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Stability of global strong solutions of the Navier-Stokes equations

川中子 正
(Tadashi KAWANAGO)
Osaka University

0. Introduction

We consider the following Navier-Stokes system.

$$\text{(NS)} \begin{cases} u_t - \Delta u + (u \cdot \nabla) u + \nabla \pi = 0 \quad \text{in} \quad \mathbf{R}^N \times \mathbf{R}^+, \\ \nabla \cdot u = 0 \quad \text{in} \quad \mathbf{R}^N \times \mathbf{R}^+, \\ u(x,0) = u_0(x) \quad \text{in} \quad \mathbf{R}^N. \end{cases}$$

Let P be the Helmholtz projection. We denote by $\|\cdot\|_p$ the norm of $L^p(\mathbf{R}^N)$. Kato [K] showed that for any $u_0 \in PL^N$ the problem (NS) has a unique local (strong) solution $u(t;u_0) \in C([0,T);PL^N) \cap L^{N+2}((0,T);PL^{N+2})$ (, where $T = T(\|u_0\|_N) > 0$) and that $T = \infty$ and $u(t;u_0) \in C_0([0,\infty);PL^N) := \{u \in C([0,\infty);PL^N); \lim_{t\to\infty} \|u(t)\|_N = 0\}$ if $\|u_0\|_N$ is sufficiently small.

We study the stability of global solutions of (NS) belonging to $C_0([0,\infty);PL^N)$. This class of solutions are very important since all strong global solution belongs to $C_0([0,\infty);PL^N)$ provided $2 \le N \le 4$ and $u_0 \in PL^2 \cap PL^N$ (see Section 3).

1. Navier-Stokes system

First we will characterize the global solutions belonging to $C_0([0,\infty);PL^N)$.

Proposition 1.1. *Let u be a global solution of (NS) with the initial value $u_0 \in PL^N$. Then we have the following.*

(i) If $u \in C_0([0,\infty);PL^N)$ then we have $u \in L^q(\mathbf{R}^+;PL^r)$ with $1/q = 1/2 - N/2r$, where $q > N$ and $r > N$.

(ii) Let r be a constant such that $N < r \le 2N$. If $u \in L^q(\mathbf{R}^+;PL^r)$ with $1/q = 1/2 - N/2r$ then we have $u \in C_0([0,\infty);PL^N)$.

Proof. We can easily derive (i) from [K, Theorems 1 and 2] by Kato. We omit the proof of (ii) since it was implicitly given in the proof of [K, Theorem 2']. ∎

Remark 1.1. When $N = 3$, Ponce et al [PRST] obtained a similar result under an assumption: $u_0 \in PL^2 \cap H^1$.

Theorem 1.1. *Let $u(t\,; u_0) \in C_0([0,\infty)\,; PL^N)$ be a global solution of (NS). Then there exists a constant $\delta \in \mathbf{R}^+$ depending only on N and u_0 such that if*

$$v_0 \in PL^N \quad \text{and} \quad \|v_0 - u_0\|_N \le \delta \tag{1.1}$$

then (NS) has a unique global solution $u(t\,; v_0)$ satisfying

$$\|u(t\,; v_0) - u(t\,; u_0)\|_N \le \|v_0 - u_0\|_N \exp\Big(C_1 \int_0^t \|u(s\,; u_0)\|_{N+2}^{N+2} ds\Big) \quad \text{for} \quad t \ge 0, \tag{1.2}$$

where the constant $C_1 \in \mathbf{R}^+$ depends only on N.

We have an immediate corollary of Theorem 1.1:

Corollary 1.1. *For (NS) we set*

$$A = \{\, u_0 \in PL^N\,; u(t\,; u_0) \in C_0([0,\infty)\,; PL^N)\,\}.$$

Then A is open in PL^N.

Remark 1.2. When $N = 3$, the set A is unbounded in PL^3 (see [UI]).

Our Theorem 1.1 extends [Wi, Theorem 1] and [PRST, Theorem 1]. Wiegner [Wi] obtained a $L^2 \cap L^r$-stability result with $r > N$. Ponce et al [PRST] obtained a H^1-stability result for $N = 3$.

Proof of Theorem 1.1. Our proof is close to the argument in [N] and [Ka1] for the porous media equations. We denote $\partial_j := \partial/\partial x_j$. We have a (unique) local strong solution $u(t\,; v_0) \in C([0,T)\,; PL^N)$. We will derive the estimate (1.2). Set $w(t) := u(t\,; v_0) - u(t\,; u_0)$ and $u := u(t\,; u_0)$ for simplicity. Then w satisfies

$$\begin{cases} w_t - \Delta w + (w\cdot\nabla)w + (u\cdot\nabla)w + (w\cdot\nabla)u + \nabla\pi = 0 & \text{in} \quad \mathbf{R}^N \times \mathbf{R}^+, \\ \nabla\cdot w = 0 \quad \text{in} \quad \mathbf{R}^N \times \mathbf{R}^+. \end{cases} \tag{1.3}$$

By integration by parts,

$$\frac{1}{p}\frac{d}{dt}\int_{\mathbf{R}^N}|w(t)|^p = -A_p(w)^2-(p-2)B_p(w)^2-I_1-I_2-I_3-I_4, \tag{1.4}$$

where we set

$$\begin{aligned}
A_p(w) &= (\int |\nabla w|^2|w|^{p-2})^{1/2},\\
B_p(w) &= (\int |\nabla |w|\,|^2\,|w|^{p-2}\,)^{1/2},\\
I_1 &= \int |w|^{p-2}w\cdot(w\cdot\nabla)w,\\
I_2 &= \int |w|^{p-2}w\cdot(w\cdot\nabla)u,\\
I_3 &= \int |w|^{p-2}w\cdot(u\cdot\nabla)w,\\
I_4 &= \int |w|^{p-2}w\cdot\nabla\pi.
\end{aligned}$$

With the aid of Gagliardo - Nirenberg inequality,

$$\|w\|_{p+2}\le C_p\|w\|_N^{2/(p+2)}A_p(w)^{2/(p+2)}. \tag{1.5}$$

In what follows, we always set $p=N$. We will estimate I_j with $j=1,2,3,4$. By (1.5),

$$\begin{aligned}
|I_1| &\le \int |w|^N|\nabla w| \le C_\varepsilon\int |w|^{N+2}+\varepsilon A_p(w)^2\\
&\le (C\|w\|_N^2+\varepsilon)A_N(w)^2.
\end{aligned} \tag{1.6}$$

It follows from the integration by parts and (1.5) that

$$\begin{aligned}
|I_2|+|I_3| &\le N\int |u||w|^{N-1}|\nabla w|\\
&\le \varepsilon\int |w|^{N-2}|\nabla w|^2+C_\varepsilon\int |u|^2|w|^N\\
&\le \varepsilon A_N(w)^2+C\|u\|_{N+2}^2\|w\|_{N+2}^N\\
&\le \varepsilon A_N(w)^2+C\|u\|_{N+2}^2\|w\|_N^{2N/(N+2)}A_N(w)^{2N/(N+2)}\\
&\le 2\varepsilon A_N(w)^2+C\|u\|_{N+2}^{N+2}\|w\|_N^N.
\end{aligned} \tag{1.7}$$

By similar argument in [VS] we will estimate I_4. In view of (1.3) we have

$$-\Delta\pi=\sum_{i,j}\partial_j w^i\cdot\partial_i(2u^j+w^j)=\sum_{i,j}\partial_i\partial_j[w^i(2u^j+w^j)]. \tag{1.8}$$

By the Calderon - Zygmund inequality and Hölder's inequality,

$$(1.9) \quad \|\pi\|_{(N+2)/2}^2 \le C \sum_{i,j} \|w^i(2u^j + w^j)\|_{(N+2)/2}^2 \le C\|w\|_{N+2}^2(\|u\|_{N+2}^2 + \|w\|_{N+2}^2).$$

It follows from the integration by parts, (1.5), (1.7) and (1.9) that

$$(1.10) \quad \begin{aligned} |I_4| &\le (N-2)\int |\pi||w|^{N-2}|\nabla w| \\ &\le C_\varepsilon \int |\pi|^2|w|^{N-2} + \varepsilon \int |w|^{N-2}|\nabla w|^2 \\ &\le C\|\pi\|_{(N+2)/2}^2\|w\|_{N+2}^{N-2} + \varepsilon A_N(w)^2 \\ &\le C\|w\|_{N+2}^N(\|u\|_{N+2}^2 + \|w\|_{N+2}^2) + \varepsilon A_N(w)^2 \\ &\le C_\varepsilon\|u\|_{N+2}^{N+2}\|w\|_N^N + (2\varepsilon + C\|w\|_N^2)A_N(w)^2. \end{aligned}$$

Therefore, we have

$$(1.11) \quad \frac{1}{N}\frac{d}{dt}\|w(t)\|_N^N \le -(\frac{1}{2} - C_0\|w\|_N^2)A_N(w)^2 + C_1\|u\|_{N+2}^{N+2}\|w\|_N^N.$$

Set $\delta := (2C_0)^{-1/2}\exp(-C_1\int_0^\infty \|u(s\,;\,u_0)\|_{N+2}^{N+2})$. Let $\|v_0 - u_0\|_N \le \delta$. Then we obtain (1.2) from (1.11). ∎

Remark 1.2. Although we can obtain some similar results for the Dirichlet problem for the Navier-Stokes system, we omit here. See [K3] for the details.

2. Scalar semilinear heat equation

We can obtain some results of a semilinear heat equation in the similar argument in Section 1. We consider the following problem (H).

$$(\mathrm{H}) \begin{cases} u_t = \Delta u + |u|^{p-1}u & \text{in} \quad \mathbf{R}^N \times \mathbf{R}^+, \\ u(x,0) = u_0(x) & \text{in} \quad \mathbf{R}^N, \end{cases}$$

where $p \in (1 + 2/N, \infty)$. We set $p_0 := N(p-1)/2$ (> 1). Giga [G] showed that for any $u_0 \in L^{p_0}$ the problem (H) has a unique local solution $u(t\,;\,u_0) \in C([0,T)\,;\,L^{p_0}) \cap L^{p_0+p-1}((0,T)\,;\,L^{p_0+p-1})$ (, where $T = T(\|u_0\|_{p_0}) > 0$) and that $T = \infty$ and $u(t\,;u_0) \in C_0([0,\infty)\,;\,L^{p_0})$ if $\|u_0\|_{p_0}$ is sufficiently small.

We state our results without proofs. See [Ka3] for the proofs.

Proposition 2.1. *Let u be a global solution of (H) with the initial value $u_0 \in L^{p_0}$. Then we have the following.*

(i) If $u \in C_0([0,\infty)\,;\,PL^N)$ then we have $u \in L^q(\mathbf{R}^+\,;\,PL^r)$ with $1/q = (1/p_0 - 1/r)N/2$, where $q > \max(p_0, p)$ and $r > p_0$.

(ii) Let r be a constant such that $r > p_0$ and $p \le r \le p_0 p$. If $u \in L^q(\mathbf{R}^+\,;\,PL^r)$ with $1/q = (1/p_0 - 1/r)N/2$ then we have $u \in C_0([0,\infty)\,;\,L^{p_0})$.

Theorem 2.1. *Let $u(t\,;u_0) \in C_0([0,\infty)\,;\,L^{p_0})$ be a global solution of (H). Then there exists a constant $\delta \in \mathbf{R}^+$ depending only on N, p and u_0 such that if*

$$v_0 \in L^{p_0} \quad \text{and} \quad \|v_0 - u_0\|_{p_0} \le \delta$$

then (H) has a unique global solution $u(t\,;\,v_0)$ satisfying

$$\|u(t\,;\,v_0) - u(t\,;u_0)\|_{p_0} \le \|v_0 - u_0\|_{p_0} \exp\Big(C_1 \int_0^t \|u(s\,;\,u_0)\|_{p_0+p-1}^{p_0+p-1} ds\Big) \quad \text{for} \quad t \ge 0,$$

where the constant $C_1 \in \mathbf{R}^+$ depends only on N.

We have an immediate corollary of Theorem 2.1:

Corollary 2.1. *For (H) we set*

$$A = \{u_0 \in L^{p_0}\,;\,u(t\,;\,u_0) \in C_0([0,\infty)\,;\,L^{p_0})\}.$$

Then A is open in L^{p_0}.

By [Ka2] and [Wa] we see that the set A is unbounded in L^{p_0}.

Remark 2.1. Our Theorem 2.1 extends [Ka2, Proposition 6] by the author, where we assumed $u_0 \in L^1 \cap L^\infty$ and $u_0 \ge 0$ in $\mathbf{R}^N$.

3. Structure of space of solutions for (NS) and (H)

We mention the topological structure for the space of solutions of (NS) and (H). We set

$$B := \{u_0 \in PL^N\,;\,\|u(t\,;u_0)\|_N \ \text{ blows up in finite time }\} \quad \text{for (NS)}$$

and

$$B := \{u_0 \in L^{p_0}\, ;\ \|u(t\,;u_0)\|_{p_0} \quad \text{blows up in finite time } \} \quad \text{for (H)}.$$

For (NS) we have $A = PL^2$ for $N = 2$ (see [KM], [M] and [Wi]). However, we can easily derive this well-known result from our Proposition 1.1 and the energy equality. Indeed, by the energy equality:

$$\text{(3.1)} \qquad \|u(t)\|_2^2 + 2\int_0^t \|\nabla u(s)\|_2^2 ds = \|u_0\|_2^2,$$

$u(t\,;u_0)$ is global for any $u_0 \in PL^2$. By Gagliardo-Nirenberg inequality

$$\text{(3.2)} \qquad \|u(t)\|_4 \le C\|u(t)\|_2^{1/2}\|\nabla u(t)\|_2^{1/2}.$$

In view of (3.1) and (3.2) we have $u(t\,;u_0) \in L^4(\mathbf{R}^+\,;\,PL^4)$. Therefore, we immediately obtain $u(t\,;u_0) \in C_0([0,\infty)\,;L^2)$ from our Proposition 1.1 (ii). For (NS) we have $(A \cup B) \cap PL^2 = PL^2 \cap PL^N$ for $N = 3$ and $N = 4$, which is due to the energy equality. Therefore, $B \cap PL^2$ is closed in $PL^2 \cap PL^N$. Since $PL^2 \cap PL^N$ is dence in PL^N, we see that for $N = 3$ and $N = 4$ the set B is empty or B is not open in PL^N.

It seems to be interesting to compare (NS) with (H). If u is a solution of (H) with $p = 3$ then $\lambda u(\lambda x, \lambda^2 t)$ is also a solution of (H) for $\lambda > 0$. We remark that the solution of (NS) has just the same property with respect to the same self-similar transformation. For (H) with $p = 3$ we have the following (see [Ka2]): the set $B \cap L^2$ is not empty and is open in $L^2 \cap L^N$. If we set $S := \{u_0 \in L^2 \cap L^N - (A \cup B)\,;\, u_0(x) \ge 0 \quad \text{in} \quad \mathbf{R}^N\}$ then S is not empty and $S \subset \partial A$, where ∂A is the boundary of A in L^N.

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