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Betti numbers of minimal free resolutions of Stanley-Reisner rings

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Introduction

A *simplicial complex* Δ on the *vertex set* $V = \{x_1, x_2, \ldots, x_v\}$ is a collection of subsets of V such that (i) $\{x_i\} \in \Delta$ for every $1 \leq i \leq v$ and (ii) $\sigma \in \Delta$, $\tau \subset \sigma \Rightarrow \tau \in \sigma$. Each element σ of Δ is called a *face* of Δ. Let $\sharp(\sigma)$ denote the cardinality of a finite set σ. We set $d = \max\{\sharp(\sigma) \mid \sigma \in \Delta\}$ and define the *dimension* of Δ to be $\dim \Delta = d - 1$.

Let $A = k[x_1, x_2, \ldots, x_v]$ be the polynomial ring in v-variables over a field k. Here, we identify each $x_i \in V$ with the indeterminate x_i of A. Define I_Δ to be the ideal of A which is generated by square-free monomials $x_{i_1} x_{i_2} \cdots x_{i_r}$, $1 \leq i_1 < i_2 < \cdots < i_r \leq v$, with $\{x_{i_1}, x_{i_2}, \cdots, x_{i_r}\} \notin \Delta$. We say that the quotient algebra $k[\Delta] := A/I_\Delta$ is the *Stanley-Reisner ring* of Δ over k. In what follows, we consider A to be the graded algebra $A = \bigoplus_{n \geq 0} A_n$ with the standard grading, i.e., each $\deg x_i = 1$, and may regard $k[\Delta] = \bigoplus_{n \geq 0} (k[\Delta])_n$ as a graded module over A with the quotient grading.

We are interested in a minimal free resolution of $k[\Delta]$.

Let $\mathbf{Z}$ (resp. $\mathbf{Q}$) denote the set of integers (resp. rational numbers). We write $A(j)$, $j \in \mathbf{Z}$, for the graded module $A(j) = \bigoplus_{n \in \mathbf{Z}} [A(j)]_n$ over A with $[A(j)]_n := A_{n+j}$. When $k[\Delta]$ is regarded as a graded module over A with the quotient grading, it has a graded *finite free resolution*

$$0 \longrightarrow \bigoplus_{j \in \mathbf{Z}} A(-j)^{\beta_{h_j}} \xrightarrow{\varphi_h} \cdots \xrightarrow{\varphi_2} \bigoplus_{j \in \mathbf{Z}} A(-j)^{\beta_{1_j}} \xrightarrow{\varphi_1} A \xrightarrow{\varphi_0} k[\Delta] \longrightarrow 0; \quad (1)$$

where each $\bigoplus_{j \in \mathbf{Z}} A(-j)^{\beta_{i_j}}$, $1 \leq i \leq h$, is a graded free module of rank $0 \neq \sum_{j \in \mathbf{Z}} \beta_{i_j} < \infty$, and where every φ_i is degree-preserving. Moreover, there exists a unique such resolution which minimizes each β_{i_j}; such a resolution is called *minimal.* If a finite free resolution (1) is minimal, then the *homological dimension* $\mathrm{hd}_A(k[\Delta])$ of $k[\Delta]$ over A is the non-negative integer h and

$\beta_i = \beta_i^A(k[\Delta]) := \sum_{j\in\mathbf{Z}} \beta_{i_j}$ is called the i-th *Betti number* of $k[\Delta]$ over A.

Even though $\mathrm{hd}_A(k[\Delta])$ may depend on the base field k, (with a fixed field k) the integer $v - \mathrm{hd}_A(k[\Delta])$ is topological [Mun], i.e., it depends only on the geometric realization of Δ. Since the first Betti number $\beta_1^A(k[\Delta])$ is equal to the minimal number of generators of the ideal I_Δ, $\beta_1^A(k[\Delta])$ is independent of the base field k. However, in general, $\beta_i^A(k[\Delta])$ may depend on k. It is known, e.g., [Bru–Her$_2$] that the second Betti number $\beta_2^A(k[\Delta])$ does not depend on the base field k. We give a short proof of this result by using the Alexander duality theorem of topology. Moreover, when the ideal I_Δ is generated by square-free monomials of degree two (e.g., Δ is the order complex of a finite partially ordered set), we show that both the third and fourth Betti numbers of $k[\Delta]$ over A are independent of k. On the other hand, it would be of interest to find a natural class of simplicial complexes Δ for which all Betti numbers $\beta_i^A(k[\Delta])$ are independent of k. We show that, for example, if the geometric realization of Δ is either a 3-sphere or a 3-ball, then all Betti numbers of $k[\Delta]$ are independent of k.

Finally we give a ring-theoretical short proof of the following result, which was first proved by Barnette.

Theorem. *The 1-skelton of a simplicial $(d-1)$-sphere is d-connected.*

§1. Hochster's formula

We first recall some notation on simplicial complexes and Hochster's topological formula on Betti numbers of Stanley-Reisner rings. We refer the reader to, e.g., [Bru–Her$_1$], [H$_1$], [Hoc] and [Sta$_1$] for the detailed information about combinatorial and algebraic background.

(1.1) Given a subset W of V, the *restriction* of Δ to W is the subcomplex

$$\Delta_W = \{\sigma \in \Delta \mid \sigma \subset W\}$$

of Δ. In particular, $\Delta_V = \Delta$ and $\Delta_\emptyset = \{\emptyset\}$. On the other hand, if σ is a face of Δ, then we define the subcomplexes $\mathrm{link}_\Delta(\sigma)$ and $\mathrm{star}_\Delta(\sigma)$ to be

$$\mathrm{link}_\Delta(\sigma) = \{\tau \in \Delta \mid \sigma \cap \tau = \emptyset, \sigma \cup \tau \in \Delta\};$$

$$\text{star}_\Delta(\sigma) = \{\tau \in \Delta \mid \sigma \cup \tau \in \Delta\}.$$

Thus, in particular, $\text{link}_\Delta(\emptyset) = \text{star}_\Delta(\emptyset) = \Delta$.

Let $\tilde{H}_i(\Delta; k)$ denote the i-th *reduced simplicial homology group* of Δ with the coefficient field k. Note that $\tilde{H}_{-1}(\Delta; k) = 0$ if $\Delta \neq \{\emptyset\}$ and

$$\tilde{H}_i(\{\emptyset\}; k) = \begin{cases} 0 & (i \geq 0) \\ k & (i = -1). \end{cases}$$

(1.2) In the above notation Hochster's formula [Hoc, Theorem (5.1)] is that

$$\beta_{i_j} = \sum_{W \subset V,\ \sharp(W)=j} \dim_k \tilde{H}_{j-i-1}(\Delta_W; k). \tag{2}$$

Thus, in particular,

$$\beta_i^A(k[\Delta]) = \sum_{W \subset V} \dim_k \tilde{H}_{\sharp(W)-i-1}(\Delta_W; k). \tag{3}$$

Some combinatorial and algebraic applications of Hochster's formula have been studied. Munkres [Mun] proved that $v - \text{hd}_A(k[\Delta])$ depends only on the geometric realization of Δ. Moreover, if Δ is the order complex of a modular lattice, then the last Betti number of $k[\Delta]$ can be computed by means of the Möbius function of the lattice ([H_2], [H_3]). See also [Bac], [B–H_1], [B–H_2], [Frö] and [H_4] for related topics and results.

§2. Second Betti numbers of Stanley–Reisner rings

It is known, e.g., [Bru–Her_2] that the second Betti number of a Stanley-Reisner ring is independent of the base field. By virtue of Hochster's formula together with the Alexander duality theorem of topology, we give a short proof of this result. Let $\mid \Delta \mid$ denote the geometric realization of a simplicial complex Δ.

(2.1) LEMMA. *Let Δ be a simplicial complex on the vertex set V with $\sharp(V) = v$ and k a field. Then $\dim_k \tilde{H}_{v-3}(\Delta; k)$ is independent of k.*

Proof. Let 2^V denote the set of all subsets of V. Thus, the geometric realization X of the simplicial complex $2^V - \{V\}$ is the $(v-2)$-sphere. We may assume that $V \notin \Delta$; in particular, $\mid \Delta \mid$ is a subspace of X. Note that $H_{v-3}(\mid \Delta \mid; k) \cong H^{v-3}(\mid \Delta \mid; k)$ since k is a field. Now, the Alexander duality theorem guarantees that $H^{v-3}(\mid \Delta \mid; k) \cong \tilde{H}_0(X - \mid \Delta \mid; k)$. On the other hand, $\dim_k \tilde{H}_0(X- \mid \Delta \mid; k) + 1$ is equal to the number of connected components of $X- \mid \Delta \mid$. Thus, $\dim_k H_{v-3}(\Delta; k) = \dim_k \tilde{H}_0(X- \mid \Delta \mid; k)$ is independent of the base field k as required. Q. E. D.

(2.2) THEOREM. *The second Betti number $\beta_2^A(k[\Delta])$ of the Stanley-Reisner ring $k[\Delta] = A/I_\Delta$ of a simplicial complex Δ is independent of the base field k.*

Proof. By virtue of Hochster's formula (4), the second Betti number $\beta_2^A(k[\Delta])$ is equal to $\sum_{W\subset V} \dim_k \tilde{H}_{\sharp(W)-3}(\Delta_W; k)$, which is independent of k by Lemma (2.1) as desired. Q. E. D.

§3. Ideals I_Δ generated by monomials of degree two

The purpose of this section is to show that the third and fourth Betti numbers of a Stanley-Reisner ring $k[\Delta] = A/I_\Delta$ are independent of the base field k when the ideal I_Δ is generated by square-free monomials of degree two. For example, the ideal I_Δ associated with a simplicial complex Δ is generated by square-free monomials of degree two when, e.g., Δ is the order complex ([Sta$_3$, p.120]) of a finite partially ordered set.

Let Δ (resp. Δ') be a simplicial complex on the vertex set V (resp. V') and suppose that $V \cap V' = \emptyset$. Recall that the *simplicial join* $\Delta * \Delta'$ of Δ and Δ' is the simplicial complex on the vertex set $V \cup V'$ which consists of all subsets of $V \cup V'$ of the form $\sigma \cup \tau$ with $\sigma \in \Delta$ and $\tau \in \Delta'$.

(3.1) LEMMA. *Let Δ be a simplicial complex on the vertex V with $\sharp(V) = v$ and suppose that the ideal I_Δ is generated by square-free monomials of degree two. Then $\tilde{H}_n(\Delta; k) = 0$ if $v < 2(n+1)$. Moreover, if $v = 2(n+1)$, then $\tilde{H}_n(\Delta; k) \neq 0$ if and only if Δ is the simplicial join of $n+1$ copies of the 0-sphere $S^0 (= \bullet \;\; \bullet)$.*

(3.2) COROLLARY. *Suppose that the ideal I_Δ is generated by square-free monomials of degree two and that a finite free resolution (2) of $k[\Delta] = A/I_\Delta$ over A is minimal. Then, $\beta_{i_j} = 0$ for all i and j with $j > 2i$.*

Proof. By Lemma (3.1), we have $\tilde{H}_{\sharp(W)-i-1}(\Delta_W; k) = 0$ if $\sharp(W) < 2(\sharp(W) - i)$, i.e., $\sharp(W) > 2i$. Hence, thanks to Hochster's formula (3), $\beta_{i_j} = 0$ for all i and j with $j > 2i$. Q. E. D.

Taylor [Tay] constructed an explicit (not nesessarily minimal) finite free resolution of $k[\Delta] = A/I_\Delta$ over A. The above Corollary (3.2) also follows immediately from Taylor resolutions.

(3.3) LEMMA. *Let Δ be a simplicial complex on the vertex set V with $\sharp(V) = 7$. Suppose that I_Δ is generated by square-free monomials of degree two and that $\tilde{H}_2(\Delta; k) \neq 0$. Then, one of the following conditions* (i) *and* (ii) *is satisfied:*

(i) *Δ is the simplicial join of the cycle of length* 5 *and* 0*-sphere* $\mathbf{S}^0$;

(ii) *there exists $x \in V$ such that* $\Delta_{V-\{x\}} = \mathbf{S}^0 * \mathbf{S}^0 * \mathbf{S}^0$.

We are now in the position to state the main result of this section.

(3.4) THEOREM. *Let Δ be a simplicial complex and suppose that the ideal I_Δ is generated by square-free monomials of degree two. Then, both the third Betti number $\beta_3^A(k[\Delta])$ and the fourth Betti number $\beta_4^A(k[\Delta])$ of $k[\Delta] = A/I_\Delta$ over A are independent of the base field k.*

Proof. First, we study the third Betti number $\beta_3^A(k[\Delta])$ of $k[\Delta]$ over A. Let V be the vertex set of Δ. Thanks to Proposition (3.2), what we must prove is that β_{3_j} is independent of the base field k for every $j \leq 6$. Thus, by virtue of Hochster's formula (3), what we must prove is that $\dim \tilde{H}_{\sharp(W)-4}(\Delta_W; k)$ is independent of k for every $W \subset V$ with $\sharp(W) \leq 6$. If $\sharp(W) = 5$, then $\tilde{H}_i(\Delta_W; k) = 0$ for every $i \geq 2$ by Lemma (3.1). Thus, since the reduced Euler characteristic $\tilde{\chi}(\Delta)$ and $\dim_k \tilde{H}_0(\Delta_W; k)$ are independent of k, it follows from Euler-Poincaré formula that $\dim \tilde{H}_1(\Delta_W; k)$ is independent of k. On the other hand, if $\sharp(W) = 6$, then $\dim \tilde{H}_2(\Delta_W; k) = 0$ unless Δ_W is the simplicial join of three copies of the 0-sphere by Lemma (3.1). Moreover, if Δ_W is the simplicial join of three copies of the 0-sphere, then $\dim \tilde{H}_2(\Delta_W; k) = 1$ for an arbitrary field k.

Secondly, we show that the fourth Betti number $\beta_4^A(k[\Delta])$ of $k[\Delta]$ over A is independent of the base field k. We must prove that $\dim \tilde{H}_{\sharp(W)-5}(\Delta_W; k)$ is independent of k for every $W \subset V$ with $\sharp(W) \leq 8$. If either $\sharp(W) = 6$ or $\sharp(W) = 8$, then we can show that $\dim \tilde{H}_{\sharp(W)-5}(\Delta_W; k)$ is independent of k by the similar technique with Lemma (3.1) as above. Let $\sharp(W) = 7$ and suppose that $\tilde{H}_2(\Delta_W; k) \neq 0$. Then, by Lemma (3.3), we easily see that Δ_W has the homotopy type of one of the following spaces: (i) the 2-sphere; (ii) the disjoint union of the 2-sphere and a single point; (iii) the space $X \cup Y$, where X is the 2-sphere and Y is either the 1-sphere or the 2-sphere, such that $X \cap Y$ consists of a single point. Hence, $\dim_k \tilde{H}_2(\Delta_W; k)$ is independent of the base field k as desired. Q. E. D.

§4. Finite free resolutions of the n-sphere

In general, it is possible to define the Stanley-Reisner ring $\mathbf{Z}[\Delta] = A/I_\Delta$ of Δ over the commutative ring $\mathbf{Z}$. However, a minimal free resolution of $\mathbf{Z}[\Delta]$ over the polynomial ring $A = \mathbf{Z}[x_1, x_2, \ldots, x_v]$ does not necessarily exist. On the other hand, there exists a minimal free resolution of $\mathbf{Z}[\Delta]$ over A if and only if all Betti numbers $\beta_i^A(k[\Delta])$ are independent of the base field k (see, e.g., [H–K]). Thus, it might be of interest to find a natural class of simplicial complexes Δ for which all Betti numbers $\beta_i^A(k[\Delta])$ are independent of k. The main purpose of this section is to show that if $|\Delta|$ is the n-sphere $\mathbf{S}^n$ (or the n-ball $\mathbf{B}^n$) with $n \leq 3$, then all Betti numbers $\beta_i^A(k[\Delta])$ of $k[\Delta]$ are independent of k. Moreover, we construct a shellable simplicial complex Δ with $|\Delta| = \mathbf{S}^4$ such that some Betti number $\beta_i^A(k[\Delta])$ does depend on the base field k.

(4.1) PROPOSITION. (a) *Let Δ be a simplicial complex and suppose that the geometric realization $|\Delta|$ of Δ is a connected 3-manifold without boundary. Then, all Betti numbers $\beta_i^A(k[\Delta])$ are independent of the base field k if $|\Delta|$ is orientable and $\tilde{H}_1(\Delta; \mathbf{Z}) = 0$.*

(b) *Let Δ be a simplicial complex such that $|\Delta|$ is a connected 2-manifold without boundary. Then, all Betti numbers $\beta_i^A(k[\Delta])$ are independent of the base field k if and only if $|\Delta|$ is orientable.*

Proof. By virtue of Hochster's formula, in order for all Betti numbers $\beta_i^A(k[\Delta])$ to be independent of the base field k, it is necessary and sufficient that $\dim_k \tilde{H}_j(\Delta_W; k)$ is independent of k for every subset W of the vertex set V and for each integer $j \geq -1$.

On the other hand, it follows easily that, for a simplicial complex Δ on the vertex set V, all Betti numbers $\beta_i^A(k[\Delta])$ are independent of k if one of the following conditions is satisfied: (i) $\dim \Delta \leq 1$; (ii) Δ is a 2-manifold with non-empty boundary; (iii) $\sharp(V) \leq 5$.

(4.2) THEOREM. *Let Δ be a simplicial complex and suppose that the geometric realization $|\Delta|$ of Δ is the n-sphere $\mathbf{S}^n$ (or the n-ball $\mathbf{B}^n$) with $n \leq 3$. Then, the Betti number $\beta_i^A(k[\Delta])$ is independent of the base field k for every $i \geq 0$.*

Proof. If $|\Delta| = \mathbf{S}^n$, then the above Proposition (4.1) guarantees that all Betti numbers $\beta_i^A(k[\Delta])$ are independent of the base field k.

On the other hand, suppose that $|\Delta| = \mathbf{B}^n$ and define Δ' to be the simplicial complex $\Delta \cup (\partial\Delta * \{$ a single point $\})$. Thus, $|\Delta'| = \mathbf{S}^n$. Let V denote the vertex set of Δ. Then $\Delta'_V = \Delta$. Hence, it follows that, for every subset W of V and for each integer $j \geq -1$, $\dim_k \tilde{H}_j(\Delta_W; k)$ is independent of the base field k as required. Q. E. D.

(4.3) EXAMPLE. Let Γ denote the simplicial complex on the vertex set $V = \{1, 2, 3, 4, 5, 6\}$ which is the minimal triangulation of the real projective plane (see [Rei]). Let Δ denote the simplicial complex which consists of all subsets σ of V with $\sigma \neq V$. Thus, $|\Delta|$ is the 4-sphere. We consider Γ to be a subcomplex of Δ in the obvious way. Let $\mathrm{Sd}(\Delta)$ denote the barycentric subdivision of Δ. If W is the vertex set of $\mathrm{Sd}(\Gamma)$, then $\sharp(W) = 31$ and $\mathrm{Sd}(\Delta)_W = \mathrm{Sd}(\Gamma)$. Thus, we have

$$\dim_{\mathbf{Z}/2\mathbf{Z}} \tilde{H}_{31-28-1}(\mathrm{Sd}(\Delta)_W; \mathbf{Z}/2\mathbf{Z}) > \dim_{\mathbf{Q}} \tilde{H}_{31-28-1}(\mathrm{Sd}(\Delta)_W; \mathbf{Q});$$

$$\dim_{\mathbf{Z}/2\mathbf{Z}} \tilde{H}_{31-29-1}(\mathrm{Sd}(\Delta)_W; \mathbf{Z}/2\mathbf{Z}) > \dim_{\mathbf{Q}} \tilde{H}_{31-29-1}(\mathrm{Sd}(\Delta)_W; \mathbf{Q}).$$

Hence

$$\beta_{28}^A((\mathbf{Z}/2\mathbf{Z})[\mathrm{Sd}(\Delta)]) > \beta_{28}^A(\mathbf{Q}[\mathrm{Sd}(\Delta)]);$$
$$\beta_{29}^A((\mathbf{Z}/2\mathbf{Z})[\mathrm{Sd}(\Delta)]) > \beta_{29}^A(\mathbf{Q}[\mathrm{Sd}(\Delta)]).$$

Note that $\mathrm{hd}_A(k[\mathrm{Sd}(\Delta)]) = 57$ and $\beta_{28}^A(k[\mathrm{Sd}(\Delta)]) = \beta_{29}^A(k[\mathrm{Sd}(\Delta)])$. Since Δ is the boundary complex of the 5-simplex, it follows that Δ is shellable (defined in, e.g., [B–M]). Hence, thanks to [Bjö$_1$], $\mathrm{Sd}(\Delta)$ is also shellable.

(4.4) EXAMPLE. Let Δ denote the simplicial complex as in Example (4.3) and define Δ' to be $\Delta - \{\{1, 2, 3, 4, 5\}\}$. Then $|\Delta'|$ is the 4-ball. The similar technique as in Example (4.3) enables us to see that some Betti numbers $\beta_i^A(k[\mathrm{Sd}(\Delta')])$ of the Stanley-Reisner ring $k[\mathrm{Sd}(\Delta')]$ of the barycentric subdivision $\mathrm{Sd}(\Delta')$ of Δ' depend on the base field k. The simplicial complex $\mathrm{Sd}(\Delta')$ is also shellable.

The above Examples (4.3) and (4.4) illustrate the following

(4.5) PROPOSITION. *Fix an integer $n \geq 4$ and let V denote the finite set $\{1, 2, \ldots, n, n+1, n+2\}$. Define Δ_n to be the simplicial complex which consists of all subsets σ of V with $\sigma \neq V$. Moreover, let Δ'_n denote the simplicial complex $\Delta_n - \{\{1, 2, \ldots, n+1\}\}$. Then, there exist integers i*

and j such that $\beta_i^A(k[\mathrm{Sd}(\Delta_n)])$ and $\beta_j^A(k[\mathrm{Sd}(\Delta'_n)])$ depend on the base field k. Note that both $\mathrm{Sd}(\Delta_n)$ *and* $\mathrm{Sd}(\Delta'_n)$ *are shellable with* $|\mathrm{Sd}(\Delta_n)| = \mathbf{S}^n$ *and* $|\mathrm{Sd}(\Delta'_n)| = \mathbf{B}^n$.

§5. 1-skeltons of simplicial spheres

In this section we give a ring-theoretical short proof of the following result, which was first proved by Barnette.

Theorem. *The 1-skelton of a simplicial $(d-1)$-sphere is d-connected.*

Proof. Suppose Δ is a simplicial $d-1$-sphere on the vertex set V with $\sharp(V) = v$. Since $k[\Delta]$ is Gorenstein, $\beta_{i_{i+1}}(k[\Delta]) = 0$ for $i \geq v - d$. Thus we have $\tilde{H}_0(\Delta_{V-W}; k) = 0$ for every subset W of V with $\sharp(W) \leq d-1$ by Hochster's formula. That is, Δ_{V-W} is connected. Hence, the 1-skelton of Δ is d-connected. Q.E.D.

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