various sufficient conditions for the finiteness of the above intertwining space. Because we need further terminology to describe his results, we do not give the details of his results here. They cover various important cases in application. I recommend the readers to consult with the original paper. Here we quote a theorem of [B-O] which is easy to state.

Theorem. *(Bien-Oshima [B-O]) Set* $k = \mathbf{R}$. *Assume that* $R_{\mathbf{C}}$ *has open orbits on the flag manifold* $G_{\mathbf{C}}/P_{\mathbf{C}}$. *Then for any finite-dimensional representation* η *of* R *and any irreducible admissible* $(\mathfrak{g}, K)$*-module* π, *the interwining space* $Hom_{\mathfrak{g},K}(\pi, C_\eta^\infty(R\backslash G))$ *is of finite dimension.*

Now let me explain the titles of some talks given here. But before that recall the following. When $G = S_p(2 : \mathbb{R})$, Miyazaki-Oda[ ] computed explicitely the holonomic system for the radial part of the principal series Whittaker functions with smallest K-types. Oda[ ] and Miyazaki-Oda[ ] give integral expressions of Whittaker function with the corner K-type for the discrete series representations and generalized principal series for the cuspidal parabolic subgroup corresponding to long root.

The talk of Iida (Masatoshi) will discuss the holomorphic system of the radial part of the matrix coefficients with smallest k-types of the principal series representation of $Sp(2; \mathbb{R})$, and this shift operates. Oda's talk is on the holonomic system of the matrix coefficients with minimal K-type of the large discrete series representations of $Sp(2; \mathbb{R})$.

Takuya Miyazaki will talk about the holonomic system for the radial part of generalized Whittaker functions on $Sp(2; \mathbb{R})$ with respect to the Siegel parabolic subgroup.

Hayata (Takahiro) will treat principal series Whittaker function on $SU(2, 2)$ with smallest K-type.

Taniguchi (Kenji) works out completely the Whittaker functions with minimal K-types of the discrete series on real unitary group $SU(n,1)$ of split-rank 1 (and recently he settled also the case of $Spin(2n,1)$).

Tuzuki considers the case when $G = SU(2,1)$ and $H = S(U(1,1)\times U(1) \cong U(1,1)$ when n the representations η of H are of *infinite-dimension.*

References

On spherical subgroups

The following references are noticed by Prof. T. Matsuki

[Br1] Brion, M.: Quelques propriétés des espaces homogènes sphériques, Manuscrip Math. **55** (1986) 191–198.

[Br2] Brion, M.: Classification des espaces homogénes sphériques, Comp. Math. **63** (1987) 189–208.

[K] Krämer, M.: Sphärische Untergruppen in Kompakten zusammenhängenden Liegruppen Comp. Math. **38** (1979) 129–153.

The section 4 of

[M] Matsuki, T.: Orbits on Flag Manifolds, Proc. of Intern. Congress of Math., Kyoto (1990), 807–813.

is illuiminating. Here is a much simpler proof of the main theorem of [Br1]. The conjecture 1,2 on real spherical subgroups given there are solved by B. Kimelfeld and F. Bien.

On multiplicity finiteness

[W] Wallach, n.: Asymptotic expansions of generalized matrix entries of representations of real reductive groups. Lie Group Representations I. Lecture Notes in Math. **1024**, Springer (1984), 284–369.

[B-O] Bien, F. and Oshima, T.: Multiplicities of induced representations of semisim ple his groups, preprint.

[Y] Yamashita, H.: Criteria for the finiteness of restinction on - modules to subalgebras and application to Harish-Chandra modules. J. Funct. Annual. **121** (1994), 296–329.