数理解析研究所講究録
911 巻 1995 年 75-83



# Existence of Selfsimilar Shrinking Curves for Anisotropic Curvature Flow Equations

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## 1 Introduction

This is a joint work with Prof. C. Dohmen and Prof. Y. Giga.

We consider a simple looking ordinary differential equation of the form

$$u_{xx}+u-\frac{a(x)}{u}=0 \quad \text{in } \mathbf{R} \tag{1}$$

with a given positive function $a$. This equation arises in describing a selfsimilar solution of anisotropic curvature flow equations. Since $x$ is the argument of the normal of the curve it is natural to impose $2\pi$-periodicity for $a$ in (1) and to ask for existence of $2\pi$-periodic solutions. To simplicity the notation we notice that a $2\pi$-periodic function can be regarded as a function on the flat torus $\mathbf{T}=\mathbf{R}/2\pi\mathbf{Z}$. For example the space $C^m(\mathbf{T})$ is the space of all $2\pi$-periodic $C^m$-functions on $\mathbf{R}$. Let $C_+^m(\mathbf{T})$ denote the set of all positive functions in $C^m(\mathbf{T})$. In particular

$$C_+^2(\mathbf{T})=\{u\in C^2(\mathbf{R}): u(x+2\pi)=u(x) \quad \text{for} \quad x\in\mathbf{R}, u>0\}. \tag{2}$$

Using this notations, we want to investigate the existence of solutions of (1) in $C_+^2(\mathbf{T})$. As to this, we have the following

**Theorem 1.** Assume that $a$ is a positive, continuous function on $\mathbf{T}$. Then there is a function $u\in C_+^2(\mathbf{T})$ solving (1).

The key step to prove this result is to derive a priori bounds for solutions of (1) :

**Theorem 2.** Let $0 < A_1 < A_2$ be two constants. Then there are two positive constants $m$ and $M$, depending only on $A_1$ and $A_2$, such that if $u \in C^2_+(\mathbf{T})$ solves (1) on $\mathbf{T}$ with

$$A_1 \le a \le A_2 \tag{3}$$

then

$$m \le u \le M \quad \text{on } \mathbf{T}. \tag{4}$$

The proof of this a priori estimate actually shows that the continuity of $a$ is not needed.

**Corollary 1.** Let $a \in L^\infty(\mathbf{T})$ and satisfy (3). Then there is a function $u \in C^{1,1}_+(\mathbf{T})$ solving (1).

Here $C^{1,1}_+(\mathbf{T})$ denotes the space of all positive, $2\pi$-periodic functions whose derivative is Lipschitz continuous. The differential equation is solved in the sense of distributions and almost everywhere.

To prove this corollary, we approximate $a$ by continuous functions $a_j$, keeping the bounds (3) and $a_j \to a$ in $L^p_{loc}$-sense for $p > 1$ as $j \to \infty$. Let $u_j$ be the solution of (1) taking $a_j$ instead of $a$. By the a priori bounds (4) and the equation (1) the sequence $u_j$ is bounded in $L^\infty$ along with $u_{jx}$ and $u_{jxx}$. Thus a subsequence of the $u_j$ converges to some function $u$ in $C^1_+(\mathbf{T})$ ; it is not difficult to show $u \in C^{1,1}_+(\mathbf{T})$ and that $u$ solves (1).

To get a better understanding of the mechanisms we will carry out the proof of the a priori bounds considering the slightly more general equation

$$u_{xx} + u - a(x)g(u) = 0 \quad \text{in } I \subset \mathbf{R} \tag{5}$$

instead of (1). Here again $a$ satisfies (3) on the interval $I$ and $g$ is assumed to be a positive, continuous, nonincreasing function on $(0,\infty)$. Defining

$$G(p) = \int_1^p g(s)ds, \tag{6}$$

we consider impose the following conditions on $g$ :

$$\lim_{p\to 0} G(p) = -\infty, \quad \lim_{p\to\infty} G(p)p^{-2} = 0, \tag{7}$$

$$\lim_{p\to 0, q\to\infty} \frac{G(p)p}{g(q)q^2} = 0, \tag{8}$$

$$\lim_{p\to\infty} g(p) = 0. \tag{9}$$

Note that the second condition in (7) is automatically satisfied by (6) and the non-increasing property of $g$. Examples for functions satisfying these conditions are given by

$$g(p) = p^{-\sigma}, \quad 1 \le \sigma < 2. \tag{10}$$

Our main existence theorem has an application for evolution equations for embedded colsed curves $\{\Gamma_t\}_{t>0}$ in $\mathbf{R}^2$ derived in [10].

Let $V$ be the inward velocity of $\Gamma_t$ in the direction of its unit inward normal vector

$$n(\theta) = (\cos\theta, \sin\theta).$$

Let $k$ be the inward curvature of $\Gamma_t$ and let $f$ and $\beta$ be positive functions on $\mathbf{R}$, which are $2\pi$-periodic. we consider an equation for $\Gamma_t$ of the form

$$V = a(\theta)k, \quad a(\theta) = \frac{f''(\theta) + f(\theta)}{\beta(\theta)}.$$

Here $f'' + f$ is assumed to be positive so that the equation is parabolic. Such an equation arises in a model describing the motion of phase boundaries in an anisotropic medium (see [10]). The function $f$ is called the surface energy density and $\beta$ is called the cinetic coefficient.

If $a(\theta)$ is constant, the equation becomes the curvature flow equation and the evolution of $\Gamma_t$ is well studied. No matter what initial curve is given, the solution stays smooth and embedded and eventually becomes convex ([10]). It then stays convex and shrinks to a point in finite time ([8]). The type of shrinking is asymptotically similar to that of a shrinking circle $\{C_t\}$ ([6], [7], [8]), which is self-similar in the sense that

$$C_t = (t_* - t)^{1/2} C,$$

where $C$ denotes the unit circle centered at the origin, the time $t_*$ is the extinction time and $\lambda C$ denotes the dilatation of $C$ with multiplier $\lambda$ . Selfsimilar solutions are classified even for immersed curves ([2]) and the asymptotic shape of singularities of this type is classified ([1]). We are interested in finding such selfsimilar solutions

$$\Gamma_t = (t_* - t)^{1/2} \Gamma$$

for general $a(\theta)$. Such solutions exist in the case that $\beta(\theta)^{-1}$ equals a constant multiple of $f(\theta)$. Then $\Gamma$ is the boundary of the so-called Wulff-shape $W$ of $f$, i.e.,

$$W = \{x \in \mathbf{R}^2 : x \cdot n(\theta) \leq f(\theta) \quad \text{for all } \theta \in \mathbf{R}\}.$$

This is explicitely stated in [12], including the multidimensional case where $\beta$ and the second differential $f''$ are assumed continuous, so also $a$ is continuous. It is not difficult to see that such results extend to $f \in C^{1,1}$, provided that $f'' + f$ is still bounded away from zero and if the definition of a solution is given in some appropriate sense.

Our main existence theorem yields the existence of selfsimilar solutions for arbitrary bounded $a$. Indeed every equation $V = a(\theta)k$ can be rewritten as

$$V = u(u'' + u)k,$$

where $u$ is a solution of (1) with $\theta$ replacing $x$.

## 2 A priori extimates

To simplify the terminology let us difine the following terms. A solution $u \in C^2_+(\mathbf{T})$ of (1) or (5) is called a singlepeak-solution if the set of points not being local extrema consists of two connedted components in $\mathbf{T}$. Otherwise $u$ is called a multipeak-solution.

To prove the a priori bounds these two types of solutions need essentially different techniques. Thus let us state the results separately.

**Lemma 1.** Let $u \in C^2_+(I)$ be a solutions of (5) on some open interval $I$ and let (3) be satisfied. If $u$ attains local minima in $\alpha, \beta \in I, \alpha < \beta$ and $n_x$ changes its sign only once in $(\alpha, \beta)$, then there is a positive constant $M_0$ depending only on $A_1, A_2$ and $g$ such that

$$u \leq M_0 \quad \text{in } (\alpha, \beta) \tag{11}$$

provided that $\beta - \alpha \leq \pi$.

**Lemma 2.** Let $u \in C^2_+(\mathbf{T})$ be a singlepeak-solutions of (5) and let (3) be satisfied. Then there is a positive constant $M_1$ depending only on $A_1, A_2$ and $g$ such that

$$u \leq M_1 \quad \text{in } \mathbf{T}. \tag{12}$$

**Proposition 1.** Let $u \in C^2_+(\mathbf{T})$ be a solution of (5) and let (3) be satisfied.

i) If there is a constant $\tilde{M}$ depending only on $A_1, A_2$ and $g$ such that one local maximum $u(\gamma)$ is estimated by $u(\gamma) \leq \tilde{M}$, then there are two other constants $0 < m < M$, also depending only on $A_1, A_2$ and $g$ such that

$$m \leq u \leq M \quad \text{on } \mathbf{T}.$$

ii) The conclusion in i) also holds if there is a constant $\tilde{m} > 0$ depending only on $A_1, A_2$ and $g$ such that one local minimum $u(\alpha)$ is estimated by $u(\alpha) \geq \tilde{m}$ .

See the proofs of Lemmas 1, 2 and Proposition 1 in [4]. Theorem 2 is an immediate consequence of Lemma 1, 2 and Proposition 1 as can be seen as follows. If $u$ is a multipeak solution, there exists at least one pair of local minima with a distance less or equal $\pi$. On these intervals Lemma 1 can be applied and due to Proposition 1 all extrema are estimated in terms of one extremum. The situation needed to apply Lemma

1 fails to exist only if $u$ has exactly one local minimum, i.e., is a singlepeak solution. But in this case Lemma 2 yields the upper bound and due to Proposition 1 we again have a lower bound; thus the theorem is proved.

The results above also show that the set of all $2\pi$-periodic solutios of (1) or (5) is bounded uniformly in the set of all $a$ that satiafy (3).

# 3 Existence of solutions

In this chapter, we will prove the existence of a solution of (1) using the Leray-Schauder degree. Herein we make use of the uniform boundedness of solutions of (1) with respect to functions $a$ satisfying (3) stated in Theorem 2. We define

$$E = \{v \in C^0_+(\mathbf{T}) : \frac{m}{2} \le v \le 2M \quad \text{in } \mathbf{T}\}. \tag{13}$$

Let $F$ be a continuous mapping from $E \times [0,1]$ into $C^0_+(\mathbf{T})$ defined by

$$F(u,\tau) = 2u - \frac{\tau a(x) + (1-\tau)a_0}{u} \tag{14}$$

with a constant $a_0$ satisfying the bounds imposed on $a$ in (3).

Let $T$ denote a linear compact operator from $C^0_+(\mathbf{T})$ into itself given by $w = T(f)$, where $w$ is the unique solution of

$$-w_{xx} + w = f \quad \text{in } \mathbf{T}.$$

Setting $S_\tau = S(\cdot,\tau) = T \circ F(\cdot,\tau)$, we have a continuous, compact mapping from $E$ into $C^0_+(\mathbf{T})$. Clearly $u$ is a fixed point of $S_\tau$ if and only if $u \in E$ solves

$$u_{xx} - u + 2u - \frac{\tau a(x) + (1-\tau)a_0}{u} = 0 \quad \text{in } \mathbf{T},$$

which is (1) in case of $\tau = 1$. The a priori bounds in Theorem 2 now imply that $S_\tau$ has no fixed point on the boundary of $E$, in other words

$$(I - S_\tau)u \neq 0 \quad \text{on } \partial E, \quad 0 \le \tau \le 1.$$

Thus the homotopy invarianve of the Leray-Schauder degree yields

**Proposition 2.**

$$\deg\,(I - S_1, E, 0) = \deg\,(I - S_0, E, 0).$$

To show the existence of a solution of (5) it now suffices to prove that this degree is not equal zero.

**Lemma 3.** The number

$$\deg\,(I - S_0, E, 0) \tag{15}$$

is not zero ; in fact, it equals $-1$.

**Proof.** As proved by Gage and Hamilton in [8] (see also [2], [5]), there is a unique solution $u \in E$ of

$$u_{xx} + u - \frac{a_0}{u} = 0 \quad \text{in } \mathbf{T},$$

which is given by the constant $a_0^{1/2}$. (Actually in [8] the setting is $a_0 = 1/2$, but our problem here reduces to theirs by changing from $u$ to $(2a_0)^{1/2}u$. )

So $u_0 = a_0^{1/2}$ is the only zero of $I - S_0$ in $E$ ; thus

$$\deg\,(I - S_0, E, 0) = \deg\,(I - S_0, B_\delta(u_0), 0)$$

for some sufficiently small $\delta$. At $u_0$ the mapping $I - S_0$ is nondegenerate in the sense that the derivative $I - S_0'(u_0)$ is injective. Indeed, suppose that

$$(I - S_0'(u_0))v = 0.$$

Since $S_0'(u_0) = T \circ F'(u_0, 0)$, this implies

$$-v_{xx} + v = 2v + \frac{a_0}{u_0^2}v$$

or, using the definition of $u_0$

$$v_{xx} + 2v = 0.$$

But this problem has no nontrivial $2\pi$-peiodic solution. This nondegeneracy enables us to apply a standard degree theory result (see [11], Theorem 2.8.1, p.66 or [3], Example 2.8.3, p.65), which states

$$\deg\,(I - S_0, B_\delta(u_0), 0) = (-1)^\beta,$$

where $\beta$ is the number of eigenvalue of $S_0'$ (counting algebraic multiplicity) greater than one.

We show the elementary computation of $\beta$. A number $\lambda$ is an eigenvalue of $S_0'(u_0)$ if and only if there is a nontrivial solution $v \in C_+^0(\mathbf{T})$ of

$$\lambda v = S_0'(u_0)v$$

or equivalently

$$-v_{xx} = \frac{3-\lambda}{\lambda}v.$$

Thus $\beta$ equals the number of $\lambda > 1$ (counted with multiplicity) that solve $\dfrac{3-\lambda}{\lambda} = n^2$ for some integer $n \geq 0$. As these $\lambda$ are given by $\lambda = 3$ and $\lambda = 3/2$ with multiplicity 1 and 2, respectively, we have

$$\deg\,(I - S_0, B_\delta(u_0), 0) = (-1)^3 = -1. \quad \square$$

**Remark 1.** Concerning the uniqueness of solutions of (1) in $C_+^2(\mathbf{T})$, the implicit function theorem implies that the zero of $I - S_\tau$ is unique provided $\tau$ is small since no bifurcation from $(u_0, 0)$ occurs due to the nondegeneracy of the unique zero $u_0$ of $I - S_0$.

# References

[1] S. Angenent, On the formation of singularities in the cruve shortening flow, J. Diff. Geometry 33 (1991), 601-633.

[2] U. Abresch and J. Langer, The normalized curve shortening flow and homothetic solutions, J. Diff. Geometry 23 (1986), 175-196.

[3] K. Deinling, Nonlinear Functional Analysis, Springer, Heidelberg, Berlin, New York (1985).

[4] C. Dohmen, Y. Giga and N. Mizoguchi, Existence of selfsimilar shrinking curves for anisotropic curvature flow equations, preprint.

[5] C. Epstein and M. Weinstein, A stable manifold theorem for the curve shortening equation, Comm. Pure Appl. Math. 40 (1987), 119-139.

[6] M. Gage, An isoperimetric inequality with application to curve shortening, Duke M. J. 50 (1983), 1225-1229.

[7] M. Gage, Curve shortening makes convex curves circular, Inv. Math. 76 (1984), 357-364.

[8] M. Gage and R. S. Hamilton, The heat equation shrinking convex plane curves, J. Diff. Geometry 23 (1986), 69-96.

[9] The heat equation shrinks embedded plane curves to points, J. Diff. Geometry 26 (1987), 285-314.

[10] M. E. Gurtin, Thermodynamics of evolving phase boundaries in the plane, Clarendon Press, Oxford (1993).

[11] L. Nirenberg, Topics in nonlinear functional analysis, Lecture Notes 1973/74, Courant Inst. of Math. Sciences.

[12] H. M. Soner, Motion of a set by the curvature of its boundary, J. Diff. Eq. 101 (1993), 313-392.