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On the state constraint boundary value problem for Isaacs equations

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§1. Introduction

We consider the following Isaacs equations of first-order:

$$\min_{b\in B}\max_{a\in A}\{\lambda u(x)-\langle g(x,a,b),Du(x)\rangle-f(x,a,b)\}=0. \tag{1}$$

for $x\in\Omega$, where $\Omega\subset\mathbf{R}^n$ is an open bounded set, A and B are compact sets in $\mathbf{R}^N$ for some $N\in\mathbf{N}$, f and g are given continuous real-valued and $\mathbf{R}^n$-valued functions, respectively, on $\overline{\Omega}\times A\times B$ and $\lambda>0$ is a constant.

Soner [5] first characterized the value function associated with the state constraint (SC in short) problem arising in deterministic optimal control (i.e. $\#B=1$) as the unique viscosity solution of (1) (i.e. Bellman equation) among continuous viscosity solutions under a boundary condition, which he proposed via dynamic programming principle.

Recently, for first-order Bellman equations, Ishii and myself in [3] have proposed a new boundary condition, which is naturally derived from the

SC requirement. In [3], it was shown that the value function is the unique viscosity solution among possibly discontinuous viscosity solutions under the new boundary condition.

Our question here is what is the natural boundary condition for the Isaacs equations (1) under the SC requirement.

Our aim here is to present the definition of value for the SC problem of differential games in a reasonable way

§2. Value function

We shall define the *value* for the SC problem associated with (1).

In what follows, we suppose the continuity assumptions:

$$(A1)\quad \begin{cases} \exists\omega_0\text{: a modulus of continuity s.t.} \\ (i)|g(x,a,b)-g(x,\hat{a},\hat{b})| \le \omega_0(|a-\hat{a}|+|b-\hat{b}|), \\ (ii)|f(x,a,b)-f(\hat{x},\hat{a},\hat{b})| \le \omega_0(|x-\hat{x}|+|a-\hat{a}|+|b-\hat{b}|) \\ \text{for } \forall x,\hat{x}\in\overline{\Omega}, \forall a,\hat{a}\in A, \forall b,\hat{b}\in B, \text{ and} \\ (iii)\ \sup_{(a,b)\in A\times B}\{\|f(\cdot,a,b)\|_{C(\overline{\Omega})}+\|g(\cdot,a,b)\|_{C^{0,1}(\overline{\Omega})}\}<\infty. \end{cases}$$

Notations

【Controls by the player I】 $\mathcal{A}=\{\alpha:[0,\infty)\to A \mid \alpha:\text{ measurable }\}$

【Controls by the player II】 $\mathcal{B}=\{\beta:[0,\infty)\to B \mid \beta:\text{ measurable }\}$

【States for $(\alpha,\beta,x)\in\mathcal{A}\times\mathcal{B}\times\overline{\Omega}$】 $X(\cdot;x,\alpha,\beta)$ is the unique solution of

$$\begin{cases} \dfrac{dX}{dt}(t) = g(X(t), \alpha(t), \beta(t)) & \text{for } t > 0 \\ X(0) = x. \end{cases}$$

Roughly speaking, our SC problem is as follows: For each $x \in \overline{\Omega}$, the players I and II, respectively, "minimize and maximize" a functional $J(x, \alpha, \beta)$ over $\alpha \in \mathcal{A}$ and $\beta \in \mathcal{B}$ for which $X(t; x, \alpha, \beta)$ stays in $\overline{\Omega}$ for any time $t \geq 0$. Then, we will derive a function depending on the x, which we will call the value function.

We shall characterize the boundary value problem which the value function satisfies in the sense of viscosity solutions.

Notations

【Admissible pairs of controls for $(x, s) \in \overline{\Omega} \times (0, \infty]$】

$$\mathcal{AD}_s(x) = \{(\alpha, \beta) \in \mathcal{A} \times \mathcal{B} \mid X(t; x, \alpha, \beta) \in \overline{\Omega} \text{ for } t \in [0, s]\}.$$

We will suppose that $\mathcal{AD}_\infty(x) \neq \emptyset$ for all $x \in \overline{\Omega}$.

【Admissible controls by I for $(x, s) \in \overline{\Omega} \times (0, \infty]$】

$$\mathcal{A}_s(x) = \{\alpha \in \mathcal{A} \mid \exists \beta \in \mathcal{B} \text{ s.t. } (\alpha, \beta) \in \mathcal{AD}_s(x)\}$$

【Admissible controls by II for $(x, s) \in \overline{\Omega} \times (0, \infty]$】

$$\mathcal{B}_s(x) = \{\beta \in \mathcal{B} \mid \exists \alpha \in \mathcal{A} \text{ s.t. } (\alpha, \beta) \in \mathcal{AD}_s(x)\}$$

【Strategies】

$$\Delta_s = \left\{ \delta : \mathcal{B} \to \mathcal{A} \;\middle|\; \begin{array}{l} \text{For } \forall t \in (0, s], \text{ if } \beta = \hat{\beta} \text{ a.e. in } (0, t) \text{ for} \\ \beta, \hat{\beta} \in \mathcal{B}, \text{ then } \delta[\beta] = \delta[\hat{\beta}] \text{ a.e. in } (0, t). \end{array} \right\}$$

【Admissible strategies】

$$\Delta_s(x) = \{\delta \in \Delta_s \mid (\delta[\beta], \beta) \in \mathcal{AD}_s(x) \text{ for } \forall \beta \in \mathcal{B}_s(x)\}$$

Note that, for $\forall x \in \Omega$, by $(A1)$, $\exists s > 0$ s.t. $\mathcal{A}_s(x) = \mathcal{A}$ and $\mathcal{B}_s(x) = \mathcal{B}$.

Now we define the *value* V on $\overline{\Omega}$ by

$$V(x) \equiv \inf_{\delta \in \Delta_\infty(x)} \sup_{\beta \in \mathcal{B}_\infty(x)} \int_0^\infty e^{-\lambda t} f(X(t; x, \alpha, \beta), \alpha(t), \beta(t)) dt.$$

Note that V coincides with that for the control problem in [3] if $\#B = 1$.

§3. SC problem for (1)

For each $(x, b) \in \overline{\Omega} \times B$, we define the following subsets of A:

$$A(x,b) = \left\{ a \in A \;\middle|\; \begin{array}{l} \exists r > 0 \text{ s.t. } X(t; y, a, b) \in \overline{\Omega} \\ \\ \text{for } t \in [0, r] \text{ if } y \in \overline{\Omega} \cap B_r(x). \end{array} \right\}$$

We shall use the following assumptions:

$$(A2) \qquad \begin{cases} \exists r, s > 0 \text{ s.t., if } b \in B \text{ and } \beta \in \mathcal{B} \text{ satisfy } |\beta(t) - b| < r \\ \\ \text{for a.a. } t \in [0, s], \text{ and } x \in \partial\Omega, \text{ then } A(x, b) \neq \emptyset, \\ \\ \text{and } X(t; x, a, \beta) \in \overline{\Omega} \text{ for } t \in [0, s] \text{ and } a \in A(x, b). \end{cases}$$

Notations

$$H(x, r, p; a, b) = \lambda r - \langle g(x, a, b), p \rangle - f(x, a, b)$$

for $(x, r, p, a, b) \in \overline{\Omega} \times \mathbf{R} \times \mathbf{R}^n \times A \times B$.

$$H(x, r, p) = \min_{b \in B} \max_{a \in A} H(x, r, p; a, b)$$

$$H_{in}(x, r, p) = \inf_{b \in B} \sup_{a \in A(x,b)} H(x, r, p; a, b)$$

for $(x,r,p) \in \overline{\Omega} \times \mathbf{R} \times \mathbf{R}^n$.

We shall adapt the following definition of viscosity solutions for the SC problems of (1).

Definition. *We call u a subsolution (resp., supersolution and solution) for the SC problem of* (1) *if it is a viscosity subsolution (resp., supersolution and solution) of*

$$H_{in}(x, u(x), Du(x)) = 0 \text{ for } x \in \overline{\Omega}. \tag{SC}$$

Remarks. $H_{in}(x,r,p) = H(x,r,p)$ for $(x,r,p) \in \Omega \times \mathbf{R} \times \mathbf{R}^n$.

$$H_{in}^*(x,r,p) = H(x,r,p) \text{ for } (x,r,p) \in \partial\Omega \times \mathbf{R} \times \mathbf{R}^n.$$
$$H_*(x,r,p) = H_{in}(x,r,p) \text{ for } x \in \partial\Omega \times \mathbf{R} \times \mathbf{R}^n.$$

Theorem 1. ([4] cf. [2]) *Assume $(A1)$ and $(A2)$. Then, the value function V is a viscosity solution of (SC).*

§4. Comparison and uniqueness results

Now, we present our comparison result for (SC), which implies that the value function constructed in section 2 is the unique viscosity solution of (SC) and that it is continuous.

We first introduce the following subsets: For $b \in B$ and $x \in \overline{\Omega}$, we let

$$G(x,b) \equiv \text{the convex hull of } \{g(x,a,b) \mid a \in A(x,b)\}.$$

We use the following hypotheses concerning on the vector fields.

$$(A3) \quad \begin{cases} \text{For } z \in \partial\Omega, \exists\theta \in (0,1), r > 0 \text{ and } \xi \in S^{n-1} \text{ s.t.} \\ (i) \quad \cup_{0<t<r} B_{t\theta}(x+t\xi) \subset \Omega \text{ for } x \in \overline{\Omega} \cap B_r(z). \\ (ii) \quad G(z,b) \cap \cup_{t>0} B_{t\theta}(t\xi) \neq \emptyset \text{ for } b \in B. \end{cases}$$

We remark that assumption $(A3)$ allows us to treat the case when the vector fields are tangential to $\partial\Omega$ if a convex combination of those directs inside rather perpendicularly.

We shall also suppose the nondegeneracy of the convex combinations of the vector fields appearing in our boundary condition;

$$(A4) \qquad \inf_{x\in\partial\Omega, b\in B} \{|\eta| \mid \eta \in G(x,b)\} > 0.$$

We first present a key lemma:

Lemma. ([4] cf. [2], [3]) *Let $z \in \partial\Omega$, $r > 0$, $\xi \in S^{n-1}$ and $\theta \in (0,1)$ satisfy that*

$$\cup_{0<t<r} B_{t\theta}(x+t\xi) \subset \Omega \text{ for } x \in \overline{\Omega} \cap B_r(z).$$

Then, there are constants $C_0, C_1 \geq 1$, $\sigma \in (0, 1-\theta)$ and a function $\psi \in C^1(\overline{\Omega} \times \overline{\Omega})$ such that, for $x, y \in \overline{\Omega} \cap B_r(z)$,

$$C_0^{-1}|x-y|^2 \leq \psi(x,y) \leq C_0|x-y|^2, \tag{3}$$

$$\langle \xi', D_x\psi(x,y)\rangle \leq 0 \textit{ provided } x \in \partial\Omega \textit{ and } \xi' \in B_{\theta+\sigma}(\xi), \tag{4}$$

$$|D_x\psi(x,y)| \leq C_1|x-y| \textit{ and } D_x\psi(x,y) + D_y\psi(x,y) = 0.$$

Remark. Note that (4) is stronger than the associated requirement in [3].

Theorem 2. ([5]) *Assume* $(A1)-(A4)$. *Let* u *and* v *be a viscosity sub- and supersolution of* (SC), *respectively. Then, we have*

$$u^* \leq v_* \ in \ \overline{\Omega}.$$

Outline of proof. For simplicity, we suppose that u and v are upper and lower semicontinuous, respectively.

It is enough to get a contradiction when there exist $\Theta > 0$ and $z \in \partial\Omega$ such that $\Theta = u(z) - v(z) > u(x) - v(x)$ for any $x \in \partial\Omega\backslash\{z\}$.

Choose the C^1-function ψ from Lemma for this z. For $\alpha, \mu > 0$, we set

$$\Psi_\alpha(x,y) = u(x) - v(y) - \alpha\psi(x,y) + \mu\langle\xi, x - y\rangle,$$

where $\mu > 0$ will be fixed later and $\alpha > 0$ will be sent to ∞. Let (x_α, y_α) satisfy

$$\Psi_\alpha(x_\alpha, y_\alpha) = \max_{x,y\in\overline{\Omega}} \Psi_\alpha(x,y) \geq \Theta.$$

A standard observation using $\Psi_\alpha(x_\alpha, y_\alpha) \geq \Psi_\alpha(z,z)$ together with (3) implies

$$\lim_{\alpha\to\infty} x_\alpha = \lim_{\alpha\to\infty} y_\alpha = z, \lim_{\alpha\to\infty} u(x_\alpha) = u(z), \lim_{\alpha\to\infty} v(y_\alpha) = v(z). \qquad (5)$$

We shall write x and y in place of x_α and y_α, respectively.

A difficulty arises only when $x \in \partial\Omega$;

$$H_{in}(x, u(x), \alpha D_x\psi(x,y) - \mu\xi) \leq 0.$$

Thus, there is $\hat{b} \in B$, for some $l \in \mathrm{N}$, such that

$$H(x, u(x), \alpha D_x\psi(x,y) - \mu\xi; a, \hat{b}) \leq 0$$

for all $a \in A(x,\hat{b})$. In view of $(A3)-(ii)$, we can choose $\{t_k > 0\}_{k=1}^{l}$ and $\{a_k \in A(z,\hat{b})\}_{k=1}^{l}$ such that

$$\sum_{k=1}^{l} t_k = 1 \text{ and } \sum_{k=1}^{l} t_k g(z,a_k,\hat{b}) \in \cup_{t>0} B_{t\theta}(t\xi).$$

Set $\eta(x) \equiv \sum_{k=1}^{l} t_k g(x,a_k,\hat{b})$. Taking a convex combination over $a \in A(z,\hat{b}) (\subset A(x,\hat{b}))$ in the above inequality, we see that

$$-\langle \eta(x), \alpha D_x\psi(x,y) - \mu\xi \rangle \leq C \tag{6}$$

for a constant $C > 0$ independent of α, μ. In view of $(A1)-(iii)$ and $(A4)$, there is a contant $k > 0$ independent of α, μ such that $k\eta(x) \in B_\theta(\xi)$. By (5), for large $\alpha > 1$, we see that $k\eta(x) \in B_{\theta+\sigma}(\xi)$. Hence, by (6), we have

$$\mu k |\eta(x)| \sqrt{1-(\theta+\sigma)^2} \leq C$$

for some $C > 0$. Therefore, we get a contradiction for a large $\mu > 1$. ■

Now, according to Theorem 2, it is easy to show the uniqueness and continuity of the lower and upper value functions.

Corollary. *Assume* $(A1)-(A4)$. *Then,* V *is the unique viscosity solution of* (SC). *Moreover,* $V \in C(\overline{\Omega})$.

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