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A FORMULA SEPARATING TWO THEORIES

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Abstract

If theory T+$\neg\psi$ is incomplete, then there is a formula φ such that $T \subsetneq T + \varphi \subsetneq T+\psi$. In special case ψ =Con(T), Rosser sentence of T is a separating formula.

0. Preliminaries

We consider only formulas in Arithmetic. A theory is a set of sentences that contains all of its logical consequences. T+φ denotes the least theory which contains T$\cup\{\varphi\}$ (T is a theory, φ is a sentence).

$T \vdash \varphi$ means "φ is provable in T".

1. A formula separating two theories

Definition Let T, S be theories.

A sentence φ separates T from S $\overset{\text{def}}{\Leftrightarrow}$ $T \subsetneq T+\varphi \subsetneq S$

Lemma 1 Let T be a theory, ψ be a sentence, and $T \subsetneq T+\psi$. The following conditions on φ are equivalent.

1. φ separates T from T+ψ.
2. (i) $T \nvdash \varphi$, and
   (ii) $T \vdash \psi \rightarrow \varphi$, $T \nvdash \varphi \rightarrow \psi$.

Proof) $T \subsetneq T + \varphi$ is equivalent to (i). $T + \varphi \subsetneq T + \psi$ is equivalent to (ii). □

Remark $T \subsetneq T + \varphi \subsetneq T + \psi \Leftrightarrow T + \neg\psi \subsetneq T + \neg\varphi \subsetneq$ Incon. (Incon is the inconsistent theory.)

Theorem If $T+\neg\psi$ is incomplete, then there is a formula φ separating T from $T+\psi$.
Proof) There is a sentence φ' such that $T+\neg\psi \nvdash \varphi'$ and $T+\neg\psi \nvdash \neg\varphi'$ because $T+\neg\psi$ is incomplete. Let $\varphi \equiv \varphi' \vee \psi$. It is easily verified that φ satisfy the conditions of the above remark. So this φ is a separating sentence T from $T+\psi$. □

Remark If T is a consistent primitive recursive theory, then $T \nvdash \mathrm{Con}(T)$ by second incompleteness theorem [2]. So $T+\neg\mathrm{Con}(T)$ is a consistent primitive recursive theory, too. We have that $T+\neg\mathrm{Con}(T)$ is incomplete by Rosser's theorem [2]. So there is a theory between T and T+Con(T) by above theorem.

2. On a formula separating T from T+Con(T)

In the previous section, we showed that there is a theory between T and T+Con(T) (T is consistent and primitive recursive). Actually, let φ be a Rosser sentence of $T+\neg\mathrm{Con}(T)$, then $\neg\varphi\vee\mathrm{Con}(T)$ and $\varphi\vee\mathrm{Con}(T)$ are separating sentences. In this section, we show that Rosser sentence of T itself is a sentence separating T from T+Con(T). We will use certain results on the provability predicate. See [1], [2].

Definition
Rosser sentence φ of T is a sentence such that

$$T \vdash \varphi \leftrightarrow \neg\mathrm{Pr}^*(\ulcorner\varphi\urcorner) \tag{1}$$

where $\mathrm{Pr}^*(x) \equiv \exists y(\mathrm{Prov}_T(x,y) \wedge \forall z \leq y(\neg\mathrm{Prov}_T(\mathrm{not}(x),z)))$.

Theorem Rosser sentence φ is a sentence separating T from T+Con(T).
Proof) We show (i) $T \nvdash \varphi$, (ii) $T \vdash \mathrm{Con}(T) \to \varphi$, (iii) $T \nvdash \varphi \to \mathrm{Con}(T)$, and then lemma 1 implies that φ is the separating sentence.

(i) $T \nvdash \varphi$ is well-known as Rosser's theorem.

(ii) By (1), we have

$$T+\neg\varphi \vdash \exists y(\mathrm{Prov}_T(\ulcorner\varphi\urcorner, y)).$$

That is

$$T+\neg\varphi \vdash \mathrm{Pr}(\ulcorner\varphi\urcorner). \tag{2}$$

On the other hand, we have

$$T \vdash \mathrm{Con}(T) \to \neg\mathrm{Pr}(\ulcorner\varphi\urcorner). \tag{3}$$

$$T \vdash \mathrm{Con}(T) \to \neg\mathrm{Pr}(\ulcorner\neg\varphi\urcorner). \tag{4}$$

by formalized Rosser's theorem. From (2) and (3), we have

$$T+\neg\varphi \vdash \neg\mathrm{Con}(T),$$
$$T \vdash \neg\varphi \to \neg\mathrm{Con}(T),$$
$$T \vdash \mathrm{Con}(T) \to \varphi.$$

(iii) Assume that $T \vdash \varphi \rightarrow \mathrm{Con}(T)$. By the derivability conditions of the provabiity predicate Pr(), we have

$$T \vdash \Pr(\ulcorner \neg \mathrm{Con}(T) \rightarrow \neg\varphi \urcorner),$$

$$T \vdash \Pr(\ulcorner \neg \mathrm{Con}(T) \urcorner) \rightarrow \Pr(\ulcorner \neg\varphi \urcorner). \tag{5}$$

By (4) and (5),

$$T \vdash \Pr(\ulcorner \neg \mathrm{Con}(T) \urcorner) \rightarrow \neg \mathrm{Con}(T).$$

Then we have

$$T \vdash \neg \mathrm{Con}(T)$$

by Löb's theorem, and by assumption

$$T \vdash \neg\varphi .$$

This contradicts that φ is Rosser sentence. □

References

[1] G. E. Boolos and R. C. Jeffery. *Computability and Logic 3rd ed.* . CAMBRIDGE UNIVERSITY PRESS 1989

[2] C. Smoryński. *The Incompleteness Theorems*, Handbook of Math. Logic (J. Barwise ed.) . North-Holland 1977. pp.821-865