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On heat flows for a variational functional of degenerate type

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1 Introduction.

Let M, N be compact, smooth orientable Riemannian manifolds of dimension m, l with metrics g, h respectively and suppose that $\partial M,\ \partial N=\emptyset$. Since N is compact, N may be isometrically embedded into a Euclidean space R^{n} for some n. For a $C^{1}-$map $u: M\rightarrow N\subset R^{n}$, we introduce a variational functional $I(u)$ given by

$$I(u)=\int_{M}f(|Du|^{2})dM, \tag{1.1}$$

where, in local coordinates on M,

$$dM=\sqrt{|g|}dx,\quad |Du|^{2}=\sum_{\alpha,\beta=1}^{m}\sum_{i=1}^{n}g^{\alpha\beta}D_{\alpha}u^{i}D_{\beta}u^{i},\quad D_{\alpha}=\partial/\partial x^{\alpha}\quad(\alpha,\beta=1,\cdots,m)$$

with $(g^{\alpha\beta})=(g_{\alpha\beta})^{-1},\quad |g|=\det(g_{\alpha\beta})$. and f is a scalar valued $C^{2}-$function defined on $[0,+\infty)$ satisfying the following relations with uniform positive constants γ_{i} $(i=1,2,3)$ and $p\geq 2$

$$\text{(A1)}\quad\begin{cases}\gamma_{1}|Q|^{p-2}|\xi|^{2}\leq\frac{\partial^{2}f}{\partial Q_{\alpha}^{i}\partial Q_{\beta}^{j}}(|Q|^{2})\xi_{\alpha}^{i}\xi_{\beta}^{j}\leq\gamma_{2}|Q|^{p-2}|\xi|^{2}, & \text{all }\xi,Q\in R^{nm},|Q|\leq 1,\\ \gamma_{2}|\xi|^{2}\leq\frac{\partial^{2}f}{\partial Q_{\alpha}^{i}\partial Q_{\beta}^{j}}(|Q|^{2})\xi_{\alpha}^{i}\xi_{\beta}^{j}\leq\gamma_{3}|\xi|^{2}, & \text{all }\xi,Q\in R^{nm},|Q|>1.\end{cases}$$

Here and in what follows, the summation notation over repeated indices is adopted. From (A1), we obtain the relations:

$$\begin{cases}(\gamma_{1}/2)\tau^{p/2-1}\leq f'(\tau)+2f''(\tau)\tau\leq(\gamma_{2}/2)\tau^{p/2-1}, & 0\leq\tau\leq 1,\\ \gamma_{2}/2\leq f'(\tau)+2f''(\tau)\tau\leq\gamma_{3}/2, & \tau>1\end{cases} \tag{1.2}$$

and

$$\begin{cases}(\gamma_{1}/2(p-1))\tau^{p/2-1}\leq f'(\tau)\leq(\gamma_{2}/2(p-1))\tau^{p/2-1}, & 0\leq\tau\leq 1,\\ (\gamma_{1}/2(p-1))(1/\sqrt{\tau})+(\gamma_{2}/2)(1-1/\sqrt{\tau})\leq f'(\tau) & \\ \leq(\gamma_{2}/2(p-1))(1/\sqrt{\tau})+(\gamma_{3}/2)(1-1/\sqrt{\tau}), & \tau>1.\end{cases} \tag{1.3}$$

(1.2) and (1.3) imply that there exists positive numbers γ_{23}, $\gamma_{2}/2\leq\gamma_{23}\leq\gamma_{3}/2$, and $\tilde{\gamma}_{23}$ such that

$$f'(\tau)\rightarrow\gamma_{23},\quad f(\tau)\rightarrow\gamma_{23}\tau-\tilde{\gamma}_{23}\quad(\tau\rightarrow\infty). \tag{1.4}$$

We here assume that, with uniform positive constants a and τ_0,

$$\text{(A2)} \qquad |1 - f'(\tau)/\gamma_{23}| \le \tau^{-a/2}, \quad \text{all } \tau > \tau_0.$$

By (1.2) and (1.3), we also observe that, with positive constants $\bar{\tilde{\gamma}}_{23} \ge \tilde{\gamma}_{23}$, and $\bar{\gamma}_{23} \ge \gamma_{23}$,

$$\gamma_{23}\tau - \bar{\tilde{\gamma}}_{23} \le f(\tau) \le \bar{\gamma}_{23}\tau, \quad \tau \ge 0. \tag{1.5}$$

The Euler-Lagrange equation of a variationsl functinal I is given by

$$-\Delta_M^f u + A^f(u)(Du, Du) = 0, \tag{1.6}$$

where Δ_M^f denotes the differential operator on M :

$$\Delta_M^f u = \frac{1}{\sqrt{|g|}} D_\alpha \left(\sqrt{|g|} g^{\alpha\beta} f'(|Du|^2) D_\beta u \right)$$

and, by the second fundamental form $A(u)$ of N in R^n at u, $A^f(u)$ is given as follows:

$$A^f(u)(Du, Du) = f'(|Du|^2) \sum_{\alpha,\beta=1}^{m} g^{\alpha\beta} A(u)(D_\alpha u, D_\beta u).$$

For $q > 1$, we now define a space of Sobolev mappings between M and N, denoted by $W^{1,q}$ (M, N), as a space of maps belonging to usual Sobolev space $W^{1,q}(M, R^n)$ such that $u \in N$ almost everywhere on M. To look for maps belonging to $W^{1,2}(M, N)$ satisfying (1.6) in the distribution sence, we are concerned with heat flows $u(t) \in W^{1,2}(M, N)$, $0 \le t < \infty$, for a variational functional (1.1) with a given map $u_0 \in W^{1,2}(M, N)$ where the heat flows are prescribed by a system of nonlinear second order partial differential equations of parabolic type:

$$\begin{aligned} \partial_t u - \Delta_M^f u + A^f(u)(Du, Du) &= 0 \quad \text{in} \quad (0, \infty) \times M, \\ u(0, x) &= u_0(x), \quad x \in M. \end{aligned} \tag{1.7}$$

The partial regularity of minimizing harmonic maps was achieved in [17,24]. The results were generalized to obtain the partial regularity of minimizing p-harmonic maps ($p > 1$) in [19] and similar results were also treated in [15](also see references in [15,19]). These results become fundamental to the regularity theory of harmonic maps. The partial regularity of $p-$harmonic maps of C^1-class ($p \ge 2$) was also investigated in [14, 23]. On the other hand, Chen and Struwe established the global exstence and partial regularity for heat flows for harmonic maps, based on a decay estimate analogous to the monotonicity formula for minimizing harmonic maps (see [4,25]). The heat flows for p-harmonic maps are prescribed by nonlinear degenerate parabolic system. The regularity of weak solutions of degenerate parabolic systems with only principal terms was discussed and the $C^{1,\mu}-$regularity of solutions was accomplished in [11,12,13] (also see [6,8] and [18,26] for corresponding elliptic systems). The global existence of a weak solution to the heat flow for p-harmonic maps has recently shown in the case that the target manifold is a sphere[1]. However the partial regularity of heat flows for p-harmonic maps remains a difficult problem to be settled (for the scalar case see [10]). In this paper we make an extension of Struwe's results[25], which may be of some use for attacking the partial regularity problem for heat flows for p-harmonic maps.

Now take an arbitrary positive number T. We are now interested in weak solutions of (1.7): $u \in L^\infty((0,T); L^2(M)) \cap L^2((0,T); W^{1,2}(M,N))$ satisfying, for all t_1, t_2, $0 \le t_1 < t_2 < T$, and $\varphi \in L^2((0,T); W_0^{1,2}(M)) \cap L^\infty((0,T) \times M)$ the support of which is contained in a coordinate chart for M,

$$\int_{M\times\{t\}} u\varphi\Big|_{t=t_1}^{t=t_2} dM + \int_{t_1}^{t_2}\int_M \{-u\partial_t\varphi + f'(|Du|^2) g^{\alpha\beta} D_\alpha u D_\beta \varphi + \varphi A^f(u)(Du, Du)\} dx dt = 0. \tag{1.8}$$

To state our results, we need some preminalies: Let us introduce the parabolic metric

$$\delta(z_1, z_2) = \max\{|x_1 - x_2|,\ |t_1 - t_2|^{1/2}\}, \quad z_i = (t_i, x_i),\ i = 1, 2 \tag{1.9}$$

and denote by $\mathrm{dist}_\delta(z, A)$ and $H^k(\cdot, \delta)$ a distance between a point z and a set A and the k-dimensional Hausdorff measure with respect to δ respectively.

Then our main theorem is the following:

Theorem. *Let $\{u_k\}$ be a sequence of weak solutions $u_k \in C^0_{\mathrm{loc}}((0,T); C^1_{\mathrm{loc}}(M))$ to (1.7). Then there exist a subsequence $\{u_k\}$ and a map $u : [0,T) \times M \to R^n$ such that*

$$\sup_{0\le t\le T} I(u(t)) \le I(u_0), \quad \partial_t u \in L^2((0,T) \times M), \tag{1.10}$$

$$u(t,x) \in N \quad \textit{almost everywhere } (t,\ x) \ \in (0,T) \times M \tag{1.11}$$

and

$$\begin{aligned} &Du_k \to Du \quad \textit{weak-star in } L^\infty([0,T); L^2(M)),\\ &\partial_t u_k \to \partial_t u \quad \textit{weakly in } L^2([0,T) \times M),\\ &u_k \to u \quad \textit{weakly in } L^2([0,T); W^{1,2}(M)). \end{aligned} \tag{1.12}$$

Moreover u is a weak solution to (1.7) and there exists an open set $Q_0 \subset (0,T) \times M$ (with respect to a metric δ) and a positive number $\alpha, 0 < \alpha < 1$ such that u, Du are locally Hölder continuous in Q_0 with an exponent α with respect to δ and it holds that

$$\partial_t u - \Delta_M^f u + A^f(u)(Du, Du) = 0 \quad \textit{almost everywhere in } Q_0 \tag{1.13}$$

and that

$$H^m((0,T) \times M \setminus Q_0, \delta) < \infty. \tag{1.14}$$

In the forthcoming paper[22] we will treat the existence of weak solutions to (1.7) based on such an approximate scheme as stated above.

For simplicity we restrict ourselves to the case $M = R^m$. Then note that, for $u : [0,T) \times R^m \to N \subset R^n$,

$$|Du| = (D_\alpha u^i D_\alpha u^i)^{1/2}, \quad \Delta_M^f u = \mathrm{div}(f'(|Du|^2)Du),$$
$$A^f(Du, Du) = f'(|Du|^2)A(u)(D_\alpha u, D_\alpha u). \tag{1.15}$$

Some standard notations: For $z_0 = (t_0, x_0) \in (0,T) \times M$ and $r, \tau > 0$

$$B_r(x_0) = \{x \in R^n : |x - x_0| < r\}, \quad Q_{r,\tau}(z_0) = (t_0 - \tau, \ t_0) \times B_r(x_0)$$

and $Q_r(z_0) = Q_{r,r^2}(z_0)$. The center points x_0, z_0 are omitted when no confusion may arise.

2 Energy estimates and monotonicity formula.

In this section we assume, for an initial data u_0, $\int_{R^m} f(|Du_0|^2)dx < +\infty$ and we give a-priori estimates valid for weak solutions of (1.7) belonging to $C^0_{\mathrm{loc}}((0,T); C^1_{\mathrm{loc}}(R^m))$ and satisfying $\int_0^T \int_{R^m} f(|Du|^2)dxdt < +\infty$. Throughout this section let $u \in C^0_{\mathrm{loc}}((0,T);\ C^1_{\mathrm{loc}}(R^m))$ be a weak solution to (1.7) with $\int_0^T \int_{R^m} f(|Du|^2)dxdt < +\infty$.

First of all we have the following estimate (refer to [6,11,14]), the proof of which is performed by Caccippoli estimate with the quotient method (see [16,20]).

Lemma 2.1. *A function* $(\min\{|Du|^{p-2}, 1\})^{1/2}Du$ *has weak derivatives which lie in* $L^2_{\mathrm{loc}}((0,T) \times R^m)$ *and there exists a positive constant* γ *depending only on* m, p, γ_i $(i = 1,2,3)$ *and the geometry of* N *such that, for all* $Q_{2r} = Q_{2r}(t_0, x_0) \subset (0,T) \times R^m$,

$$\begin{aligned} \sup_{t_0 - r^2 \le t \le t_0} \int_{B_r \times \{t\}} |Du|^2 dx + \int_{Q_r} |D((\min\{|Du|^{p-2}, 1\})^{1/2} Du)|^2 dz \\ \le \gamma r^{-2}(1 + |Du|^2_{L^\infty(Q_{2r})})\left(1 + \int_{Q_{2r}} |Du|^p dz\right). \end{aligned} \tag{2.1}$$

The following estimate is fundamental(refer [1,4,25]).

Lemma 2.2. (Energy inequality) *It holds*

$$\sup_{0 \le t \le T} I(u(t)) + \int_0^T \int_{R^m} |\partial_t u|^2 dxdt \le I(u_0). \tag{2.2}$$

We also need the monotonicity type inequality(refer to [4,25]). This is the main estimate in our arguments. Let us take $z_0 = (t_0, x_0) \in (0,T) \times R^m$ arbitrarily and fix it. We also set, for $0 < R < (t_0)^{1/2}$,

$$\Phi(R, z_0, u) = R^2 \int_{R^m \times \{t = t_0 - R^2\}} f(|Du|^2) G_{z_0} dx \tag{2.3}$$

and, for $0 < R < (t_0)^{1/2}/2$,

$$\Psi(R, z_0, u) = \int_{t_0-(2R)^2}^{t_0-R^2} \int_{R^m \times \{t\}} f(|Du|^2) G_{z_0} dx dt, \tag{2.4}$$

where, with a positive constant γ_{23} in (A2),

$$G_{z_0}(t, x) = (4\pi(t_0 - t))^{-m/2} \exp(-|x - x_0|^2 / 4\gamma_{23}(t_0 - t)), \quad t < t_0.$$

Lemma 2.3 (Monotonicity formula) *There exists a positive constant γ depending only on m, p, γ_2, γ_3 and γ_{23} such that, for any $0 < R_0 \le R_1 < (t_0)^{1/2}$,*

$$\Phi(R_0, z_0, u) \le \exp(\gamma(R_1 - R_0)) \Phi(R_1, z_0, u) + \gamma I(u_0)(R_1 - R_0), \tag{2.5}$$

and, for any $0 < R_0 \le R_1 < (t_0/2)^{1/2}$,

$$\Psi(R_0, z_0, u) \le \exp(\gamma(R_1 - R_0)) \Psi(R_1, z_0, u) + \gamma E(u_0)(R_1 - R_0). \tag{2.6}$$

Proof. We give the proof of (2.5). (2.6) is similarily proven. Let us fix $z_0 = (t_0, x_0) \in (0, T) \times R^m$. For each $0 < R < (t_0)^{1/2}$, note the following facts. Using a scaling transformation: $(t, x) \to (s, y)$ such that

$$t = t_0 + R^2 s, \quad x = x_0 + Ry \tag{2.7}$$

and setting

$$u_R(s, y) = u(t_0 + R^2 s, x_0 + Ry), \tag{2.8}$$

the equation (1.7) on $(0, t_0) \times R^m$ is rewritten as follows: On $(-t_0/R^2, 0) \times R^m$,

$$\partial_s u_R - \operatorname{div}(f'(R^{-2}|Du_R|^2) Du_R) + f'(R^{-2}|Du_R|^2) A(u_R)(Du_R, Du_R) = 0. \tag{2.9}$$

Since Lemmata 2.1 and 2.2 implies that u satisfies (1.7) almost everywhere in $(0, T) \times R^m$, (2.9) holds almost everywhere in $(-t_0/R^2, 0) \times R^m$. Also note that

$$\begin{aligned} \Phi(R, z_0, u) &= R^2 \int_{R^m \times \{t = t_0 - R^2\}} f(|Du|^2) G_{z_0} dx \\ &= R^2 \int_{R^m \times \{s=-1\}} f(R^{-2}|Du_R|^2) G dy. \end{aligned} \tag{2.10}$$

We now calculate $\frac{d}{dR}\Phi(R, z_0, u)$ for any $0 < R < (t_0)^{1/2}$. We demonstrate only formal calculations for simplicity, the justification of which is made in [21].

$$
\begin{aligned}
&\frac{d}{dR}\Phi(R, z_0, u) \\
&= 2R \int_{R^m\times\{s=-1\}} f(R^{-2}|Du_R|^2)G \\
&\quad + R^2 \int_{R^m\times\{s=-1\}} f'(R^{-2}|Du_R|^2)\frac{d}{dR}(R^{-2}|Du_R|^2)Gdy \\
&= 2R \int_{R^m\times\{s=-1\}} f(R^{-2}|Du_R|^2)Gdy - 2R\int_{R^m\times\{s=-1\}} R^{-2}|Du_R|^2 f'(R^{-2}|Du_R|^2)Gdy \\
&\quad + 2\int_{R^m\times\{s=-1\}} D_\alpha \frac{d}{dR}u_R^i D_\alpha u_R^i f'(R^{-2}|Du_R|^2)Gdy \\
&= I_1 + I_2 + I_3
\end{aligned}
\tag{2.11}
$$

We now make an estimation of I_1 and I_2. Split the integrations into four parts:

$$
\begin{aligned}
I_1 + I_2 =& 2R\int_{R^m\times\{s=-1\}\cap\{R^{-2}|Du_R|^2\le\widetilde{\tau}_0\}} f(R^{-2}|Du_R|^2)Gdy \\
&+ 2R\int_{R^m\times\{s=-1\}\cap\{R^{-2}|Du_R|^2>\widetilde{\tau}_0\}} f(R^{-2}|Du_R|^2)Gdy \\
&- 2R\int_{R^m\times\{s=-1\}\cap\{R^{-2}|Du_R|^2\le\widetilde{\tau}_0\}} R^{-2}|Du_R|^2 f'(R^{-2}|Du_R|^2)Gdy \\
&- 2R\int_{R^m\times\{s=-1\}\cap\{R^{-2}|Du_R|^2>\widetilde{\tau}_0\}} R^{-2}|Du_R|^2 f'(R^{-2}|Du_R|^2)Gdy \\
=& I_{11} + I_{12} + I_{21} + I_{22}.
\end{aligned}
\tag{2.12}
$$

where $\widetilde{\tau}_0$ is a positive constant determined later. In virtue of (1.3), we have

$$
\begin{aligned}
I_{11} &\ge 0, \\
I_{21} &= -2R\int_{R^m\times\{s=-1\}\cap\{R^{-2}|Du_R|^2\le\widetilde{\tau}_0\}} R^{-2}|Du_R|^2 f'(R^{-2}|Du_R|^2)Gdy \\
&\ge -2R\int_{R^m\times\{s=-1\}\cap\{R^{-2}|Du_R|^2\le\widetilde{\tau}_0\}} \widetilde{\tau}_0\left(\frac{\gamma_2}{2(p-1)} + \frac{\gamma_3}{2}\right)Gdy.
\end{aligned}
\tag{2.13}
$$

To estimate $I_{12} + I_{22}$, we note by (1.4) that, for all positive numbers ε_0, there exists a positive constant $\tau_1 = \tau_1(\varepsilon_0)$ such that

$$
|f(\tau) - (\gamma_{23}\tau - \widetilde{\gamma}_{23})| < \varepsilon_0, \quad \tau > \tau_1 \tag{2.14}
$$

and, by (A2), for all positive numbers ε_1,

$$|\gamma_{23}-f'(\tau)|\tau=\gamma_{23}|1-f'(\tau)/\gamma_{23}|\tau\le\gamma_{23}\tau^{-a/2}\tau\le\gamma_{23}\varepsilon_1 R^{\delta}\tau,$$
$$\tau>\max\{\tau_0,(\varepsilon_1 R^{\delta})^{-2/a}\}.$$

By (1.4) again, for all positive numbers $\varepsilon, 0<\varepsilon<\gamma_{23}$, we are able to take $\tau_2=\tau_2(\varepsilon)$ such that

$$\varepsilon\tau<f(\tau),\quad \tau>\tau_2. \tag{2.15}$$

Thus we find that, for all $\tau\ge\max\{\tau_0,\tau_1(\varepsilon_0),\ \tau_2(\varepsilon_1),\ \varepsilon_1^{-2/a}R^{-2\delta/a}\}$,

$$\begin{aligned}f(\tau)-\tau f'(\tau)&=f(\tau)-(\gamma_{23}\tau-\widetilde{\gamma}_{23})+(\gamma_{23}\tau-\widetilde{\gamma}_{23})-\tau f'(\tau)\\&=f(\tau)-(\gamma_{23}\tau-\widetilde{\gamma}_{23})+(\gamma_{23}-f'(\tau))\tau-\widetilde{\gamma}_{23}\\&\ge-\varepsilon_0-\gamma_{23}R^{\delta}f(\tau)-\widetilde{\gamma}_{23}\end{aligned}$$

so that, putting

$$\widetilde{\tau_0}=\max\{\tau_0,\tau_1(\varepsilon_0),\tau_2(\varepsilon_1),(\varepsilon_1)^{-2/a}R^{-2\delta/a}\}, \tag{2.16}$$

we have

$$\begin{aligned}&I_{12}+I_{22}\\&\ge-2(\varepsilon_0+\widetilde{\gamma}_{23})R\int_{R^m\times\{s=-1\}}Gdy-2\gamma_{23}R^{\delta-1}R^2\int_{R^m\times\{s=-1\}}f(R^{-2}|Du_r|^2)Gdy.\end{aligned} \tag{2.17}$$

Substituting (2.13) and (2.17) into (2.12), we have

$$\begin{aligned}&I_1+I_2\\&\ge-2R\int_{R^m\times\{s=-1\}\cap\{R^{-2}|Du_R|^2\le\widetilde{\tau_0}\}}\widetilde{\tau_0}\left(\frac{\gamma_2}{2(p-1)}+\frac{\gamma_3}{2}\right)Gdy\\&\quad-2(\varepsilon_0+\widetilde{\gamma}_{23})R\int_{R^m\times\{s=-1\}}Gdy-2\gamma_{23}R^{\delta-1}R^2\int_{R^m\times\{s=-1\}}f(R^{-2}|Du_R|^2)Gdy.\end{aligned} \tag{2.18}$$

We now treat I_3 in (2.11). Using (2.9) and noting $D_\alpha G=-y^\alpha G/2\gamma_{23}(-s)$, we have, by integration by parts,

$$\begin{aligned}I_3=&-2\int_{R^m\times\{s=-1\}}\frac{d}{dR}u_R^iD_\alpha(f'(R^{-2}|Du_R|^2)D_\alpha u_R^i)Gdy\\&-2\int_{R^m\times\{s=-1\}}\frac{d}{dR}u_R^if'(R^{-2}|Du_R|^2)D_\alpha u_R^iD_\alpha Gdy\\=&-2\int_{R^m\times\{s=-1\}}\frac{d}{dR}u_R^i\partial_s u_R^iGdy\\&+\int_{R^m\times\{s=-1\}}\frac{d}{dR}u_R^if'(R^{-2}|Du_R|^2)D_\alpha u_R^i\left(\frac{y^\alpha}{\gamma_{23}(-s)}\right)Gdy\\=&R^{-1}\int_{R^m\times\{s=-1\}}\frac{1}{(-s)}|2s\partial_s u_R+y^\alpha D_\alpha u_R|^2Gdy\\&-R^{-1}\int_{R^m\times\{s=-1\}}\frac{1}{(-s)}(2s\partial_s u_R^i+y^\alpha D_\alpha u_R^i)(1-f'(R^{-2}|Du_R|^2)/\gamma_{23})y\cdot Du_R^iGdy.\end{aligned}$$

The latter is bounded from below by

$$\frac{R^{-1}}{2}\int_{R^m\times\{s=-1\}}\frac{1}{(-s)}|2s\partial_s u_R+y^\alpha D_\alpha u_R|^2 G dy$$
$$-\frac{R^{-1}}{2}\int_{R^m\times\{s=-1\}}\frac{1}{(-s)}|1-f'(R^{-2}|Du_R|^2)/\gamma_{23}|^2|yDu_R|^2 G dy. \tag{2.19}$$

For the purpose of an evaluation of the last term in (2.19), take positive numbers δ and ε_2, $0<\varepsilon_2<\min\{R^{-2\delta}\tau_0^{-a},\gamma_{23}\}$ with a positive constant τ_0 in (A2) and let $\tau_1\geq(\varepsilon_2 R^{2\delta})^{-1/a}$. In virtue of (1.3),

$$R^{-1}\int_{R^m\times\{s=-1\}\cap\{R^{-2}|Du_R|^2\leq\tau_1\}}\frac{1}{(-s)}|1-f'(R^{-2}|Du_R|^2)/\gamma_{23}|^2|yDu_R|^2 G dy$$
$$\leq\left\{1+\frac{1}{\gamma_{23}}\left(\frac{\gamma_2}{2(p-1)}+\frac{\gamma_3}{2}\right)\right\}^2\tau_1 R\int_{R^m\times\{s=-1\}\cap\{R^{-2}|Du_R|^2\leq\tau_1\}}|y|^2 G dy. \tag{2.20}$$

Noting that $\tau_1>\tau_0$, we have, by (A2),

$$R^{-1}\int_{R^m\times\{s=-1\}\cap\{R^{-2}|Du_R|^2>\tau_1\}}\frac{1}{(-s)}|1-f'(R^{-2}|Du_R|^2)/\gamma_{23}|^2|yDu_R|^2 G dy$$
$$\leq R^{-1}\int_{R^m\times\{s=-1\}\cap\{R^{-2}|Du_R|^2>\tau_1\}}\frac{1}{(-s)}((\varepsilon_2)^{1/2}R^\delta)^2|y|^2|Du_R|^2 G dy$$
$$=R^{-1+2\delta}R^2\int_{R^m\times\{s=-1\}\cap\{R^{-2}|Du_R|^2>\tau_1\}}\frac{1}{(-s)}\varepsilon_2 R^{-2}|Du_R|^2|y|^2 G dy.$$

Since $0<\varepsilon_2<\gamma_{23}$, similarily as in (2.15) we find that

$$\varepsilon_2\tau<f(\tau),\quad \tau>\tau_2(\varepsilon_2),$$

so that, if τ_1 is taken as

$$\tau_1=\max\{(\varepsilon_2 R^{2\delta})^{-1/a},\tau_2(\varepsilon_2)\}, \tag{2.21}$$

the latter is bounded from above by

$$R^{-1+2\delta}R^2\int_{R^m\times\{s=-1\}}\frac{1}{(-s)}f(R^{-2}|Du_R|^2)|y|^2 G dy. \tag{2.22}$$

We are able to proceed as follows: for a positive number $\bar\delta>0$ which is determined later,

$$\int_{R^m\times\{s=-1\}}\frac{1}{(-s)}f(R^{-2}|Du_R|^2)|y|^2 G dy$$
$$=\int_{R^m\times\{s=-1\}\cap\{|y|\leq R^{-\bar\delta}\}}\frac{1}{(-s)}f(R^{-2}|Du_R|^2)|y|^2 G dy$$
$$+\int_{R^m\times\{s=-1\}\cap\{|y|>R^{-\bar\delta}\}}\frac{1}{(-s)}f(R^{-2}|Du_R|^2)|y|^2 G dy. \tag{2.23}$$

The first term in (2.23) is estimated from above by

$$R^{-2\bar{\delta}}\int_{R^m\times\{s=-1\}\cap\{|y|\le R^{-\bar{\delta}}\}}\frac{1}{(-s)}f(R^{-2}|Du_R|^2)|y|^2Gdy. \tag{2.24}$$

Noting that, if $|y|>R^{-\bar{\delta}}$,

$$|y|^2G\le\gamma(m)\exp\{-R^{-2\bar{\delta}}/16\gamma_{23}\}$$

and exploiting our energy inequality (2.2), we have, for the second term in (2.23),

$$\begin{aligned}\int_{R^m\times\{s=-1\}\cap\{|y|>R^{-\bar{\delta}}\}}&\frac{1}{(-s)}f(R^{-2}|Du_R|^2)|y|^2Gdy\\ &\le\gamma(m)\exp\{-R^{-2\bar{\delta}}/16\gamma_{23}\}R^{-m}\int_{R^m\times\{t=t_0-R^2\}}f(|Du|^2)dx\\ &\le\gamma(m)\int_{R^m\times\{t=t_0-R^2\}}f(|Du_0|^2)dx.\end{aligned} \tag{2.25}$$

Substituting (2.24) and (2.25) into (2.23) and combining the resulting ineqaulity with (2.22), we obtain from (2.19) and (2.20) the estimate for I_3 in (2.11):

$$\begin{aligned}I_3\ge&-\frac{1}{2}\left\{1+\frac{1}{\gamma_{23}}\left(\frac{\gamma_2}{2(p-1)}+\frac{\gamma_3}{2}\right)\right\}^2\tau_1R\int_{R^m\times\{s=-1\}\cap\{R^{-2}|Du_R|^2\le\tau_1\}}|y|^2Gdy\\ &-\frac{1}{2}R^{-1+2\delta}R^2R^{-2\bar{\delta}}\int_{R^m\times\{s=-1\}\cap\{|y|\le R^{-\bar{\delta}}\}}\frac{1}{(-s)}f(R^{-2}|Du_R|^2)Gdy\\ &-\gamma(m)R^{-1+2\delta}R^2\int_{R^m\times\{t=t_0-R^2\}}f(|Du_0|^2)dx.\end{aligned} \tag{2.26}$$

Gathering the estimates (2.18) and (2.26) with (2.11), we obtain

$$\begin{aligned}&\frac{d}{dR}\Phi_R(R,z_0,u)\\ &\ge-2\gamma_{23}R^{\delta-1}\Phi_R(R,z_0,u)-R^{-1+2\delta-2\bar{\delta}}\Phi_R(R,z_0,u)/2\\ &\quad-R^{1+2\delta}\gamma(m)I(u_0)-\gamma(p,\gamma_2,\gamma_3)(\varepsilon_0+\widetilde{\gamma_{23}}+\widetilde{\tau}_0)R\int_{R^m\times\{s=-1\}}Gdy\\ &\quad-\gamma(p,\gamma_2,\gamma_3,\gamma_{23})\tau_1R\int_{R^m\times\{s=-1\}}|y|^2Gdy\end{aligned}$$

from which the desired estimate follows, if δ and $\bar{\delta}$ are taken so small with recalling settings (2.16) and (2.21) of $\widetilde{\tau}_0$ and τ_1.

We have so-called Bochner estimate, the proof of which is similar as in [14,23,25].

Lemma 2.4(Bochner estimate) *It holds, for* $\varphi \in L^2((0,T); W_0^{1,2}(R^m)) \cap W^{1,2}((0,T); L^2_{\mathrm{loc}}(R^m))$ *with* $\varphi \geq 0$ *in* $(0,T) \times R^m$ *and all* t_1, t_2, $0 < t_1 < t_2 < T$,

$$
\begin{aligned}
&\int_{R^m \times \{t\}} |Du|^2 \varphi dx \Big|_{t=t_1}^{t=t_2} - \int_{R^m \times (t_1,t_2)} |Du|^2 \partial_t \varphi dz \\
&+ \int_{R^m \times (t_1,t_2)} (\delta^{\alpha\beta} f'(|Du|^2) + 2f''(|Du|^2) D_\alpha u^i D_\beta u^i) D_\beta |Du|^2 D_\alpha \varphi dz \\
&+ \int_{R^m \times (t_1,t_2)} 2f'(|Du|^2)|D^2 u|^2 \varphi dz + \int_{R^m \times (t_1,t_2)} f''(|Du|^2)|D|Du|^2|^2 \varphi dz \\
\leq &\int_{R^m \times (t_1,t_2)} f'(|Du|^2)(D_\beta u^l \frac{dA^i(u)}{du^l}(Du, Du) + 2A^i(u)(D_\beta Du, Du)) \varphi D_\beta u^i dz.
\end{aligned}
\tag{2.27}
$$

3 Partial regularity

We adapt ideas of Schoen-Uhlenbeck and Schoen to have a decay estimate(refer to [23,24,25]), where we need to derive Harnack type estimate by the technique of De-Giorgi(Proposition 3.2).

Lemma 3.1 (ε−regularity theorem) For all $\bar{t}$, $0 < \bar{t} < T$, there exists a constant $\varepsilon_0 > 0$ depending only on m, p, γ_i $(i = 1,2,3)$, γ_{23}, $\bar{t}$ and $I(u_0)$ such that, for any weak solution u to (1.7) belonging to $C^0_{\mathrm{loc}}((0,T); C^1_{\mathrm{loc}}(R^m))$ and satisfying $\int_0^T \int_{R^m} f(|Du|^2)\, dxdt < +\infty$ with an intial data u_0, $\int_{R^m} f(|Du_0|^2)\, dx < +\infty$, the following holds: If, for some R, $0 < R < \min\{(\bar{t})^{1/2}/2, \varepsilon_0\}$, there holds

$$
\Psi(R, (\bar{t}, \bar{x}), u) = \int_{\bar{t}-(2R)^2}^{\bar{t}-R^2} \int_{R^m \times \{t\}} f(|Du|^2) G_{(\bar{t},\bar{x})} dxdt < \varepsilon_0, \tag{3.1}
$$

then

$$
\sup_{Q_{\delta R/2}(\bar{t},\bar{x})} |Du|^2 \leq 16(\delta R)^{-2} \tag{3.2}
$$

with constants $\delta > 0$ depending only on m, p, γ_1, γ_3, γ_{23}, $\bar{\bar{\gamma}}_{23}$ and $\min\{\bar{t}^{1/2}/2, \varepsilon_0\}$.
Proof. We proceed our investigations similarily as in [4,25]. For simplicity we translate $(\bar{t}, \bar{x})$ to the origin. Set $r_1 = \delta R$ with δ, $0 < \delta < 1/2$, determined later. For positive numbers r, σ, $0 < r, \sigma < r_1$ and $r + \sigma < r_1$, and $z_0 \in P_r$,

$$
\begin{aligned}
\sigma^{-m} \int_{Q_\sigma(z_0)} f(|Du|^2) dz &\leq \gamma(m, \gamma_{23}) \int_{Q_\sigma(z_0)} f(|Du|^2) G_{(t_0+2\sigma^2, x_0)} dz \\
&\leq \gamma(m, \gamma_{23}) \int_{(t_0-2\sigma^2, t_0+\sigma^2)\times R^m} f(|Du|^2) G_{(t_0+2\sigma^2, x_0)} dz \\
&= \gamma(m, \gamma_{23}) \Psi(\sigma, t_0 + 2\sigma^2, u).
\end{aligned}
\tag{3.3}
$$

We proceed to an estimation of the right hand side of (3.3). If $t_0 + 2\sigma^2 \leq 0$, then we are able to take ρ, $0 < \rho < R$, such that

$$t_0 + 2\sigma^2 - \rho^2 = -R^2.$$

Then, by Monotonicity formula (2.6), we have

$$\begin{aligned} &\Psi(\sigma, t_0 + 2\sigma^2, u) \\ &\leq \exp(\gamma(\rho - \sigma)) \int_{t_0+2\sigma^2}^{t_0+2\sigma^2-\rho^2} \int_{R^m \times \{t\}} f(|Du|^2) G_{(t_0+2\sigma^2, x_0)} dz + \gamma I(u_0)(\rho - \sigma) \\ &\leq \exp(\gamma(\rho - \sigma)) \Psi(R, t_0 + 2\sigma^2, u) + \gamma I(u_0)(\rho - \sigma). \end{aligned} \tag{3.4}$$

If $t_0 + 2\sigma^2 > 0$, Monotonicity formula (2.6) gives

$$\begin{aligned} \Psi(\sigma, t_0 + 2\sigma^2, u) &\leq \exp(\gamma(\rho - \sigma)) \Psi(R, t_0 + 2\sigma^2, u) + \gamma I(u_0)(\rho - \sigma) \\ &= \{ \int_{-R^2}^{t_0+2\sigma^2-R^2} + \int_{t_0+2\sigma^2-(2R)^2}^{-R^2} \} \int_{R^m \times \{t\}} f(|Du|^2) G_{(t_0+2\sigma^2, x_0)} dx dt. \end{aligned} \tag{3.5}$$

We now make an estimation of the first term in the right hand side of (3.5). For all $\tau \in (-R^2, t_0 + 2\sigma^2 - R^2)$, Monotonicity formula (2.5) implies

$$\begin{aligned} &(t_0 + 2\sigma^2 - \tau) \int_{R^m \times \{t=\tau\}} f(|Du|^2) G_{(t_0+2\sigma^2, x_0)} dx = \Phi(t_0 + 2\sigma^2 - \tau, t_0 + 2\sigma^2, u) \\ &\quad \leq \exp(\gamma(2(t_0 + 2\sigma^2) - \tau)^{1/2} - (t_0 + 2\sigma^2 - \tau)^{1/2}) \Phi(2(t_0 + 2\sigma^2) - \tau, t_0 + 2\sigma^2, u) \\ &\qquad + \gamma I(u_0)((2(t_0 + 2\sigma^2) - \tau)^{1/2} - (t_0 + 2\sigma^2 - \tau)^{1/2}), \end{aligned}$$

so that

$$\begin{aligned} &\int_{-R^2}^{t_0+2\sigma^2-R^2} \int_{R^m \times \{t\}} f(|Du|^2) G_{(t_0+2\sigma^2, x_0)} dx dt \\ &= \int_{-R^2}^{t_0+2\sigma^2-R^2} (t_0 + 2\sigma^2 - \tau)^{-1} \Psi(t_0 + 2\sigma^2 - \tau, t_0 + 2\sigma^2, u) d\tau \\ &\leq \int_{-R^2}^{t_0+2\sigma^2-R^2} (t_0 + 2\sigma^2 - \tau)^{-1} \\ &\cdot \{ \exp(\gamma(2(t_0 + 2\sigma^2) - \tau)^{1/2} - (t_0 + 2\sigma^2 - \tau)^{1/2}) \Phi(2(t_0 + 2\sigma^2) - \tau, t_0 + 2\sigma^2, u) \\ &+ \gamma I(u_0)((2(t_0 + 2\sigma^2) - \tau)^{1/2} - (t_0 + 2\sigma^2 - \tau)^{1/2}) \} d\tau. \end{aligned} \tag{3.6}$$

Since $2(t_0+2\sigma^2)-\tau \le 3R^2$, by changing of variables: $\tau-(t_0+2\sigma^2)\to\tau$, the right hand of (3.6) is bounded from above by

$$\begin{aligned}
&2\exp(\gamma 3^{1/2}R)\int_{-R^2-(t_0+2\sigma^2)}^{-R^2}\int_{R^m\times\{t=\tau\}} f(|Du|^2)G_{(t_0+2\sigma^2,x_0)}dxd\tau\\
&\quad+\gamma 2^{1/2}I(u_0)\int_{-R^2-(t_0+2\sigma^2)}^{-R^2}(-\tau)^{-1/2}d\tau\\
&=2\exp(\gamma 3^{1/2}R)\int_{-R^2-(t_0+2\sigma^2)}^{-R^2}\int_{R^m\times\{t\}} f(|Du|^2)G_{(t_0+2\sigma^2,x_0)}dxdt+\gamma 3^{1/2}RI(u_0).
\end{aligned}$$

Substituting this estimate into (3.6) and combining the resulting inequality with (3.5), we have

$$\begin{aligned}
&\Psi(\sigma,t_0+2\sigma^2,u)\\
&\le(2\exp(\gamma 3^{1/2}R)+1)\int_{-(2R)^2}^{-R^2}\int_{R^m\times\{t\}} f(|Du|^2)G_{(t_0+2\sigma^2,x_0)}dxdt+\gamma 3^{1/2}RI(u_0).
\end{aligned}\tag{3.7}$$

Now we shall make an estimation of $G_{(t_0+2\sigma^2,x_0)}$ (refer to [4,5,25]). For $(t,x)\in(-4R^2,\ -R^2)\times R^m$, if $|x|\le kR$,

$$G_{(t_0+2\sigma^2,x_0)}\le(3\pi)^{-m/2}R^{-m}\le 3^{-m/2}4^m\exp(k^2/4\gamma_{23})G.\tag{3.8}$$

If $|x|>kR$, note that $G_{(t_0+2\sigma^2,x_0)}=\frac{G_{(t_0+2\sigma^2,x_0)}}{G_{(R^2,x_0)}}G_{(R^2,x_0)}$ and that

$$\begin{aligned}
\frac{G_{(t_0+2\sigma^2,x_0)}}{G_{(R^2,x_0)}}&\le\frac{(R^2-t)^{m/2}}{(t_0+2\sigma^2-t)^{m/2}}\exp\{-\frac{|x-x_0|^2}{4\gamma_{23}(t_0+2\sigma^2-t)}+\frac{|x-x_0|^2}{4\gamma_{23}(R^2-t)}\}\\
&\le\frac{(R^2+4R^2)^{m/2}}{(3R^2/4-t)^{m/2}}\exp\{-\frac{1}{4\gamma_{23}}|x-x_0|^2\frac{R^2-t-(t_0+2\sigma^2-t)}{(t_0+2\sigma^2-t)(R^2-t)}\}\\
&\le(4/3)^{m/2}5^{m/2}\exp\{-\frac{k^2}{180\gamma_{23}}\}.
\end{aligned}\tag{3.9}$$

By (3.8) and (3.9) we have, for $(t,x)\in(-4R^2,R^2)\times R^m$,

$$\begin{aligned}
&G_{(t_0+2\sigma^2,x_0)}(t,x)\\
&\le 3^{-m/2}2^{2m}\exp(k^2/4\gamma_{23})G(t,x)+(4/3)^{m/2}5^{m/2}\\
&\quad\cdot\exp\{-k^2/180\gamma_{23}\}G_{(R^2,x_0)}(t,x).
\end{aligned}\tag{3.10}$$

Applying (3.10) to (3.7), making an estimation of $\int_{-(2R)^2}^{-R^2}\int_{R^m\times\{t\}} f(|Du|^2)G_{(R^2,x_0)}dz$ with monotonicity formula (2.5) in the following manner:

$$\begin{aligned}
&\int_{-(2R)^2}^{-R^2}\int_{\{|x|\geq KR\}} f(|Dv|^2)G_{(R^2,x_0)}dz \\
&\leq \int_{-(2R)^2}^{-R^2}(R^2-t)^{-1}\Phi((R^2-t)^{1/2},(R^2,x_0),v) \\
&\leq \int_{-(2R)^2}^{-R^2}(R^2-t)^{-1} \\
&\quad\cdot\{\exp(\gamma\{(R^2+\bar{t})^{1/2}-(R^2-t)^{1/2}\})\Phi((R^2+\bar{t})^{1/2},(R^2,x_0),v) \\
&\quad\ + \gamma I(u_0)((R^2+\bar{t})^{1/2}-(R^2-t)^{1/2})\}dt \\
&\leq (\exp(\gamma(R^2+\bar{t})^{1/2})(R^2+\bar{t})^{1-m/2}+\gamma(R^2+\bar{t})^{1/2})I(u_0)\int_{-(2R)^2}^{-R^2}(R^2-t)^{-1}dt \\
&\leq I(u_0)\log(5/2)(\exp(\gamma(5^{1/2}/2)\bar{t}^{1/2})\bar{t}^{1-m/2}+\gamma(5^{1/2}/2)\bar{t}^{1/2})
\end{aligned} \tag{3.11}$$

and substituting the resulting inequality into (3.3), we have, for r,σ, $0<r,\sigma<r_1$ and $r+\sigma<r_1$, and $z_0\in P_r$,

$$\begin{aligned}
&\sigma^{-m}\int_{Q_\sigma(z_0)} f(|Du|^2)dz \\
&\quad\leq \gamma(m,\gamma_{23})\Big\{(2\exp(\gamma(3^{1/2}/2)\bar{t}^{1/2})+1)3^{-m/2}4^m\exp(k^2/4\gamma_{23}) \\
&\qquad\cdot\int_{-(2R)^2}^{R^2}\int_{R^m\times\{t\}} f(|Du|^2)Gdxdt \\
&\qquad+(2\exp(\gamma(3^{1/2}/2)\bar{t}^{1/2})+1)(4/3)^{m/2}5^{m/2}\exp(-k^2/180\gamma_{23}) \\
&\qquad\quad\cdot I(u_0)\log(5/2)(\exp(\gamma(5^{1/2}/2)\bar{t}^{1/2})\bar{t}^{1-m/2}+\gamma(5^{1/2}/2)\bar{t}^{1/2}) \\
&\qquad\qquad+\gamma 3^{1/2}RI(u_0)\Big\}.
\end{aligned}$$

For any positive number ε, take k as so large dependently on $m, p, \gamma_2, \gamma_3, \gamma_{23}, \bar{t}, I(u_0)$ and ε, so that we derive from the assumption (3.1) and the above inequality, with a positive constant γ depending only on $m, p, \gamma_2, \gamma_3, \gamma_{23}, \bar{t}, I(u_0)$ and ε,

$$\sigma^{-m}\int_{Q_\sigma(z_0)} f(|Du|^2)dz \leq \gamma\varepsilon_0+\varepsilon. \tag{3.12}$$

Since $u\in C^0_{\text{loc}}((0,T);C^1_{\text{loc}}(R^m))$, there exists $\sigma_0, 0\leq\sigma_0\leq r_1$, such that

$$(r_1-\sigma_0)^2\sup_{Q_{\sigma_0}}|Du|^2 = \max_{0\leq\sigma\leq r_1}\{(r_1-\sigma)^2\sup_{Q_\sigma}|Du|^2\}. \tag{3.13}$$

Here, if $\sigma_0 = r_1$, the desired estimate (3.2) immediately follows, so that we assume $\sigma_0 < r_1$. We find that there exists $(t_0, x_0) \in \overline{Q_{\sigma_0}}$ such that

$$\sup_{Q_{\sigma_0}} |Du|^2 = |Du|^2(t_0, x_0).$$

Now set $e_0 = |Du|^2(t_0, x_0)$ and $\rho_0 = (1/2)(r_1 - \sigma_0)$. Noting a choice of σ_0 and (t_0, x_0), we have, by (3.13),

$$\sup_{Q_{\rho_0}(t_0,x_0)} |Du|^2 \le \sup_{Q_{\sigma_0+\rho_0}} |Du|^2 \le 4e_0. \tag{3.14}$$

Introduce

$$\begin{aligned} r_0 &= \sqrt{e_0}\rho_0, \\ v(s,y) &= u(t_0 + s/e_0, x_0 + y/\sqrt{e_0}). \end{aligned} \tag{3.15}$$

We now show that $r_0 \le 1$. First note that, by Lemmata 2.1 and 2.2, the equation (1.7) holds almost everywhere in $Q_{\rho_0}(t_0, x_0)$, so that v satisfies, almost everywhere in Q_{r_0},

$$\partial_s v - \operatorname{div}(f'(e_0|Dv|^2)Dv) + f'(e_0|Dv|^2)A(v)(Dv, Dv) = 0. \tag{3.16}$$

Moreover (3.14) and (3.15) imply

$$e(v)(0,0) = 1, \quad \sup_{Q_{r_0}} |Dv|^2 \le 4. \tag{3.17}$$

Similarily as in Lemma 2.4 with (3.16), we have Bochner estimate for v : Set $B = B_{r_0}$. v satisfies, for $\varphi \in L^2((-(r_0)^2, 0);\ W^{1,2}(B_{r_0})) \cap W^{1,2}((-(r_0)^2, 0);\ L^2(B_{r_0}))$ with $\varphi \ge 0$ in Q_{r_0} and all intervals $(t_1, t_2) \subset (-(r_0)^2, 0)$,

$$\begin{aligned} &\int_{B\times\{s\}} |Dv|^2 \varphi dx \Big|_{s=t_1}^{s=t_2} - \int_{(t_1,t_2)\times B} |Dv|^2 \partial_s \varphi dyds \\ &+ \int_{(t_1,t_2)\times B} (\delta^{\alpha\beta} f'(e_0|Dv|^2) + 2e_0 f''(e_0|Dv|^2) D_\alpha v^i D_\beta v^i) D_\beta |Dv|^2 D_\alpha \varphi dyds \\ &+ \int_{(t_1,t_2)\times B} 2f'(e_0|Dv|^2)|D^2 v|^2 \varphi dyds + \int_{(t_1,t_2)\times B} e_0 f''(e_0|Dv|^2)|D|Dv|^2|^2 \varphi dyds \\ &\le \int_{(t_1,t_2)\times B} f'(e_0|Dv|^2)(D_\beta v^l \frac{dA(v)}{dv^l}(Dv, Dv) + 2A(v)(D_\beta Dv, Dv)) \cdot \varphi D_\beta v dyds. \end{aligned} \tag{3.18}$$

Now we assume that $r_0 > 1$. Then, we are able to derive Harnack type estimate from (3.18) (see [21] for the proof).

Proposition 3.2. *There exists a positive constant γ depending only on γ_1, γ_3, p and m such that*

$$\sup_{Q_1} |Dv|^4 \le \gamma(\gamma_1, \gamma_3, p, m)(k_0)^{-(p+2)(m+2)/4} \int_{Q_1} |Dv|^4 dz + 4(k_0)^2 \tag{3.19}$$

holds for any k_0, $0 < k_0 < 1$.

Thus we have, by (3.19),

$$1 = |Dv|^4(0,0) \leq \gamma(\gamma_1, \gamma_3, p, m)(k_0)^{-(p+2)(m+2)/4} \int_{Q_1} |Dv|^4 dz + 4(k_0)^2 \tag{3.20}$$

holds any for k_0, $0 < k_0 < 1$. Noting that $r_0 > 1$ implies $\sigma_0 + 1/\sqrt{e_0} \leq \sigma_0 + \rho_0 < r_1$ and adopting (3.12) with $\sigma = 1/\sqrt{e_0}$, we have, by (3.17) and scaling back,

$$\begin{aligned}
\int_{Q_1} |Dv|^4 dz &\leq 4 \int_{Q_1} |Dv|^2 dz = 4(\sqrt{e_0})^m \int_{Q_{1/\sqrt{e_0}}(t_0,x_0)} |Du|^2 dz \\
&= (4/\gamma_{23})(\sqrt{e_0})^m \int_{Q_{1/\sqrt{e_0}}(t_0,x_0)} (\gamma_{23}|Du|^2 - \bar{\bar{\gamma}}_{23}) dz + (4/\gamma_{23})\bar{\bar{\gamma}}_{23}(1/\sqrt{e_0})^2 \\
&\leq (4/\gamma_{23})(\sqrt{e_0})^m \int_{Q_{1/\sqrt{e_0}}(t_0,x_0)} f(|Du|^2) dz + (4/\gamma_{23})\bar{\bar{\gamma}}_{23}(\delta R/2)^2 \\
&\leq (4/\gamma_{23})(\gamma\varepsilon_0 + \varepsilon) + (4/\gamma_{23})\bar{\bar{\gamma}}_{23}(\delta R/2)^2,
\end{aligned} \tag{3.21}$$

where we used (1.5)

$$\gamma_{23}\tau - \bar{\bar{\gamma}}_{23} \leq f(\tau) \quad \text{for all } \tau \geq 0$$

and an estimation

$$1/\sqrt{e_0} \leq \rho_0 \leq r_1/2 \leq \delta R/2.$$

Taking $\varepsilon_0, \varepsilon > 0$, $\delta > 0$ and $k_0 > 0$ as so small, we obtain from (3.20) and (3.21) the contradiction. Therefore we conclude that $r_0 \leq 1$. By choice of σ_0, this implies

$$\max_{0 \leq \sigma \leq r_1} \{(r_1 - \sigma)^2 \sup_{Q_\sigma} |Du|^2\} \leq 4\rho^2 e_0 = 4r_0^2 \leq 4. \tag{3.22}$$

We choose $\sigma = (1/2)r_1 = (\delta/2)R$ in (3.22) and divide the both side of the resulting inequality by $(\delta R/2)^2$ to obtain (3.2).

Let $\{u_k\} \subset C^0_{\mathrm{loc}}((0,T); C^1_{\mathrm{loc}}(R^m))$ be a sequence of weak solutions to (1.7) with $\int_0^T \int_{R^m} f(|Du|^2)\, dxdt < +\infty$. Recall that, for all $t_0 \in (0,T)$, ε_0 is determined in Lemma 3.1, depending on t_0. For $t_0 \in (0,T)$ and $0 < R < \min\{(t_0)^{1/2}/2, \varepsilon_0(t_0)\}$,

$$\begin{aligned}
\Sigma_R^{t_0} &= \{x_0 \in R^m : \varliminf_{k\to\infty} \Psi(R, z_0, u_k) \geq \varepsilon_0(t_0)\}, \\
\Sigma^{t_0} &= \bigcap_{0<R<\min\{(t_0)^{1/2}/2, \varepsilon_0(t_0)\}} \Sigma_R^{t_0}.
\end{aligned} \tag{3.23}$$

Now we also put

$$\Sigma = \cup_{t_0\in(0,T)} \Sigma^{t_0}. \tag{3.24}$$

We now give an estimation on Hausdorff measure of a set Σ.

Lemma 3.2

$$\mathcal{H}^m_{\text{loc}}(\Sigma) < +\infty, \tag{3.25}$$

in addition, for $t_0 \in (0, T)$,

$$\mathcal{H}^{m-2}_{\text{loc}}(\Sigma^{t_0}) < +\infty. \tag{3.26}$$

Proof. We proceed as in [5,25]. Let $z_0 = (t_0, x_0)$ be a point in $(0,T) \times R^m$ and R, $0 < R < \min\{(t_0)^{1/2}/2, \varepsilon_0(t_0)\}$. Set $v = u_k$.

$$\begin{aligned}
\Psi(R, z_0, v) &= \int_{(t_0-4R^2, t_0-R^2)\times R^m} f(|Dv|^2) G_{z_0} dz \\
&\le \int_{t_0-(2R)^2}^{t_0-R^2} \int_{B_{KR}(x_0)} f(|Dv|^2) G_{z_0} dz + \int_{t_0-(2R)^2}^{t_0-R^2} \int_{\{|x-x_0|\ge KR\}} f(|Du|^2) G_{z_0} dz \\
&\le (4\pi)^{-m/2} R^{-m} \int_{t_0-(2R)^2}^{t_0-R^2} \int_{B_{KR}(x_0)} f(|Dv|^2) dz \\
&\quad + 5^{m/2} \exp\{-K^2/80\gamma_{23}\} \int_{t_0-(2R)^2}^{t_0-R^2} \int_{\{|x-x_0|\ge KR\}} f(|Dv|^2) G_{(t_0+R^2, x_0)} dz \\
&\le R^{-m} \int_{t_0-(2R)^2}^{t_0-R^2} \int_{B_{KR}(x_0)} f(|Dv|^2) dz \\
&\quad + 5^{m/2} \exp\{-K^2/80\gamma_{23}\} \int_{t_0-(2R)^2}^{t_0-R^2} \int_{\{|x-x_0|\ge KR\}} f(|Dv|^2) G_{(t_0+R^2, x_0)} dz.
\end{aligned} \tag{3.27}$$

We now evaluate $\int_{t_0-(2R)^2}^{t_0-R^2} \int_{\{|x-x_0|\ge KR\}} f(|Du|^2) G_{(t_0+R^2,x_0)} dz$ similarily as in (3.11) and take K as large, depending on $m, p, \gamma_2, \gamma_3, \gamma_{23}, t_0$ and $I(u_0)$ to have, by (3.27),

$$\Psi(R, z_0, u_R) \le R^{-m} \int_{t_0-4R^2}^{t_0-R^2} \int_{B_{kR}(x_0)} f(|Du_k|^2) dz + \varepsilon_0/2. \tag{3.28}$$

The validity of (3.26) is shown by the arguments similar as in [5,Page 172-173] with (3.28). (3.25) is shown similarly as the proof of (3.26).

Now we give the proof of our Theorem.

Proof of Theorem. We now demonstrate the proof of Theorem in the case $M = R^m$. Let $\{u_k\} \subset C^0_{\text{loc}}((0,T); C^1_{\text{loc}}(R^m))$ be a sequence of weak solutions to (1.7) with $\int_0^T \int_{R^m} f(|Du_k|^2)\, dxdt < \infty$. The validity of (1.11) and (1.12) immediately follows from our energy inequality Lemma 2.2 and Sobolev imbedding theorem (see [20, Theorem 2.1, Page 61]). (1.10) is obtained from (1.12).

We now consider the validity of the latter statement in Theorem. For $z_0 = (t_0, x_0) \notin \Sigma$, there exists $R > 0$ such that

$$\int_{(t_0-4R^2, t_0-R^2)\times R^m} f(|Du_k|^2) G_{z_0} dz \le \varepsilon_0$$

holds for infinitely many $k \in N$. By Lemma 3.1, we have that, with a uniform constant γ,

$$|Du_k| \leq \gamma \text{ in a uniform neighborhood } Q \text{ of } z_0. \tag{3.29}$$

By Lemmata 2.1 and 2.2 available for $\{u_k\}$, we find that

$$\text{each } u_k \text{ satisfies the equation (1.7) almost everywhere in } (0,T) \times R^m. \tag{3.30}$$

We are able to proceed to the estimations with (3.29) and (3.30) similarily as in the proof of Theorem 1.1 in [11] (see also [6,8] and the proof of Theorem 2 in [2]) to observe that each Du_k is locally Hölder continuous in Q, independently on an approximating number k. On the other hand, similarily as the proof of Theorem 1 in [2] (also see the proof of Theorem 1 in [7]) with (3.29) and (3.30), we also find that each u_k is locally Hölder continuous in Q, independently on k. Thus we see by Ascoli-Arzela theorem that u, Du are locally Hölder continuous in Q and we are able to pass to the limit $k \to \infty$ in (1.8) for $\{u_k\}$ with φ the support of which is contained in Q, so that u is a weak solution to (1.7) in Q. We apply (3.29) to (2.1) for $\{u_k\}$ with $Q_{2r} \subset Q$ to see that u satisfies (2.1) for $Q_{2r} \subset Q$ (with a subsequence $\{u_k\}$ if necessary) and then we have (1.13). The validity of (1.14) is shown by Lemma 3.2. At last, by a standard covering lemma(see[4,25]) with (1.13) and (1.14), we find that u is a weak solution to (1.7).

Remark. If the domain M is compact, smooth orientable Riemannian manifold, we are able to make simple modifications of the above arguments to have the validity of Theorem. Here we observe by (1.5) that, if $u \in L^2((0,T); W^{1,2}(M))$, then $\int_0^T \int_M f(|Du|^2) dM dt < \infty$ and the inverse is available.

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