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STABILITY OF SOLITARY WAVES FOR THE ZAKHAROV EQUATIONS

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1. Introduction and Result

In the present paper we consider the stability of solitary waves for the Zakharov equations:

$$i\frac{\partial}{\partial t}u+\frac{\partial^2}{\partial x^2}u=nu, \qquad t>0, \quad x\in\mathbb{R}, \tag{1.1}$$

$$\frac{\partial}{\partial t}n+\frac{\partial}{\partial x}v=0, \qquad t>0, \quad x\in\mathbb{R}, \tag{1.2}$$

$$\frac{\partial}{\partial t}v+\frac{\partial}{\partial x}n=-\frac{\partial}{\partial x}|u|^2, \quad t>0, \quad x\in\mathbb{R}, \tag{1.3}$$

where u, n and v are functions on the time-space $\mathbb{R}\times\mathbb{R}$ with values in $\mathbb{C}$, $\mathbb{R}$ and $\mathbb{R}$, respectively. From (1.2) and (1.3), we have

$$\frac{\partial^2}{\partial t^2}n-\frac{\partial^2}{\partial x^2}n=\frac{\partial^2}{\partial x^2}|u|^2. \tag{1.4}$$

The system of equations (1.1) and (1.4) was first obtained by Zakharov [20] as a model which describes the propagation of Langmuir turbulence in a plasma. In this system, u denotes the envelope of the electric field and n is the deviation of the ion density from its equilibrium. On the other hand, (1.1)–(1.3) was given by Gibbons, Thornhill, Wardrop and ter Haar [4] from a Lagrangian formalism.

It is well known that (1.1)–(1.3) has a two parameter family of solitary wave solutions:

$$u_{\omega,c}(t,x) = \sqrt{2\omega(1-c^2)}\,\mathrm{sech}\,\sqrt{\omega}(x-ct)\cdot\exp i\left(\frac{c}{2}x - \frac{c^2}{4}t + \omega t\right), \quad (1.5)$$

$$n_{\omega,c}(t,x) = -2\omega\,\mathrm{sech}^2\,\sqrt{\omega}(x-ct), \quad (1.6)$$

$$v_{\omega,c}(t,x) = -2c\omega\,\mathrm{sech}^2\,\sqrt{\omega}(x-ct), \quad (1.7)$$

where $\omega > 0$ and $-1 < c < 1$. Our purpose in this paper is to show the stability of the solitary wave solution given by (1.5)–(1.7) of (1.1)–(1.3) for any $\omega > 0$ and $-1 < c < 1$.

There are a large amount of papers concerning the stability and instability of solitary waves for the nonlinear Schrödinger equations (see, e.g., [2, 3, 7, 14, 15, 16, 18, 19]). However, to our knowledge, there are only a few results concerning the stability of solitary waves for coupled systems of Schrödinger equations and other wave equations, except the abstract theory by Grillakis, Shatah and Strauss [8] and our recent results for the coupled nonlinear Schrödinger equations [10] and for the coupled Klein–Gordon–Schrödinger equations [11].

We now state our main result.

Theorem 1.1. *For any $\omega > 0$ and $-1 < c < 1$, the solitary wave solution $(u_{\omega,c}(t), n_{\omega,c}(t), v_{\omega,c}(t))$ of (1.1)–(1.3) is stable in the following sense: for any $\varepsilon > 0$ there exists a $\delta > 0$ such that if $(u_0, n_0, v_0) \in X$ verifies*

$$\|(u_0, n_0, v_0) - (u_{\omega,c}(0), n_{\omega,c}(0), v_{\omega,c}(0))\|_X < \delta,$$

then the solution $(u(t), n(t), v(t))$ of (1.1)–(1.3) with $(u(0), n(0), v(0)) = (u_0, n_0, v_0)$ satisfies

$$\inf_{\theta, y \in \mathbb{R}} \|(u(t), n(t), v(t)) - (e^{i\theta} u_{\omega,c}(t, \cdot + y), n_{\omega,c}(t, \cdot + y), v_{\omega,c}(t, \cdot + y))\|_X < \varepsilon$$

for any $t \geq 0$, where $X = H^1(\mathbb{R}) \times L^2(\mathbb{R}) \times L^2(\mathbb{R})$.

Remark 1.2. For any $(u_0, n_0, v_0) \in X$, there exists a weak solution $(u(\cdot), n(\cdot), v(\cdot)) \in L^\infty([0, \infty); X)$ of (1.1)–(1.3) with $(u(0), n(0), v(0)) = (u_0, n_0, v_0)$ (see C. Sulem and P.L. Sulem [17]). We do not necessarily have the uniqueness and the energy identity. However, by using the method in Ginibre and Velo [5], we can find a weak solution satisfying

$$H(u(t), n(t), v(t)) \leq H(u_0, n_0, v_0), \quad t \geq 0, \tag{1.8}$$

$$N(u(t)) = N(u_0), \quad t \geq 0, \tag{1.9}$$

$$P(u(t), n(t), v(t)) = P(u_0, n_0, v_0), \quad t \geq 0, \tag{1.10}$$

where

$$H(u,n,v) = \int_{-\infty}^{\infty} \left(|\frac{\partial}{\partial x} u|^2 + n|u|^2 + \frac{1}{2} n^2 + \frac{1}{2} v^2 \right) dx,$$

$$N(u) = \int_{-\infty}^{\infty} |u|^2 dx,$$

$$P(u,n,v) = \int_{-\infty}^{\infty} \left(i\bar{u} \frac{\partial}{\partial x} u - nv \right) dx.$$

For the Cauchy problem of the Zakharov equations, see also [1], [12] and [13].

Remark 1.3. Recently, Glangetas and Merle [6] proved the strong instability (instability by blow-up) of standing waves of the Zakharov equations in two space dimensions.

In the next section, we give the proof of Theorem 1.1. We apply the variational method introduced by Cazenave and Lions [3] to the coupled system of the Schrödinger equation and the wave equations as well as in our previous papers [10] and [11]. In [3] they proved the stability of standing waves for some nonlinear Schrödinger equations. By a simple inequality in Lemma 2.3 below, we reduce our problem for the Zakharov equations to the case of the single nonlinear Schrödinger equation.

2. Proof of Theorem 1.1

In what follows, we fix the parameter $c \in (-1, 1)$. First, we briefly recall the proof by Cazenave and Lions [3] for the stability of standing wave solution $u(t,x) = e^{i\omega t}\varphi_{\omega,c}(x)$ of the nonlinear Schrödinger equation:

$$i\frac{\partial}{\partial t}u + \frac{\partial^2}{\partial x^2}u + \frac{1}{1-c^2}|u|^2 u = 0, \qquad t > 0, \quad x \in \mathbb{R}, \tag{2.1}$$

where $\varphi_{\omega,c}(x) = \sqrt{2\omega(1-c^2)}\operatorname{sech}\sqrt{\omega}x$. We consider the minimization problem:

$$I^1(\mu) = \inf\{E^1(u) : u \in H^1(\mathbb{R}),\ N(u) = \mu\}, \tag{2.2}$$

$$E^1(u) = \int_{-\infty}^{\infty}\left(|\frac{\partial}{\partial x}u|^2 - \frac{1}{2(1-c^2)}|u|^4\right)dx,$$

$$\Sigma^1(\mu) = \{u \in H^1(\mathbb{R}) : E^1(u) = I^1(\mu),\ N(u) = \mu\}.$$

We note that $E^1(u)$ and $N(u)$ are the conserved quantities of (2.1). The following two lemmas are crucial parts to prove the stability of the standing wave of (2.1). We use them in the proof of Theorem 1.1 later.

Lemma 2.1. *For any $\omega > 0$, we have*

$$\Sigma^1(\mu(\omega)) = \{e^{i\theta}\varphi_{\omega,c}(\cdot + y) : \theta, y \in \mathbb{R}\},$$

where $\varphi_{\omega,c}(x) = \sqrt{2\omega(1-c^2)}\operatorname{sech}\sqrt{\omega}x$ and

$\mu(\omega) = N(\varphi_{\omega,c}) = 4(1-c^2)\sqrt{\omega}$.

Lemma 2.2. *Let $\mu > 0$. If $\{u_j\} \subset H^1(\mathbb{R})$ satisfies $E^1(u_j) \to I^1(\mu)$ and $N(u_j) \to \mu$, then there exists $\{y_j\} \subset \mathbb{R}$ such that $\{u_j(\cdot + y_j)\}$ is relatively compact in $H^1(\mathbb{R})$.*

Lemma 2.2 is proved by using the concentration compactness method introduced by Lions [9]. For the proofs of Lemmas 2.1 and 2.2, see [3]. From the conservation laws of (2.1) and the compactness of any minimizing sequence of (2.2), Lemma 2.2, one can easily show the stability of the set of minimizers $\Sigma^1(\mu)$ for any $\mu > 0$. Moreover, the characterization of the set of minimizers, Lemma 2.1, concludes the stability of the standing wave of (2.1) (for details, see [3]).

Following Cazenave and Lions [3], we consider the following minimization problem:

$$I(\mu) = \inf\{E(u,n,v) : (u,n,v) \in X,\ N(u) = \mu\}, \tag{2.3}$$

$$E(u,n,v) = \int_{-\infty}^{\infty} \left(|\frac{\partial}{\partial x}u|^2 + n|u|^2 + \frac{1}{2}n^2 + \frac{1}{2}v^2 - cnv \right) dx,$$

$$\Sigma(\mu) = \{(u,n,v) \in X : E(u,n,v) = I(\mu),\ N(u) = \mu\},$$

where $X = H^1(\mathbb{R}) \times L^2(\mathbb{R}) \times L^2(\mathbb{R})$. We note that

$$E(e^{-icx/2}u,n,v) = H(u,n,v) + cP(u,n,v) + \frac{c^2}{4}N(u). \tag{2.4}$$

The following lemma plays an essential role in the proof of Theorem 1.1.

Lemma 2.3. *For any $(u, n, v) \in X$, we have $E^1(u) \leq E(u, n, v)$. Moreover, the equality holds if and only if $n = -(1/(1-c^2))|u|^2$ and $v = cn$.*

Proof. Since

$$0 \leq |\ |u|^2 + (1-c^2)n\ |^2 = |u|^4 + 2(1-c^2)n|u|^2 + (1-c^2)^2 n^2, \quad (2.5)$$

we have

$$\begin{aligned} E(u, n, v) &\geq \int_{-\infty}^{\infty} \left(|\frac{\partial}{\partial x} u|^2 - \frac{1}{2(1-c^2)}|u|^4 + \frac{c^2}{2}n^2 + \frac{1}{2}v^2 - cnv \right) dx \\ &= \int_{-\infty}^{\infty} \left(|\frac{\partial}{\partial x} u|^2 - \frac{1}{2(1-c^2)}|u|^4 + \frac{1}{2}(cn - v)^2 \right) dx \\ &\geq E^1(u). \end{aligned} \quad (2.6)$$

From (2.5) and (2.6), we see that the equality holds if and only if $n = -(1/(1-c^2))|u|^2$ and $v = cn$. $\square$

The following lemma follows immediately from Lemma 2.3.

Lemma 2.4. *For any $\mu > 0$, we have $I(\mu) = I^1(\mu)$ and*

$$\Sigma(\mu) = \left\{ (u, n, v) : u \in \Sigma^1(\mu), n = -\frac{1}{1-c^2}|u|^2, v = cn \right\}.$$

Proof. We set

$$\Sigma^0(\mu) = \left\{(u,n,v) : u \in \Sigma^1(\mu), n = -\frac{1}{1-c^2}|u|^2, v = cn\right\}.$$

For $u \in \Sigma^1(\mu)$, we have from Lemma 2.3

$$I(\mu) \leq E\left(u, -\frac{1}{1-c^2}|u|^2, -\frac{c}{1-c^2}|u|^2\right) = E^1(u) = I^1(\mu) \leq I(\mu).$$

Thus, we have $I(\mu) = I^1(\mu)$ and $\Sigma^0(\mu) \subset \Sigma(\mu)$.

Moreover, for $(u,n,v) \in \Sigma(\mu)$, we have

$$I(\mu) = I^1(\mu) \leq E^1(u) \leq E(u,n,v) = I(\mu),$$

which implies that $u \in \Sigma^1(\mu)$ and $E(u,n,v) = E^1(u)$. Thus, it follows from Lemma 2.3 that $\Sigma(\mu) \subset \Sigma^0(\mu)$. Hence, we have $\Sigma(\mu) = \Sigma^0(\mu)$. $\square$

We note that from (1.5)–(1.7) and Lemma 2.1, we have

$$e^{-icx/2}u_{\omega,c}(t) \in \Sigma^1(\mu(\omega)),$$

$$n_{\omega,c}(t) = -\frac{1}{1-c^2}|u_{\omega,c}(t)|^2, \quad v_{\omega,c}(t) = cn_{\omega,c}(t)$$

for any $t \in \mathbb{R}$. Therefore, from Lemma 2.4, in order to show Theorem 1.1, we have only to prove the following proposition.

Proposition 2.5. *For any $\mu > 0$, the set*

$$\mathcal{A} = \{(e^{icx/2}u, n, v) : (u, n, v) \in \Sigma(\mu)\}$$

is stable in the following sense: for any $\varepsilon > 0$ there exists a $\delta > 0$ such that if $(u_0, n_0, v_0) \in X$ verifies $\operatorname{dist}((u_0, n_0, v_0), \mathcal{A}) < \delta$, *then the solution $(u(t), n(t), v(t))$ of (1.1)–(1.3) with $(u(0), n(0), v(0)) = (u_0, n_0, v_0)$ satisfies* $\operatorname{dist}((u(t), n(t), v(t)), \mathcal{A}) < \varepsilon$ *for any $t \geq 0$, where*

$$\operatorname{dist}((u, n, v), \mathcal{A}) = \inf\{\|(u, n, v) - (u^0, n^0, v^0)\|_X : (u^0, n^0, v^0) \in \mathcal{A}\}.$$

In order to prove Proposition 2.5, we need one lemma concerning the compactness of any minimizing sequence of (2.3).

Lemma 2.6. *Let $\mu > 0$. If $\{(u_j, n_j, v_j)\} \subset X$ satisfies $E(u_j, n_j, v_j) \to I(\mu)$ and $N(u_j) \to \mu$, then there exists $\{y_j\} \subset \mathbb{R}$ such that $\{(u_j(\cdot + y_j), n_j(\cdot + y_j), v_j(\cdot + y_j))\}$ is relatively compact in X.*

Proof. From Lemma 2.3 and our assumption, we have $E^1(u_j) \to I(\mu) = I^1(\mu)$. Thus, from Lemma 2.2, there exists $\{y_j\} \subset \mathbb{R}$ such that $\{u_j(\cdot + y_j)\}$ is relatively compact in $H^1(\mathbb{R})$. Moreover, if we put $u_j^0 = u_j(\cdot + y_j)$, $n_j^0 = n_j(\cdot + y_j)$, $v_j^0 = v_j(\cdot + y_j)$, then $\{(u_j^0, n_j^0, v_j^0)\}$ is bounded in X. Therefore, for some subsequence (still denoted by the same letter), we have

$$(u_j^0, n_j^0, v_j^0) \rightharpoonup (u^0, n^0, v^0) \quad \text{weakly in } X,$$

$$u_j^0 \to u^0 \quad \text{in } H^1(\mathbb{R}).$$

Since $n^2+v^2-2cnv = (1-|c|)(n^2+v^2)+|c|\,(n-(c/|c|)v)^2$ and $|c|<1$, we obtain

$$I(\mu) \le E(u^0, n^0, v^0) \le \liminf_{j\to\infty} E(u_j^0, n_j^0, v_j^0) = I(\mu),$$

from which it follows that

$$(u_j^0, n_j^0, v_j^0) \to (u^0, n^0, v^0) \quad \text{in } X,$$

and $(u^0, n^0, v^0) \in \Sigma(\mu)$. $\square$

Proof of Proposition 2.5. In what follows, we often extract subsequences without explicitly mentioning this fact. We prove by contradiction. If $\mathcal{A}$ is not stable, then there exist a positive constant ε_0 and sequences $\{(u_{0j}, n_{0j}, v_{0j})\} \subset X$ and $\{t_j\} \subset \mathbb{R}$ such that

$$\operatorname{dist}((u_{0j}, n_{0j}, v_{0j}), \mathcal{A}) \to 0, \tag{2.7}$$

$$\operatorname{dist}((u_j(t_j), n_j(t_j), v_j(t_j)), \mathcal{A}) \ge \varepsilon_0, \tag{2.8}$$

where $(u_j(t), n_j(t), v_j(t))$ is a solution of (1.1)–(1.3) with $(u_j(0), n_j(0), v_j(0)) = (u_{0j}, n_{0j}, v_{0j})$. From the conservation laws (1.8)–(1.10), (2.4) and (2.7), we have

$$E(e^{-icx/2}u_j(t_j), n_j(t_j), v_j(t_j)) \le E(e^{-icx/2}u_{0j}, n_{0j}, v_{0j}) \to I(\mu), \quad (2.9)$$

$$N(e^{-icx/2}u_j(t_j)) = N(u_j(t_j)) = N(u_{0j}) = N(e^{-icx/2}u_{0j}) \to \mu. \quad (2.10)$$

From (2.9), (2.10) and the definition of $I(\mu)$, we have

$$E(e^{-icx/2}u_j(t_j), n_j(t_j), v_j(t_j)) \to I(\mu). \quad (2.11)$$

If we put $u_j^1(x) = e^{-icx/2}u_j(t_j, x)$, $n_j^1(x) = n_j(t_j, x)$, $v_j^1(x) = v_j(t_j, x)$, then from (2.10), (2.11) and Lemma 2.6, there exists $\{y_j\} \subset \mathbb{R}$ such that

$$(u_j^1(\cdot + y_j), n_j^1(\cdot + y_j), v_j^1(\cdot + y_j)) \to (u^1, n^1, v^1) \quad \text{in } X \quad (2.12)$$

for some $(u^1, n^1, v^1) \in \Sigma(\mu)$. Since we have

$$u_j^1(x + y_j) = e^{-icx/2}e^{-icy_j/2}u_j(t_j, x + y_j),$$

it follows from (2.12) that

$$\mathrm{dist}((u_j(t_j), n_j(t_j), v_j(t_j)), \mathcal{A}) \to 0,$$

which contradicts (2.8).

Hence, $\mathcal{A}$ is stable. This completes the proof. $\square$

3. 追記

この研究集会で講演した後しばらくして、北海道大学理学部数学教室の小澤 徹先生より Zakharov 方程式の孤立波解の安定性に関して類似の結果がすでに Y. Wu [21] により発表されていることを教えて頂きました。ここに記して、小澤先生に感謝いたします。方法としては、線形化作用素のスペクトル解析を行い、Grillakis, Shatah and Strauss [8] による抽象的理論を Zakharov 方程式に適応している。証明を比較すると今回上で紹介したような変分的方法の方が直接的であり、簡単であるように思われる。

REFERENCES

[1] H. Added and S. Added, *Existence globale de solutions fortes pour les équations de la turbulence de Langmuir en dimension 2*, C. R. Acad. Sci. Paris **299** (1984), 551–554.

[2] H. Berestycki and T. Cazenave, *Instabilité des états stationnaires dans les équations de Schrödinger et de Klein–Gordon non linéaires*, C. R. Acad. Sci. Paris **293** (1981), 489–492.

[3] T. Cazenave and P. L. Lions, *Orbital stability of standing waves for some non-linear Schrödinger equations*, Commun. Math. Phys. **85** (1982), 549–561.

[4] J. Gibbons, S. G. Thornhill, M. J. Wardrop and D. Ter Haar, *On the theory of Langmuir solitons*, J. Plasma Phys. **17** (1977), 153–170.

[5] J. Ginibre and G. Velo, *The global Cauchy problem for the non linear Schrödinger equation revisited*, Ann. Inst. Henri Poincaré, Anal. non linéaire

2 (1985), 309–327.

[6] L. Glangetas and F. Merle, *Existence of self-similar blow-up solutions for Zakharov equation in dimension two*, Commun. Math. Phys. **160** (1994), 173–215, 349–389.

[7] M. Grillakis, J. Shatah and W. A. Strauss, *Stability theory of solitary waves in the presence of symmetry I*, J. Funct. Anal. **74** (1987), 160–197.

[8] M. Grillakis, J. Shatah and W. A. Strauss, *Stability theory of solitary waves in the presence of symmetry II*, J. Funct. Anal. **94** (1990), 308–348.

[9] P. L. Lions, *The concentration-compactness principle in the calculus of variations. The locally compactness*, Ann. Inst. Henri Poincaré, Anal. non linéaire **1** (1984), 109–145, 223–283.

[10] M. Ohta, *Stability of solitary waves for coupled nonlinear Schrödinger equations*, Nonlinear Anal., T.M.A. (to appear).

[11] M. Ohta, *Stability of stationary states for the coupled Klein–Gordon–Schrödinger equations*, Nonlinear Anal., T.M.A. (to appear).

[12] T. Ozawa and Y. Tsutsumi, *Existence and smoothing effect of solutions for the Zakharov equations*, Publ. RIMS, Kyoto Univ. **28** (1992), 329–361.

[13] S. H. Schochet and M. I. Weinstein, *The nonlinear Schrödinger limit of the Zakharov equations governing Langmuir turbulence*, Commun. Math. Phys. **106** (1986), 569–580.

[14] J. Shatah and W. A. Strauss, *Instability of nonlinear bound states*, Commun. Math. Phys. **100** (1985), 173–190.

[15] A. Soffer and M. I. Weinstein, *Multichannel nonlinear scattering for nonintegrable equations*, Commun. Math. Phys. **133** (1990), 119–146.

[16] A. Soffer and M. I. Weinstein, *Multichannel nonlinear scattering for nonintegrable equations II. The case of anisotropic potentials and data*, J. Diff. Eqs. **98** (1992), 376–390.

[17] C. Sulem and P. L. Sulem, *Quelques résultats de régularité pour les équations de la turbulence de Langmuir*, C. R. Acad. Sci. Paris **289** (1979), 173–176.

[18] M. I. Weinstein, *Nonlinear Schrödinger equations and sharp interpolation estimates*, Commun. Math. Phys. **87** (1983), 567–576.

[19] M. I. Weinstein, *Lyapunov stability of ground states of nonlinear dispersive evolution equations*, Comm. Pure Appl. Math. **39** (1986), 51–68.

[20] V. E. Zakharov, *Collapse of Langmuir waves*, Sov. Phys. JETP **35** (1972), 908–914.

[21] Y. Wu, *Orbital stability of solitary waves of Zakharov system*, J. Math. Phys. **35** (1994), 2413–2422.