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Some remarks on representations of fundamental generalized inverse $*$-semigroups

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Abstract

W.D.Munn [4] described that every fundamental inverse semigroup can be faithfully represented by isomorphisms among principal ideals of the semilattice of idempotents of it. Also T.Imaoka [3] has given a generalization of the Preston-Vagner representation for generalized inverse $*$-semigroups by using a concept of a structure sandwich set of an inverse subsemigroup of the symmetric inverse semigroup on a set.

In this paper, we shall construct a fundamental regular $*$-semigroup $\mathcal{FGI}_{X(\pi)}$ on a set X with a partion $\pi : X = \Sigma\{X_i : i \in I\}$, and obtain a faithful representation of a fundamental generalized inverse $*$-semigroup into $*$-semigroup $\mathcal{FGI}_{X(\pi)}$ on a set $X(\pi)$.

1 Introduction

A semigroup S with a unary operation $* : S \to S$ is called a *regular $*$-semigroup* if it satisfies followings

$$(x^*)^* = x, \tag{1}$$

$$(xy)^* = y^*x^*, \tag{2}$$

$$xx^*x = x. \tag{3}$$

Let S be a regular $*$-semigroup. Then an idempotent e of S is called a *projection* if it satisfies $e^* = e$. For any subset A of S, denote the sets of idempotents and projections of A by $E(A)$ and $P(A)$, respectively. The following result is well-known and we use it frequently throughout this paper.

Result 1.1 (*see* [2]) *Let S be a regular $*$-semigroup. Then we have the followings:*

(1) $E(S) = P(S)^2$;

(2) *for any* $a \in S$ *and* $e \in P(S), a^*ea \in P(S)$;

(3) *each $\mathcal{L}$-class and each $\mathcal{R}$-class have one and only one projection.*

A regular $*$-semigroup S is called a *generalized inverse $*$-semigroup* if $E(S)$ forms a normal band, that is, $E(S)$ satisfies the identity $xyzx = xzyx$, equivalently, $P(S)$ satisfies the identity $xyzw = xzyw$.

Let S be a regular $*$-semigroup and ρ a congruence on S. Then ρ is called a *$*$-congruence* if $(x, y) \in \rho$ implies $(x^*, y^*) \in \rho$.

Result 1.2 (*see* [2]) *Let S be a regular $*$-semigroup. Then,*

$$\mu = \{(a, b) \in S \times S : aea^* = beb^* \text{ and } a^*ea = b^*eb \text{ for all } e \in P(S)\}$$

is the maximum idempotent separating $*$-congruence on S.

A regular $*$-semigroup S is said to be *fundamental* if μ is the equality relation on S. The notation and terminology are those of [1] and [2] unless otherwise stated.

2 Structure sandwich sets

Let $\mathcal{I}_X$ be the symmetric inverse semigroup on a set X. For any subset A of X, 1_A means the identity mapping on A. let $\mathcal{A}$ be an inverse subsemigroup of $\mathcal{I}_X$ and $\theta : \mathcal{A} \times \mathcal{A} \to \mathcal{A}$ a mapping. Denote the image $(\alpha, \beta)\theta$ of an ordered pair (α, β) by $\theta_{\alpha,\beta}$. Set $\mathcal{M} = \{\theta_{\alpha,\beta} : \alpha, \beta \in \mathcal{A}\}$. If $\mathcal{M}$ satisfies the following conditions:

$$(1) \qquad \theta^{-1}_{\alpha,\beta} = \theta_{\beta^{-1},\alpha^{-1}},$$

$$(2) \qquad \theta_{\alpha,\alpha^{-1}} = 1_{r(\alpha)},$$

$$(3) \qquad \theta_{1_{d(\alpha)},\alpha} = 1_{d(\alpha)},$$

$$(4) \qquad \theta_{\alpha,\beta}\beta\theta_{\alpha\theta_{\alpha,\beta}\beta,\gamma} = \theta_{\alpha,\beta\theta_{\beta,\gamma}\gamma}\beta\theta_{\beta,\gamma},$$

we call it the *structure sandwich set* of $\mathcal{A}$ determined by θ. The following lemma and theorem are obvious by the definition.

Lemma 2.1 *Let $\mathcal{M}$ be the structure sandwich set of $\mathcal{A}$ determined by θ. Then,*

$$\theta_{\alpha,1_{r(\alpha)}} = 1_{r(\alpha)}.$$

Theorem 2.2 (*see* [3]) *Let $\mathcal{A}$ be an inverse subsemigroup of the symmetric inverse semigroup $\mathcal{I}_X$ on a set X, and $\mathcal{M}$ the structure sandwich set of $\mathcal{A}$ determined by a mapping $\theta : \mathcal{A} \times \mathcal{A} \to \mathcal{A}$. Define a multiplication $\circ$ and a unary operation $*$ on $\mathcal{A}$ as follows:*

$$\alpha \circ \beta = \alpha\theta_{\alpha,\beta}\beta,$$
$$\alpha^* = \alpha^{-1}.$$

Then $\mathcal{A}(\circ, *)$ becomes a regular $*$-semigroup with $E(\mathcal{A}(\circ, *)) = \{\alpha \in \mathcal{A}(\circ, *) : \theta_{\alpha,\alpha} = \alpha^{-1}\}$ and $P(\mathcal{A}(\circ, *)) = \{1_{d(\alpha)} : \alpha \in A(\circ, *)\}$.

We denote $\mathcal{A}(\circ, *)$, defined above, by $\mathcal{A}(\mathcal{M})$.

3 Construction

Let X be a set and $\pi : X = \Sigma\{X_i : i \in I\}$ a partition of X. In this case, we denote X by $X(\pi)$. A subset A of X is called a *π-singleton subset of $X(\pi)$* if $|A \cap X_i| \leq 1$ for all $i \in I$. A mapping $\alpha \in \mathcal{I}_X$ is called a *π-singleton bijection* of $X(\pi)$ if $d(\alpha)$ and $r(\alpha)$ are π-singleton subsets of $X(\pi)$. Denote the set of all π-singleton bijections of $X(\pi)$ by $\mathcal{FGI}_{X(\pi)}$. The following lemma is clear.

Lemma 3.1 *The set $\mathcal{FGI}_{X(\pi)}$, defined above, is an inverse subsemigroup of $\mathcal{I}_X$.*

For any $\alpha, \beta \in \mathcal{FGI}_{X(\pi)}$, define a mapping $\theta_{\alpha,\beta}$ as follows:

$$d(\theta_{\alpha,\beta}) = \{e \in r(\alpha) : \text{there exist } i \in I \text{ and } f \in d(\beta) \text{ such that } e, f \in X_i\},$$

$$r(\theta_{\alpha,\beta}) = \{f \in d(\beta) : \text{there exist } i \in I \text{ and } e \in r(\alpha) \text{ such that } e, f \in X_i\}.$$

$$e\theta_{\alpha,\beta} = f, \text{ where } r(\alpha) \cap X_i = \{e\} \text{ and } d(\beta) \cap X_i = \{f\}.$$

Proposition 3.2 *The set $\mathcal{M} = \{\theta_{\alpha,\beta} : \alpha, \beta \in \mathcal{FGI}_{X(\pi)}\}$ is the structure sandwich set of $\mathcal{FGI}_{X(\pi)}$ determined by a mapping $\theta : (\alpha, \beta) \mapsto \theta_{\alpha,\beta}$. Therefore $\mathcal{FGI}_{X(\pi)}(\mathcal{M})$ is a regular $*$-semigroup. Furthermore $\mathcal{FGI}_{X(\pi)}(\mathcal{M})$ becomes a fundamental generalized inverse $*$-semigroup.*

Proof. We can prove the proposition by similar argument of Lemma 3.4 and Lemma 3.5 of [3].

4 Representation

Let S be a fundamental generalized inverse $*$-semigroup. In this section, we denote $E(S)$ and $P(S)$ simply by E and P, respectively. Let $E \sim \Sigma\{E_i : i \in I\}$ be the structure decomposition of the normal band E. Then $\pi : P = \Sigma\{P_i : i \in I\}$ is a partition of P, where $P_i = P(E_i)$. Then $\mathcal{FGI}_{P(\pi)}(\mathcal{M})$, constructed in the preceding section, is a fundamental generalized inverse $*$-semigroup.

Lemma 4.1 *For any $a \in S, Sa \cap P (= Sa^*a \cap P)$ is a π-singleton subset of $P(\pi)$.*

Proof. Put $a^*a = e$ and assume that $f, g \in Se \cap P_i$. Then there exist $x, y \in S$ such that $f = xe$ and $g = ye$. Hence $f = efe$ and $g = ege$. Since E_i is a rectangular band and E is a normal band, $f = fgf = efegefe = egefege = gfg = g$. This shows that $Sa \cap P$ is a π-singleton subset of $P(\pi)$.

Lemma 4.2 *For any $a \in S$, let $\tau_a : Sa^* \cap P \to Sa \cap P$ be a mapping defined by*

$$e\tau_a = a^*ea.$$

Then, for any $a \in S$, $\tau_a \in \mathcal{FGI}_{P(\pi)}(\mathcal{M})$ and $\tau_a^{-1} = \tau_{a^*}$.

Proof. By Lemma 4.1, $d(\tau_a)$ and $r(\tau_a)$ are π-singleton subsets of $P(\pi)$. Let e and f are any elements of $Sa^* \cap P$ such that $e\tau_a = f\tau_a$, that is, $a^*ea = a^*fa$. Then there exist $x, y \in S$ such that $e = xa^*$ and $f = ya^*$. Thus $e = e^*e = ax^*xa^* = aa^*ax^*xa^*aa^* = aa^*eaa^* = aa^*faa^* = aa^*f^*faa^* = aa^*ay^*ya^*aa^* = ay^*ya^* = f^*f = f$. This implies that τ_a is injective. Next, let f be any element of $Sa \cap P$. Then there exists $x \in S$ such that $f = xa$. Put $e = aa^*x^*xaa^*$. Then $e \in Sa^* \cap P$ and $e\tau_a = a^*ea = a^*(aa^*x^*xaa^*)a = a^*x^*xa = f$. This implies that τ_a is bijective. The last statement of the lemma is obvious.

Lemma 4.3 *For any* $a, b \in S, \theta_{\tau_a,\tau_b} = \tau_{a^*abb^*}$.

Proof. Let e be any element of $d(\theta_{\tau_a,\tau_b})$. Then $e \in Sa \cap P$ and there exist $i \in I$ and $f \in Sb^* \cap P$ such that $e, f \in P_i$. Hence there exist $x, y \in S$ such that $e = xa$ and $f = yb^*$. Thus $e = efe = xayb^*xa = xayb^*bb^*xaa^*a = xayb^*xabb^*a^*a \in Sbb^*a^*a \cap P = d(\tau_{a^*abb^*})$.

Conversely let e be any element of $d(\tau_{a^*abb^*}) = Sbb^*a^*a \cap P$. Then there exists $x \in S$ such that $e = xbb^*a^*a$. Hence $e \in Sa \cap P$. Put $f = bb^*a^*ax^*xa^*abb^*$. Then $f \in Sb^* \cap P$ and it is clear that e and f are contained in a same P_i. Therefore $e \in d(\theta_{\tau_a,\tau_b})$ and $e\tau_{a^*abb^*} = bb^*a^*aea^*abb^* = bb^*a^*ax^*xa^*abb^* = f = e\theta_{\tau_a,\tau_b}$.

Lemma 4.4 *Define a mapping* $\phi : S \to \mathcal{FGI}_{P(\pi)}(\mathcal{M})$ *by*

$$a\phi = \tau_a.$$

Then ϕ *is a* $*$-*monomorphism.*

Proof. To prove that ϕ is a homomorphism it is sufficient to show that, for any $a, b \in S, d(\tau_a \circ \tau_b) = d(\tau_{ab})$. Let a and b be any elements of S. Then,

$$\begin{aligned} d(\tau_a \circ \tau_b) &= d(\tau_a\tau_{a^*abb^*}\tau_b) \qquad \text{(by Lemma 4.3)} \\ &= \{Sa \cap P \cap (Sa^*abb^* \cap P \cap Sb^*)\tau_{a^*abb^*}^{-1}\}\tau_a^{-1} \\ &= \{Sa \cap P \cap (Sa^*abb^* \cap P)\tau_{a^*abb^*}^{-1}\}\tau_a^{-1} \\ &= (Sa \cap P \cap Sbb^*a^*a)\tau_a^{-1} \qquad (\text{since } \tau_{a^*abb^*}^{-1} = \tau_{bb^*a^*a}) \\ &= (Sbb^*a^*a \cap P)\tau_a^{-1}. \end{aligned}$$

Hence let e be any element of $d(\tau_a \circ \tau_b)$. Then there exists $f \in Sbb^*a^*a \cap P$ such that $e\tau_a = f$, that is $a^*ea = f$. Since $e \in Sa^*$ and $f \in Sbb^*a^*a$, there exist $x, y \in S$ such that $e = xa^*$ and $f = ybb^*a^*a$. Thus $e = e^*e = ax^*xa^* = aa^*(ax^*xa^*)aa^* = a(a^*ea)a^* = afa^* = aybb^*a^* \in Sb^*a^* \cap P = d(\tau_{ab})$.

Conversely let e be any elment of $d(\tau_{ab}) = Sb^*a^* \cap P$. Then there exists $x \in S$ such that $e = xb^*a^*$. Hence $e\tau_a = xb^*a^*\tau_a = a^*xb^*a^*a \in Sbb^*a^*a \cap P$. Thus $e \in (Sbb^*a^*a \cap P)\tau_a^{-1}$ and so $d(\tau_a \circ \tau_b) = d(\tau_{ab})$.

Let a and b be any elements of S such that $a\phi = b\phi$, that is, $\tau_a = \tau_b$. Then it is easy to show that $aa^* = bb^*$ and $a^*a = b^*b$. Hence assume that $(a, b) \notin \mu$. Then there exists $e \in P$ such that $aea^* \neq beb^*$ or $a^*ea \neq b^*eb$.

If $aea^* \neq beb^*$, we have $aea^*\tau_a = beb^*\tau_b$, since $a^*aea^*a = b^*beb^*b$. This contradicts that $\tau_a = \tau_b$ is injective. Similarly, if $a^*ea \neq b^*eb$, we have $aa^*eaa^*\tau_a = bb^*ebb^*\tau_b$, that is, $a^*ea = b^*eb$. This contradicts the hypothesis. Therefore, $(a, b) \in \mu$. Since S is fundamental, we have that $a = b$. Thus ϕ is injective.

By Lemma 4.2, ϕ is compatible with a unary operation $*$ and so ϕ is a $*$-monomorphism.

Now we have the following theorem.

Theorem 4.5 *Every fundamental generalized inverse $*$-semigroup has a faithful representation into $\mathcal{FGI}_{X(\pi)}$ on a set $X(\pi)$.*

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