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Nonlinear and Nonlocal Equations Related to Muscle Contraction

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1. Introduction

We are concerned with a nonlinear and nonlocal hyperbolic equation and its related transport-diffusion equation, both of which are related to mathematical models of muscle contraction:

$$(H)\qquad \begin{cases} u_t + z'(t)u_x = \varphi(x,t,z(t),u), & (x,t) \in \mathbb{R}\times[0,T],\\ z(t) = L\Big(\int_{\mathbb{R}} w(x)u(x,t)dx\Big), & t\in[0,T],\\ u(x,0) = u_0(x), & x\in\mathbb{R}, \end{cases}$$

$$(P)\qquad \begin{cases} u_t - \varepsilon u_{xx} + z'(t)u_x = \varphi(x,t,z(t),u), & (x,t) \in \mathbb{R}\times[0,T],\\ z(t) = L\Big(\int_{\mathbb{R}} w(x)u(x,t)dx\Big), & t\in[0,T],\\ u(x,0) = u_0(x), & x\in\mathbb{R}, \end{cases}$$

where $u : \mathbb{R}\times[0,T] \to \mathbb{R}$ and $z : [0,T] \to \mathbb{R}$ are unknown, $z' = dz/dt$, and φ, u_0, w and L are given functions specified later.

Our aim is to obtain unique solutions to both problems and investigate the convergence of the solution of (P) to that of (H) as $\varepsilon \searrow 0$. These problems arise from reological models describing the cross-bridge dynamics in the muscle contraction in physiology. See [1, 4, 5, 7, 8] and reference therein. The repeating unit of muscle structure (the sarcomere) consists of particles of myosin (thick filament) and actin (thin filament). According to the sliding filament theory of Huxley [8], the so-called cross-bridges are chemical links between myosin and actin filaments; and muscle contraction is a consequence of relative sliding between these two filaments, which occurs when the cross-bridges act like springs. The quantity $u(x,t)$ essentially represents a density of cross-bridges attached at distance x and time t. The function z is the contractile movement of filaments and it is related to the contractile force $\int_{\mathbb{R}} w(x)u(x,t)dx$. The model problem (P) having a viscosity term $-\varepsilon u_{xx}$ takes into account some "slipping effect", while (H) does not. See [2, 3].

The dynamics of the cross-bridges results from the balance of formation and breakage; and in the original model by Huxley, φ is taken as $\varphi(x,t,z,u) = \gamma(t)f(x)(1-u) - g(x)u$, where $\gamma(t)$ is the activation function, $f(x)$, $g(x)$ are the attachment rate functions. Here, we take φ more generally as

$$\varphi(x,t,z,u) = \gamma(t)f(x,z)(1-|u|^{p-1}u) - g(x,z)|u|^{q-1}u$$

having polynomial nonlinearity with p, $q \geq 1$.

In case of bounded domain in $\mathbb{R}$, Colli and Grasselli [2] have shown a local existence of a strong solution of (P) with the Dirichlet boundary condition. In case of the whole space $\mathbb{R}$, Colli and Grasselli [3] have shown a global existence of a weak solution of (P) and a strong solution of (H) for the case $\varphi(x,t,z,u) = F(x,t,z) - G(x,t,z)u$ being linear in the variable u; they have also established the convergence results and so on.

At first, we establish a global existence and uniqueness of a strong solution to (P) by using the idea in [3] combined with the theory of abstract semilinear evolution equations. Next, we show that the solution of (P) approaches to the solution of (H) when ε tends to zero. A result about the support of the solution of (H) is also investigated.

2. Existence and Convergence Results

In this section we state our assumptions and the results. In what follows, BUC stands for the space of bounded and uniformly continuous functions, $BUC^{\eta,\frac{\eta}{2}}$ the space of Hölder continuous functions of two variables which belong to BUC. The space of Hölder continuous functions will be denoted by $C^{0,\eta}$ with $0 < \eta < 1$; and by $C^{0,1}$ we mean the space of Lipschitz continuous functions.

Let $T > 0$ be fixed and we assume the following hypotheses.

(C1) $L : (a,b) \to \mathbb{R}$ is a locally Lipschitz continuous, strictly decreasing function ($-\infty \leq a < 0 < b \leq \infty$) satisfying $L(x) \nearrow \infty$ (resp. $\searrow -\infty$) as $x \searrow a$ (resp. $\nearrow b$), and $L(0) = 0$.

(C2) $w \in C^1(\mathbb{R})$ is an increasing function satisfying $w(0) = 0$ and $dw/dx \in W^{1,\infty}(\mathbb{R})$.

(C3) the functions γ, f and g are nonnegative and satisfy the following conditions: $\gamma \in C^{0,\frac{\eta}{2}}[0,T]$ $(0 < \eta \leq 1)$, f, $g \in C(\mathbb{R}^2)$, $f(x,\cdot)$, $g(x,\cdot) \in C^{0,1}_{loc}(\mathbb{R})$ uniformly for $x \in \mathbb{R}$, and $f(\cdot,z)$, $g(\cdot,z) \in BUC^{\eta}(\mathbb{R}) \cap C^{0,1}(\mathbb{R})$ uniformly for z on bounded subsets of $\mathbb{R}$. Further, $f \in L^\infty(\mathbb{R}^2)$, $x^2\|f(x,\cdot)\|_{L^\infty(\mathbb{R})} \in L^1(\mathbb{R})$ and for any $R > 0$, there is a $C(R) > 0$ such that

$$\int_{\mathbb{R}} (1+|y|)|f(y+z_1,z_1) - f(y+z_2,z_2)|dy \leq C(R)|z_1 - z_2|, \quad \forall |z_1|,\ |z_2| \leq R.$$

Our results are stated as follows:

Theorem 1. *Let the initial data u_0 belong to $BUC(\mathbb{R})$ and satisfy $0 \leq u_0 \leq 1$ on $\mathbb{R}$, $x^2 u_0 \in L^1(\mathbb{R})$ and $a < \int_{\mathbb{R}} w(x)u_0(x)dx < b$. Then there exists a unique solution $(u_\varepsilon, z_\varepsilon)$ to (P) such that $u_\varepsilon \in BUC(\mathbb{R}\times[0,T]) \cap BUC^{2+\eta,\frac{\eta}{2}}(\mathbb{R}\times[\delta,T])$ for all $\delta > 0$, $0 \leq u_\varepsilon \leq 1$, u_ε is differentiable in a.e. t uniformly for x, $wu_\varepsilon \in L^\infty(0,T;L^1(\mathbb{R}))$, $z_\varepsilon \in C^{0,1}[0,T]$ and $(u_\varepsilon, z_\varepsilon)$ satisfies the first equation in (P) for a.e. t, $\forall x$.*

Theorem 2. *In addition to the above hypotheses, suppose that* $u_0 \in W^{1,\infty}(\mathbb{R})$. *Then there exists a unique solution* (u,z) *of* (H) *such that* $u \in C([0,T];BUC(\mathbb{R})) \cap C^{0,1}(\mathbb{R} \times [0,T])$, $0 \le u \le 1$, $wu \in L^\infty(0,T;L^1(\mathbb{R}))$, $z \in C^{0,1}[0,T]$, *and* (u,z) *satisfies the first equation in* (H) *for a.e.* (x,t). *Moreover,* $u_\varepsilon \to u$ *in* $C([0,T];BUC(\mathbb{R}))$, $z_\varepsilon \to z$ *in* $C[0,T]$ *as* $\varepsilon \searrow 0$.

Theorem 3. *Assume the same hypotheses as above. In addition, suppose that*

$$\exists N > 0: \quad u_0(x) = f(x,z) = 0 \quad \text{for } |x| \ge N, \ z \in \mathbb{R}.$$

Then the solution u *of* (H) *obtained by Theorem 2 has a compact support.*

A key lemma to prove the above theorems is the following a priori estimate, whose proof is very delicate in our situation compared to the one in [3]:

Lemma 4. (a priori estimate) *There exists a* $K > 0$, *independent of* ε, *such that any solution* $(u_\varepsilon, z_\varepsilon)$ *of* (P) *as described in Theorem 1 satisfies*

$$\|z_\varepsilon\|_{C[0,T]} \le K, \quad a < L^{-1}(K) \le \int_{\mathbb{R}} w(x)u_\varepsilon(x,t)dx \le L^{-1}(-K) < b, \quad \forall t \in [0,T].$$

Remark. In Theorem 3, the support of u is contained in a strip of moving domain as specified by

$$\text{supp } u(\cdot,t) \subset [-N-K+z(t), N+K+z(t)]$$

for every $t \in [0,T]$.

3. Outline of Proofs

For the precise proofs, see [9, 10].

Proof of Theorem 1. By changing variable $x \mapsto x + z(t)$, (P) is reduced to the following problem:

$$(P') \qquad \begin{cases} v_t - \varepsilon v_{xx} = \varphi_z^*(x,t,v), \\ z(t) = L\Big(\int_{\mathbb{R}} w_z^*(x,t)v(x,t)dx\Big), \\ v(x,0) = u_0(x + z(0)), \end{cases}$$

where $\varphi_z^*(x,t,v) := \varphi(x + z(t), t, z(t), v)$ and $w_z^*(x,t) := w(x + z(t))$.

I. Solve (P') and then put $u(x,t) = v(x - z(t), t)$ to solve (P).

II. In order to solve (P'), given $z \in C[0,T]$, consider the semilinear problem

$$(P_z) \qquad \begin{cases} \partial_t v_z - \varepsilon(v_z)_{xx} = \varphi_z^*(x,t,v_z) \\ v_z(x,0) = u_0(x + z(0)). \end{cases}$$

After solving (P_z), we seek $z \in C[0,T]$ satisfying

$$(*) \qquad z(t) = L\Big(\int_{\mathbb{R}} w_z^*(x,t)v_z(x,t)dx\Big).$$

III. Finally, we find that $z \in C^{0,1}[0,T]$.

To solve (P_z), use the theory of abstract semilinear evolution equations. Let $X_0 = BUC(\mathbb{R})$ and $X_1 = \{u \in X_0 : u_{xx} \in X_0\}$ and define

$$A_\varepsilon u = \varepsilon u_{xx} \quad \text{for } u \in D(A_\varepsilon) = X_1.$$

Then A_ε is the infinitesimal generator of an analytic semigroup $\{T_\varepsilon(t)\}$ on X_0, where $T_\varepsilon(t)$ is given by

$$(T_\varepsilon(t)u)(x) = \int_{\mathbb{R}} K_\varepsilon(x-y,t)u(y)dy, \quad x \in \mathbb{R},\ t > 0 \quad \text{for } u \in X_0$$

with the heat kernel $K_\varepsilon(x,t) = (1/\sqrt{4\pi\varepsilon t})\exp(-x^2/4\varepsilon t)$. Let

$$F_z(t,u)(x) = \varphi_z^*(x,t,u(x)) \quad \text{for } t \in [0,T],\ u \in X_0.$$

Then $F_z : [0,T] \times X_0 \to X_0$ is well-defined and satisfies the following properties:

(i) For $z \in C[0,T]$, there exists an increasing function $\iota_z : [0,\infty) \to [0,\infty)$ such that for any $\rho > 0$,

$$|F_z(t,u) - F_z(t,v)|_{X_0} \le \iota_z(\rho)|u-v|_{X_0}, \quad \forall t \in [0,T],\ |u|_{X_0},\ |v|_{X_0} \le \rho.$$

(ii) If further $z \in C^{0,1}[r,T]$ for $r \ge 0$, then there is an increasing function $\iota_{r,z} : [0,\infty) \to [0,\infty)$ such that for any $\rho > 0$,

$$|F_z(t,u) - F_z(s,v)|_{X_0} \le \iota_{r,z}(\rho)(|t-s|^{\frac{\eta}{2}} + |u-v|_{X_0}), \quad \forall t,\ s \in [r,T],\ |u|_{X_0},\ |v|_{X_0} \le \rho.$$

Then (P_z) is reduced to the abstract semilinear problem in X_0; more generally, we consider the following:

$$(AP_z; r, \omega) \qquad \begin{cases} \dfrac{dv_z}{dt} = A_\varepsilon v_z + F_z(t, v_z) \\ v_z(r) = \omega \end{cases}$$

where $r \ge 0$ and $\omega \in X_0$ are given. The following Proposition plays a crucial role.

Proposition 5. *Let $r \ge 0$ and $0 \le \omega \le 1$.*

(1) If $z \in C[0,T]$, then $(AP_z; r, \omega)$ has a unique mild solution $v_z \in C([r,T]; X_0)$ satisfying $0 \le v_z \le 1$ on $\mathbb{R} \times [r,T]$ and

$$\begin{aligned} v_z(x,t) &= \int_{\mathbb{R}} K_\varepsilon(x-y,t-r)\omega(y)dy \\ &+ \int_r^t \int_{\mathbb{R}} K_\varepsilon(x-y,t-\tau)\varphi_z^*(y,\tau,v_z(y,\tau))dyd\tau, \quad (x,t) \in \mathbb{R} \times (r,T]. \end{aligned}$$

(2) If $z \in C[0,T] \cap C^{0,1}[r,T]$, then $(AP_z; r, \omega)$ has a unique classical solution $v_z \in C([r,T]; X_0) \cap C^1((r,T]; X_0)$ satisfying $(AP_z; r, \omega)$ for each $t \in (r,T]$.

(3) Moreover, suppose that $z_n \in C[0,T] \cap C^{0,1}[r,T]$, $z_n \to z$ in $C[r,T]$, and that $\omega_n \to \omega$ in X_0 and $0 \le \omega_n(x) \le 1$. Let v_n be a classical solution of $(AP_{z_n}; r, \omega_n)$, which exists by (2). Then $v_n \to v_z$ in $C([r,T]; X_0)$ as $n \to \infty$, where v_z is the mild solution of $(AP_z; r, \omega)$.

Remark. It is known ([6, Theorem 25.2, Remark 25.3(a)]) that each classical solution of $(AP_z; 0, \omega)$ is a regular solution of (P_z), i.e., $v_z \in BUC(\mathbb{R} \times [0,T]) \cap BUC^{2+\eta, 1+\frac{\eta}{2}}(\mathbb{R} \times [\delta, T])$ for all $\delta > 0$, satisfying (P_z). Notice that even if v_z is regular and $z \in C^{0,1}$, the solution $u(x,t) := v_z(x - z(t), t)$ is not enough regular in t and we have $u \in BUC(\mathbb{R} \times [0,T]) \cap BUC^{2+\eta, \frac{\eta}{2}}(\mathbb{R} \times [\delta, T])$ for all $\delta > 0$.

Now let us find $z \in C[0,T]$ satisfying $(*)$. We note that it can be shown that such z is unique if it exists (after a little long computation using Gronwall's inequality twice.) Hence the solution of (P) is uniquely determined. Let $r \in [0,T)$ be fixed arbitrarily. The equation $(*)$ is rewritten as

$$L^{-1}(z(t)) = \int_{\mathbb{R}} w(x + z(t)) \int_{\mathbb{R}} K_\varepsilon(x-y, t-r) v_z(y,r)\,dy\,dx + \int_r^t \Gamma_z(t,\tau)\,d\tau,$$

where v_z is a mild solution of $(AP_z; 0, u_0(\cdot + z(0)))$ defined by Proposition 5 and

$$\Gamma_z(t,\tau) := \begin{cases} \int_{\mathbb{R}} w_z^*(x,t) \int_{\mathbb{R}} K_\varepsilon(x-y, t-\tau) \varphi_z^*(y, \tau, v_z(y,\tau))\,dy\,dx & \text{if } 0 < \tau < t < T; \\ 0 & \text{otherwise.} \end{cases}$$

Since L^{-1} is only locally Lipschitz, we need to truncate it. Define

$$\lambda^K(\xi) := \begin{cases} L^{-1}(-2K) - \xi - 2K & \text{if } \xi < -2K\ ; \\ L^{-1}(\xi) & \text{if } |\xi| \le 2K\ ; \\ L^{-1}(2K) - \xi + 2K & \text{if } \xi > 2K, \end{cases}$$

$$\lambda_r^K(\xi, t) := \lambda^K(\xi) - \int_{\mathbb{R}} w(x+\xi) \int_{\mathbb{R}} K_\varepsilon(x-y, t-r) v_z(y,r)\,dy\,dx$$

for $(\xi, t) \in \mathbb{R} \times [r,T]$, where K is the a priori bound appeared in Lemma 4. Then we have

$$\lambda_r^K(z(t), t) = \int_r^t \Gamma_z(t,\tau)\,d\tau, \quad t \in [r,T].$$

It is shown that $\lambda_r^K(\xi, t)$ is continuous and strictly decreasing in ξ for fixed t; and denoting by $L_{r,t}^K$ the inverse function of $\xi \mapsto \lambda_r^K(\xi, t)$, $L_{r,t}^K$ becomes globally Lipschitz continuous. Now, assuming a continuous function z satisfying (*) to be known in $[0, r]$, we introduce a complete metric space

$$X_r := \{\zeta \in C[0, r+d] : \zeta = z \text{ in } [0, r],\ \|\zeta\|_{C[0,r+d]} \leq 2K\}.$$

Then we define an operator S_r^K on X_r by

$$[S_r^K(\zeta)](t) := \begin{cases} z(t) & \text{for } t \in [0, r]; \\ L_{r,t}^K\left(\int_r^t \Gamma_\zeta(t, \tau) d\tau\right) & \text{for } t \in (r, r+d], \end{cases}$$

for $\zeta \in X_r$ and seek a fixed point of S_r^K. It can be shown that for sufficiently small $d > 0$ not depending on r, $S_r^K : X_r \to X_r$ is well-defined and a contraction mapping in X_r. Hence S_r^K has a unique fixed point $\tilde{z}$ in X_r, which evidently satisfies (*) on $[0, r+d]$. Further, we can show that $\tilde{z}$ is Lipschitz continuous on $[r, r+d]$. Since r is arbitrary, we can construct step by step a Lipschitz continuous function z_ε on $[0, T]$ satisfying (*). This proves Theorem 1.

Proof of Theorem 2. Noting that the Lipschitz constant of z_ε is independent of ε, we have the estimate $\|z_\varepsilon\|_{W^{1,\infty}(0,T)} \leq C$. Then by the Ascoli-Arzela theorem, there exists a $z \in W^{1,\infty}(0, T) \subset C[0, T]$ and a subsequence $\{\varepsilon_k\}$ of $\{\varepsilon\}$ such that $z_{\varepsilon_k} \to z$ in $C[0, T]$ as $\varepsilon_k \searrow 0$. For this z, consider the ordinary differential equation

$$\begin{cases} \partial_t v_z = \varphi_z^*(x, t, v_z) \\ v_z(x, 0) = u_0(x + z(0)). \end{cases}$$

The solution exists as the following integral equation

$$v_z(t) = u_0(\cdot + z(0)) + \int_0^t F_z(s, v_z(s)) ds.$$

By the Trotter approximation theorem, it is shown that $T_\varepsilon(t)u \to u$ in X_0 uniformly for $t \in [0, T]$ for any $u \in X_0$. Hence recalling that

$$v_\varepsilon(t) = T_\varepsilon(t)u_0(\cdot + z_\varepsilon(0)) + \int_0^t T_\varepsilon(t-s) F_{z_\varepsilon}(s, v_\varepsilon(s)) ds,$$

it is shown that $v_{\varepsilon_k} \to v_z$ in $C([0,T];X_0)$. Further, from $(*)$ with $z = z_\varepsilon$, we have

$$z(t) = L\Big(\int_{\mathbb{R}} w_z^*(x,t)v_z(x,t)dx\Big).$$

Put $u(x,t) = v_z(x - z(t), t)$. Then it is easily seen that u is a weak solution of (H) in the sense of distribution. If we assume $u_0 \in W^{1,\infty}$, then the u becomes a strong solution, i.e., $u \in C([0,T];X_0) \cap C^{0,1}(\mathbb{R} \times [0,T])$ and satisfies (H) for *a.e.* (x,t). Uniqueness is shown similarly to the case (P); and consequently, we have $u_\varepsilon \to u$ in $C([0,T];X_0)$ and $z_\varepsilon \to z$ in $C[0,T]$.

Proof of Theorem 3. It is easy to see that supp $v(\cdot,t) \subset [-N-K, N+K]$; and hence

$$\text{supp } u(\cdot,t) \subset [-N-K+z(t), N+K+z(t)] \subset [-N-2K, N+2K]$$

for every $t \in [0,T]$.

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