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A non-standard proof of the Peano existence theorem in WKL_0

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This note is a sort of supplement to our paper [6], but can be read independently. Except for lacking in the proof of our self-embedding theorem that every countable non-standard model of WKL_0 has a proper initial part isomorphic to itself, our argument here is essentially self-contained. The goal of this note is to carry out the popular non-standard proof of the Peano existence theorem for solutions of ordinary differential equations within WKL_0.

The usual standard proof of Peano's theorem depends much on the Ascoli lemma, by which one can make a solution of initial value problem from a sequence of piecewise linear approximations. It was Simpson [3] who first proved the theorem within WKL_0 by avoiding the use of the Ascoli lemma. In regard to the program of Reverse Mathematics, he [3] has actually shown that Peano's theorem is provably equivalent to WKL_0 over RCA_0, while the Ascoli lemma is equivalent to the stronger system ACA_0. Subsequently, we [2] obtained another WKL_0 proof of Peano's theorem based on a version of Schauder's fixed point theorem. See [4], [5] for more information.

On the other hand, the non-standard proof of Peano's theorem is also known to be free from the Ascoli lemma. Thus, the non-standard proof and the WKL_0 proofs share the same feelings of constructivity (cf. Albeverio et al. [1, p.31]). In fact, by our self-embedding theorem, a considerable portion of non-standard analysis could be developed in WKL_0.

To begin with, recall some basic definitions and the self-embedding theorem. The system RCA_0 consists of the axioms of ordered semirings, Σ^0_1 induction and Δ^0_1 comprehension, and WKL_0 is obtained from RCA_0 by adding weak König's lemma: every infinite tree of sequences of 0's and 1's has an infinite path. A structure V of second-order arithmetic is often expressed as a pair (M, S), where M is its first-order part and S consists of subsets of (the underlying set of) M. For an initial segment I of M, let $V\lceil I = (I, S\lceil I)$ where $S\lceil I = \{X \cap I : X \in S\}$. Now, we have

The Self-Embedding Theorem. Let V = (M, S) be a countable non-standard model of WKL_0. Then there exists a proper initial part $V\lceil I = (I, S\lceil I)$ of V and an isomorphism $f: V \to V\lceil I$.

See [6] for a more general statement and its proof.

Fix a countable non-standard model V = (M, S) of WKL_0, in which we are going to develop analysis. By the above theorem, V has an initial part isomorphic to itself. Since the initial part and V are isomorphic to each other, they may exchange their roles, and so they can be regarded as V and its extension, respectively. Then, let *V = (*M, *S) denote an isomorphic extension of V, which will be used as a non-standard universe.

Following our paper [6], a *real number* in the closed unit interval [0,1] is defined as its binary expansion. Intuitively, a binary function α codes the real $\sum_i \frac{\alpha(i)}{2^{i+1}}$. Then, each real in V is an initial segment of a *V-finite sequence. A set F of pairs of finite binary sequences is said to be (a code for) a *continuous (partial) function f* from [0, 1] to itself if the following conditions hold:

1. if (s, t) ∈ F and (s, t′) ∈ F, then t extends t′ or t′ extends t;
2. if (s, t) ∈ F and s′ extends s, then (s′, t) ∈ F;
3. if (s, t) ∈ F and t extends t′, then (s, t′) ∈ F.

For a sequence s with length lh(s), we set

$$a_s = \sum_{i < lh(s)} \frac{s(i)}{2^{i+1}}, \quad b_s = a_s + \frac{1}{2^{lh(s)+1}}.$$

Then, (s, t) ∈ F intuitively means that the image of open interval (a_s, b_s) via f is included in the closed interval $[a_t, b_t]$. Finally, we write $f(\alpha) = \beta$ iff for each M-finite initial segment t of β, there exists an M-finite initial segment s of α such that (s, t) ∈ F.

Suppose that F is a code for a "total" continuous function in V. Let *F be a set of *V such that F = *F∩V. Since "F is a code for a continuous function" is a Π^0_1 predicate, by overspill, there is a p ∉ M such that *F satisfies the above three conditions for all the binary sequences with length ≤ p. Fix such a p. Let Seq(p) be the set of binary sequences with length p. We then define the function *f on Seq(p) by

*$f(\tilde{s})$ = the longest sequence $\tilde{t}$ such that $(\tilde{s}, \tilde{t})$ ∈ *F and lh($\tilde{t}$) ≤ p.

It is clear from conditions 1 and 2 that this function is well-defined. It is also obvious that for each $\tilde{s}$ ∈ Seq(p), the length of *$f(\tilde{s})$ is not in M, since f is total. Again by overspill, there is a q ∉ M such that the length of *$f(\tilde{s})$ is ≥ q for every $\tilde{s}$ ∈ Seq(p). So, by pruning, *f can be seen as a function from Seq(p) to Seq(q).

Lemma 1. Let f be a total continuous function in V. And let *f be a function from Seq(p) to Seq(q) constructed as above. Then, $f(\tilde{s} \cap M)$ = *$f(\tilde{s}) \cap M$ for each $\tilde{s}$ ∈ Seq(p).

Proof. Let $y = f(\tilde{s} \cap M)$. Choose any M-finite initial segment t of y. By the definition of $f(\alpha) = \beta$, there exists an M-finite initial segment s of $\tilde{s}$ such that $(s, t) \in F$. Hence we have $(\tilde{s}, t) \in {}^*F$ by condition 2 of the definition of continuous partial functions. So, t must be an initial segment of ${}^*f(\tilde{s})$ by condition 1. Since t is chosen as an arbitrary initial segment of y, y is also an initial segment of ${}^*f(\tilde{s})$. []

Theorem 2 (WKL_0). Any continuous function f on [0, 1] attains a maximal value.

Proof. If ${}^*f$ is maximal at $\tilde{s} \in Seq(p)$, f attains a maximal value ${}^*f(\tilde{s}) \cap M$ at $\tilde{s} \cap M$. []

Next, we show the converse to Lemma 1.

Lemma 3. Suppose we are first given a function ${}^*f$: $Seq(p) \to Seq(q)$ with $p, q \notin M$ such that for all $\tilde{s}, \tilde{t} \in Seq(p)$,

$$(*) \qquad \tilde{s} \cap M = \tilde{t} \cap M \Rightarrow {}^*f(\tilde{s}) \cap M = {}^*f(\tilde{t}) \cap M.$$

Then there exists a continuous function f in V such that $f(\tilde{s} \cap M) = {}^*f(\tilde{s}) \cap M$ for all $\tilde{s} \in Seq(p)$.

Proof. We first put

$${}^*F = \{(s,t) \in \cup_{r \leq p} Seq(r) \times \cup_{r \leq q} Seq(r) \colon \forall \tilde{s} \in Seq(p)\ (s \subseteq \tilde{s} \to t \subseteq {}^*f(\tilde{s}))\}.$$

Then it is easy to see that *F satisfies the three conditions of continuous functions with respect to sequences $s \in \cup_{r \leq p} Seq(r)$ and $t \in \cup_{r \leq q} Seq(r)$. Hence, $F = {}^*F \cap M$ is a code for a continuous (partial) function in V.

To show that F is total and $f(\tilde{s} \cap M) = {}^*f(\tilde{s}) \cap M$, take any real $\alpha \in [0, 1]$. Let $\tilde{s} \in Seq(p)$ be a sequence extending α, and t be any M-finite initial segment of ${}^*f(\tilde{s})$. By condition (*), for any $s \subseteq \tilde{s}$ such that $s \notin M$, we have $(s, t) \in {}^*F$.

So, by underspill, there is an M-finite $s \subseteq \tilde{s}$ such that $(s, t) \in {}^*F$, hence $(s, t) \in F$. This shows that $f(\alpha)$ is defined and its value is ${}^*f(\tilde{s}) \cap M$. Thus, F is a code for a desired continuous function f. []

Theorem 4 (WKL_0). Any continuous function f on [0, 1] is uniformly continuous, that is, for each $n \in M$, there exists $m \in M$ such that $\forall s \in Seq(m)\ \exists t \in Seq(n)\ (s, t) \in F$.

Proof. Fix any $n \in M$. As in the proofs of the above lemmas, we can easily see that for each $p \notin M$, $\forall s \in Seq(p)\ \exists t \in Seq(n)\ (s, t) \in F$. Hence, also by underspill, there exists $m \in M$ such that $\forall s \in Seq(m)\ \exists t \in Seq(n)\ (s, t) \in F$. []

Theorem 5 (WKL_0). Any continuous function f on $[\alpha, \beta] \subseteq [0, 1]$ is Riemann integrable.

Proof. With the help of Theorem 4, the usual argument using the upper and lower sums works. []

Remark. The Riemann integral of a continuous function f on [0, 1] is given by

$$\int_0^1 f(x)dx = \lim_{n\to\infty} \sum_{s\in Seq(n)} \max_{\alpha \supseteq s} f(\alpha)\cdot\frac{1}{2^n} = \lim_{n\to\infty} \sum_{s\in Seq(n)} \min_{\alpha \supseteq s} f(\alpha)\cdot\frac{1}{2^n}$$
$$= \left(\sum_{\tilde{s}\in Seq(p)} {}^*f(\tilde{s})\cdot\frac{1}{2^p}\right)\cap M.$$

Theorem 6 (WKL_0). Let $f(x, y)$ be a continuous function from $D = [0, 1]^2$ to [0,1]. Then the initial value problem

$$\frac{dy}{dx} = f(x,y), \qquad y(0)=0$$

has a solution y(x) on the interval [0,1]. (The Peano Existence Theorem)

Proof. Given a continuous function $f(x, y)$, we take ${}^*f$, $p \notin M$, $q \notin M$ as before so that

$f = {}^*f \cap V$, ${}^*f$: Seq(p)×Seq(p) → Seq(q).

Then define a function *y: Seq(p) → Seq(p+q) by recursion as follows:

$${}^*y(\frac{0}{2^p}) = \frac{0}{2^{p+q}},$$
$${}^*y(\frac{i+1}{2^p}) = {}^*y(\frac{i}{2^p}) + \frac{1}{2^p}\cdot {}^*f(\frac{i}{2^p}, {}^*y(\frac{i}{2^p})\lceil p),$$

where a fraction form $\frac{i}{2^p}$ denotes the binary sequence in Seq(p) encoding the real $\frac{i}{2^p}$, and ${}^*y(\frac{i}{2^p})\lceil p$ is the initial segment of ${}^*y(\frac{i}{2^p})$ with length p.

First, it is easy to see that

$$|{}^*y(\frac{i}{2^p}) - {}^*y(\frac{j}{2^p})| \leq \frac{|i-j|}{2^p},$$

since $|{}^*f(x)| \leq 1$. So, by Lemma 3, there exists a continuous function y(x) in V such that $y(\frac{i}{2^p} \cap M) = {}^*y(\frac{i}{2^p}) \cap M$. By the definition of *y,

$${}^*y(\frac{k}{2^p}) = \Sigma_{i<k}\, {}^*f(\frac{i}{2^p}, {}^*y(\frac{i}{2^p})\lceil p)\cdot \frac{1}{2^p}.$$

We also have

$$\int_0^{\frac{k}{2^p}\cap M} f(x,y)\, dx = (\Sigma_{i<k}\, {}^*f(\frac{i}{2^p}, {}^*y(\frac{i}{2^p})\lceil p)\cdot \frac{1}{2^p}) \cap M,$$

by the remark after Theorem 5. So, letting $\alpha = \frac{k}{2^p} \cap M$, we have

$$y(\alpha) = \int_0^\alpha f(x,y)dx.$$

Thus, y(x) is a solution of the differential equation. []

References

[1] S. Albeverio, J.E. Fenstad, R. Høegh-Krohn and T. Lindstrøm, Nonstandard methods in stochastic analysis and mathematical physics, Academic Press 1986.

[2] N. Shioji and K. Tanaka, Fixed point theory in weak second-order arithmetic, Ann. Pure Appl. Logic 47 (1990) 167-188.

[3] S. Simpson, Which set existence axioms are needed to prove the Cauchy/Peano theorem for ordinary differential equations?, J. Symb. Logic 49 (1984) 783-802.

[4] S. Simpson, Subsystems of Second-Order Arithmetic, to appear.

[5] K. Tanaka, Reverse Mathematics and subsystems of second-order arithmetic, American Math. Soc., Sugaku Expositions 5 (1992) 213-234.

[6] K. Tanaka, The self-embedding theorem of WKL_0 and a non-standard method, a preprint.