SINGULARITIES FOR VISCOSITY SOLUTIONS
OF HAMILTON-JACOBI EQUATIONS

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1. INTRODUCTION

In this note we study the generation and propagation of singularities (shock waves) of the solution of the Cauchy problem for Hamilton-Jacobi equations

(P)
$$\begin{cases} \dfrac{\partial y}{\partial t} + H(t, x_1, \ldots, x_n, \dfrac{\partial y}{\partial x_1}, \ldots, \dfrac{\partial y}{\partial x_n}) = 0 \\ y(0, x_1, \ldots, x_n) = \phi(x_1, \ldots, x_n), \end{cases}$$

where H and ϕ are C^∞-functions.

Hamilton-Jacobi equations play an important role in various fields e.g., calculus of variations (see e.g., [21]), optimal control theory (see e.g., [9]) and differential games (see e.g., [8] and references cited therein).

For small time t the solution of (P) is classically determined using the characteristic method. The *geometric solution* y of (P) has been defined in ([13], [14]) in the framework of *one-parameter Legendrian unfoldings* and it is constructed by the method of characteristics. Although y is initially smooth there is in general a critical time beyond which characteristics cross. The geometric solution past the critical time is multi-valued, that is singularities appear. The classification of singularities of y has been studied in [13] (see also [15]) In Section 2 we give a survey on the geometric framework ([13],[15],[16]).

The theory of *viscosity solutions* (see [5]) has provided the right weak setting for the study of (P). Existence and uniqueness of the solution of (P) in the viscosity sense have been established in [6]. The single-valued viscosity solution is continuous and coincides with the smooth geometric solution until the first critical time. After the characteristics cross, the viscosity solution develops *shock waves* i.e., curves across which the gradient of the viscosity solution is discontinuous. The shock surfaces are referred to as *singular surfaces* in the literature of optimal control and differential games (see e.g., [3], [12]).

The method of constructing the weak solution by selecting the proper single-valued branch was introduced by Tsuji ([22], [23]) for Hamilton-Jacobi equations. Nakane in [20] has constructed the weak semi-concave solution past the first critical time in case that H is convex with respect to $\nabla y = (\frac{\partial y}{\partial x_1}, \cdots, \frac{\partial y}{\partial x_n})$. The case of scalar conservation laws in $\mathbb{R}^n$ past the first critical time has been studied by Nakane in [19]. In [4] Bogaevskii has shown that the

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potential solution of the Burgers system with vanishing viscosity is given by the minimum function of a certain family of smooth functions and given a classification for $n = 1, 2, 3$. It corresponds to the viscosity solution of the Hamilton-Jacobi equation when the Hamiltonian is given by $H(p_1, \ldots p_n) = \frac{1}{2}p_1^2 + \cdots + \frac{1}{2}p_n^2$.

The viscosity solution of (P) for general Hamiltonian in a neighborhood of the first critical time has been constructed in [18] (see also [17], [20]) by selecting a continuous single-valued branch of the graph of the geometric solution. In which the shock curves of the weak solution corresponds to the intersection of the branches of the graph of the multi-valued geometric solution. In order to study the evolution of the shock curves we follow the evolution of the intersections of the branches defining the shock. After that we solve local Riemann problems for each stage.

Here, we give proofs for some of the results. Further discussions will appear in elsewhere. All maps considered here are class C^∞ unless stated otherwise.

2. GEOMETRIC SOLUTIONS

In this section we give a survey on the geometric framework and present the necessary notations which was described in ([13],[15],[16]).

Let $J^1(\mathbb{R}^n, \mathbb{R})$ be the 1-jet bundle of functions of n-variables which may be considered as $\mathbb{R}^{2n+1}$ with a natural coordinate system $(x_1, \ldots, x_n, y, p_1, \ldots, p_n)$, where $(x_1, \ldots, x_n)$ is a coordinate system of $\mathbb{R}^n$. We also have a natural projection $\pi : J^1(\mathbb{R}^n, \mathbb{R}) \to \mathbb{R}^n \times \mathbb{R}$ given by $\pi(x, y, p) = (x, y)$.

An immersion germ $i : (L_0, u_0) \to J^1(\mathbb{R}^n, \mathbb{R})$ is said to be *a Legendrian immersion germ* (i.e., Legendrian submanifold germ) if $\dim L = n$ and $i^* \theta = 0$, where $\theta = dy - \sum_{i=1}^n p_i \cdot dx_i$. The image of $\pi \circ i$ is called *the wave front set* of i and it is denoted by $W(i)$. We also consider the 1-jet bundle $J^1(\mathbb{R} \times \mathbb{R}^n, \mathbb{R})$ and the canonical 1-form Θ on that space. Let $(t, x_1, \ldots, x_n)$ be a canonical coordinate system on $\mathbb{R} \times \mathbb{R}^n$ and $(t, x_1, \ldots, x_n, y, s, p_1, \ldots, p_n)$ the corresponding coordinate system on $J^1(\mathbb{R} \times \mathbb{R}^n, \mathbb{R})$. Then, the canonical 1-form is given by $\Theta = dy - \sum_{i=1}^n p_i \cdot dx_i - s \cdot dt = \theta - s \cdot dt$.

We define the natural projection $\Pi : J^1(\mathbb{R} \times \mathbb{R}^n, \mathbb{R}) \to (\mathbb{R} \times \mathbb{R}^n) \times \mathbb{R}$ by $\Pi(t, x, y, s, p) = (t, x, y)$. We call the above 1-jet bundle *an unfolded 1-jet bundle*.

A *Hamilton-Jacobi equation* is defined to be a hypersurface

$$\text{(G-H-J)} \qquad E(H) = \left\{ (t, x, y, s, p) \in J^1(\mathbb{R} \times \mathbb{R}^n, \mathbb{R}) \,|\, s + H(t, x, p) = 0 \right\}$$

in $J^1(\mathbb{R} \times \mathbb{R}^n, \mathbb{R})$. A *geometric (multi-valued) solution* of $E(H)$ is a Legendrian submanifold L in $J^1(\mathbb{R} \times \mathbb{R}^n, \mathbb{R})$ lying in $E(H)$. In this case the wave front set $W(i)$ is "the graph" of the geometric solution which is generally a hypersurface with singularities.

In order to study (P) we need the following framework: For any $c \in (\mathbb{R}, 0)$, we define

$$E(H)_c = \left\{ (c, x, y, -H(c, x, p), p) \,|\, (x, y, p) \in J^1(\mathbb{R}^n, \mathbb{R}) \right\}.$$

Then, $E(H)_c$ is a $(2n+1)$-dimensional submanifold of $J^1(\mathbb{R} \times \mathbb{R}^n, \mathbb{R})$ and $\Theta_c = \Theta | E(H)_c = dz - \sum_{i=1}^n p_i dx_i$ gives a contact structure on $E(H)_c$. We define a mapping $\iota_c : J^1(\mathbb{R}^n, \mathbb{R}) \to$

$E(H)_c$ by $\iota_c(x,y,p) = (c,x,y,-H(c,x,p),p)$. The mapping ι_c is a contact diffeomorphism and the following diagram is commutative:

$$
\begin{CD}
J^1(\mathbb{R}^n,\mathbb{R}) @>{\iota_c}>> E(H)_c \\
@V{\pi}VV @VV{\pi_c}V \\
\mathbb{R}^n \times \mathbb{R} @= \mathbb{R}^n \times \mathbb{R}.
\end{CD}
$$

We say that *a geometric Cauchy problem (with initial condition L') associated with the time parameter(GCPT) is given for an equation $E(H)$* if there is given an n-dimensional submanifold $i : L' \subset E(H)$ with $i^*\Theta = 0$ and $i(L') \subset E(H)_c$ for some $c \in (\mathbb{R},0)$. Since $X_H \notin TE(H)_c$, we have $X_H \notin TL'$, where X_H is the characteristic vector field given by

$$
X_H = \frac{\partial}{\partial t} + \sum_{i=1}^{n} \frac{\partial H}{\partial p_i} \frac{\partial}{\partial x_i} + \left(\sum_{i=1}^{n} p_i \frac{\partial H}{\partial p_i} - H\right)\frac{\partial}{\partial y} - \frac{\partial H}{\partial t}\frac{\partial}{\partial s} - \sum_{i=1}^{n} \frac{\partial H}{\partial x_i} \frac{\partial}{\partial p_i}.
$$

By using the classical characteristic method, we can show that there exists a unique geometric solutions around L'.

We remark that Cauchy problem (P) is a GCPT. The initial submanifold is given by

$$
L_{\phi,0} = \left\{ (0,x,\phi(x),-H(0,x,\frac{\partial\phi}{\partial x}),\frac{\partial\phi}{\partial x})|x \in \mathbb{R}^n \right\} \subset E(H)_0.
$$

The problem of studying the singularities of the graph of the geometric solution is formulated as follows:

Geometric Problem. *Classify the generic bifurcations of wave fronts of*

$$
\pi_t| : L \cap E(H)_t \to \mathbb{R}^n \times \mathbb{R}
$$

with respect to the parameter t (i.e., the generic bifurcations of wave fronts of geometric solutions along the time parameter).

Following [16], in order to study the singularities of the geometric solution we identify geometric solutions with one-parameter Legendrian unfoldings. Let R be an $(n+1)$-dimensional smooth manifold, $\mu : (R,u_0) \to (\mathbb{R},t_0)$ be a submersion germ and $\ell : (R,u_0) \to J^1(\mathbb{R}^n,\mathbb{R})$ be a smooth map germ. We say that the pair (μ,ℓ) is *a Legendrian family* if $\ell_t = \ell|\mu^{-1}(t)$ is a Legendrian immersion germ for any $t \in (\mathbb{R},t_0)$. Then we have the following simple but very important lemma.

Lemma 2.1. *Let (μ,ℓ) be a Legendrian family. Then there exist a unique element $h \in C^\infty_{u_0}(R)$ such that $\ell^*\theta = h \cdot d\mu$, where $C^\infty_{u_0}(R)$ is the ring of smooth function germs at u_0.*

Define a map germ $\mathcal{L} : (R,u_0) \to J^1(\mathbb{R} \times \mathbb{R}^n, \mathbb{R})$ by

$$
\mathcal{L}(u) = (\mu(u),x \circ \ell(u),y \circ \ell(u),h(u),p \circ \ell(u)).
$$

We can easily show that $\mathcal{L}$ is a Legendrian immersion germ. If we fix 1-forms Θ and θ, the Legendrian immersion germ $\mathcal{L}$ is uniquely determined by the Legendrian family (μ, ℓ). We call $\mathcal{L}$ a *Legendrian unfolding associated with the Legendrian family* (μ, ℓ).

In order to study the evolution of the shock waves of the viscosity solutions of (P), we have to classify the generic types of the appearing singularities i.e., how a singularity is generated, how one type can change into another and how different types of singularities interact. We study how various branches of the multi-valued graph $W_t = (\{t\} \times \mathbb{R}^n \times \mathbb{R}) \cap W(i)$ intersecting at a point bifurcate in time for an arbitrary Hamiltonian $H(t, x, p)$ in [15]. We classify the bifurcations of the branches of the graph by classifying the bifurcations of singularities of multi-Legendrian unfoldings which are expressed in terms of multi-germs.

Let $\mathcal{L}_i : (R, u_0) \to (J^1(\mathbb{R} \times \mathbb{R}^n, \mathbb{R}), z_i)$ $(i = 1, \ldots, r)$ be Legendrian unfoldings with $\Pi(z_i) = 0$ where $z_1, \ldots, z_r$ are distinct. We call $(\mathcal{L}_1, \ldots, \mathcal{L}_r)$ a *multi-Legendrian unfolding*. Let $(\mathcal{L}_1, \ldots, \mathcal{L}_r)$ and $(\mathcal{L}'_1, \ldots, \mathcal{L}'_r)$ be multi-Legendrian unfoldings. We say that these are $P_{(r)}$-*Legendrian equivalent* if there exist contact diffeomorphism germs

$$K_i : (J^1(\mathbb{R} \times \mathbb{R}^n, \mathbb{R}), z_i) \to (J^1(\mathbb{R} \times \mathbb{R}^n, \mathbb{R}), z'_i) \quad (i = 1, \ldots, r)$$

of the form $K_i(t, x, y, s, p) = (\phi_1(t), \phi_2(t, x, y), \phi_3(t, x, y), \phi_4^i(t, x, y, s, p), \phi_5^i(t, x, y, s, p))$ and a diffeomorphism germ $\Psi : (R, u_0) \to (R, u'_0)$ such that $K_i \circ \mathcal{L}_i = \mathcal{L}'_i \circ \Psi$ for any $i = 1, \ldots, r$. It is clear that if two multi-Legendrian unfoldings are $P_{(r)}$-Legendrian equivalent, then there exists a diffeomorphism germ $\Phi : (\mathbb{R} \times (\mathbb{R}^n \times \mathbb{R}), 0) \to (\mathbb{R} \times (\mathbb{R}^n \times \mathbb{R}), 0)$ of the form $\Phi(t, x, y) = (\phi_1(t), \phi_2(t, x, y), \phi_3(t, x, y))$ such that $\Phi(\cup_{i=1}^r W(\mathcal{L}_i)) = \cup_{i=1}^r W(\mathcal{L}_i)$. Thus the above equivalence describes how bifurcations of wavefronts (i.e. graphs of solutions) interact. We can define the notion of stability with respect to the $P_{(r)}$-Legendrian equivalence in the same way as for the ordinary Legendrian stability (see [1],[24]). Motivated by Arnol'd-Zakalyukin's theory ([1],[24]), we can construct multi-generating families of multi-Legendrian unfoldings and give a classification of $P_{(r)}$-Legendrian stable Legendrian unfoldings by using the classification of multi-families of function germs in Zakalyukin [24]. We get a list of classifications for $n = 1, 2, 3$ in [15]. However, we only present the list of classifications for $n = 1$. For the case $n = 2, 3$, see [15].

Theorem 2.2 [15]. *Suppose that* $n = 1$. *Then a generic multi-Legendrian unfolding is* $P_{(r)}$-*Legendrian equivalent to one of the multi-Legendrian unfoldings in the following list* :

$r = 1$;

$^0A_1$: $(t, u, 0, 0, 0)$;

$^0A_2$: $(t, 3u^2, 2u^3, 0, u)$;

$^1A_3$: $(t, 4u^3 + 2ut, 3u^4 + u^2t, -u^2, u)$.

$r = 2$;

$^0(^0A_1\,^0A_1)$: $((t, u, -u, 0, -1), (t, u, u, 0, 1))$;

$^1(^0A_1\,^0A_1)$: $((t, u, t \pm u^2, 1, \pm 2u), (t, u, 0, 0, 0))$;

$^1A_2\,^0A_1$: $((t, 3u^2 - t, 2u^3, u, u), (t, u, -u, 0, -1))$.

$r = 3$;

$^0A_1\,^0A_1\,^0A_1$: $((t, u, t - u, 1, -1), (t, u, 0, 0, 0), (t, u, u, 0, 1))$.

When we consider the geometric solution, we can get rid of the germ $^1(^0A_1\,^0A_1)$ from the above list because the geometric solution is a one-to-one immersions into the unfolded 1-jet space.

On the other hand, we have to identify geometric solutions with one-parameter Legendrian unfoldings in locally, so that we prove the following realization theorems.

Theorem 2.3 [13]. (1) *The local solution of the GCPT for the Hamilton-Jacobi equation (G-H-J) is a Legendrian unfolding* $\mathcal{L} : (\mathbb{R} \times \mathbb{R}^n, 0) \rightarrow J^1(\mathbb{R} \times \mathbb{R}^n, \mathbb{R})$.

(2) *Let* $\mathcal{L} : (\mathbb{R} \times \mathbb{R}^n, 0) \rightarrow J^1(\mathbb{R} \times \mathbb{R}^n, \mathbb{R})$ *be a Legendrian unfolding associated with* (π_1, ℓ). *Then there exists a* C^∞-*function germ* $H(t, x_1, \ldots, x_n, p_1, \ldots, p_n)$ *such that* $\mathcal{L}$ *is a local solution of the generalized Cauchy problem associated with the time parameter for the Hamilton-Jacobi equation (G-H-J), where the initial condition is given by* $\ell(0, u)$.

The above theorem guarantees that the class of Legendrian unfoldings supplies the correct class to describe the geometric solutions of (GCPT) for Hamilton-Jacobi equations. Thus, generic results for the singularities of Legendrian unfoldings can be translated to generic results in the class of all Hamiltonians and all initial conditions. However, we have to also concern ourselves with what are the types of singularities that the geometric solution to a given Hamilton-Jacobi equation might exhibit. For the purpose, we need a kind of non-degeneracy condition on the Hamiltonian function. We say that a Hamiltonian function $H(t, x, p)$ is *non-degenerate at* (t_0, x_0, p_0) if it $\frac{\partial^2 H}{\partial p_i \partial p_j}(t_0, x_0, p_0) \neq 0$ for some $1 \leq i, j \leq n$. This condition is weaker than the condition that $H(t, x, p)$ is convex (or concave) with respect to $(p_1, \ldots, p_n)$-variables at (t_0, x_0, p_0) for $n \geq 2$. The following theorem is a realization theorem for generic singularities for a given Hamilton-Jacobi equation.

Theorem 2.4 ([15],[16]). *Let* $H(t, x, p)$ *be a non-degenerate Hamiltonian function germ at* (t_0, x_0, p_0) *and* $\mathcal{L} : (R, u_0) \rightarrow (J^1(\mathbb{R} \times \mathbb{R}^n, \mathbb{R}), (t_0, x_0, y_0, s_0, p_0))$ *be a* $P_{(1)}$-*Legendrian stable Legendrian unfolding associated with* (μ, ℓ). *Then there exists a Legendrian unfolding* $\mathcal{L}'$ *which is a geometric solution of the Hamilton-Jacobi equation* $s + H(t, x, p) = 0$ *such that* $\mathcal{L}$ *and* $\mathcal{L}'$ *are* $P_{(1)}$-*Legendrian equivalent.*

We remark that $^1A_3$ singularity (even for general n) describes how the singularity appears from a smooth solution. These are $P_{(1)}$-Legendrian stable Legendrian unfoldings, so that these can be realized as geometric solutions at the non-degenerated point for a given Hamilton-Jacobi equation. We can also specify the point at where the $^1A_3$-singularity appears.

Theorem 2.5 [16]. *If an* $^1A_3$-*singularity appears at* (t_0, x_0, p_0), *then* $H(t, x, p)$ *is non-degenerate at* (t_0, x_0, p_0).

3. Viscosity solutions

The viscosity solutions for nonlinear equations of first order have been introduced by Crandall and Lions [6]. Such solutions need not be differentiable everywhere, as the only regularity required in the definition is that of continuity. The function $y_v \in C(\mathcal{O})$ is a viscosity solution of

(H-J) $$\frac{\partial y}{\partial t} + H(t, x, \frac{\partial y}{\partial x_1}, \ldots, \frac{\partial y}{\partial x_n}) = 0$$

in the open domain $\mathcal{O} \subset \mathbb{R}^+ \times \mathbb{R}^n$ provided

$$\frac{\partial \psi}{\partial t}(t, x) + H(t, x, \frac{\partial \psi}{\partial x_1}(t, x), \ldots, \frac{\partial \psi}{\partial x_n}(t, x)) \leq 0, (resp. \geq 0)$$

for any $\psi \in C^1(\mathcal{O})$ for which $y_\mathfrak{v} - \psi$ attains a local maximum (resp. local minimum) at the point $(t, x) \in \mathcal{O}$. The function $y_\mathfrak{v} \in C([0, \infty) \times \mathbb{R}^n)$ is a viscosity solution of the Cauchy problem (P) if and only if it is a viscosity solution of (H-J) in the domain $(0, \infty) \times \mathbb{R}^n$ and satisfies the initial condition

$$\lim_{t \to 0+} y_\mathfrak{v}(t, x) = \phi(x).$$

The above inequality will be referred as the *viscosity criterion* at the point (t, x). We next state the viscosity criterion in a form which is more useful for the construction of the solution. To this end, assume that $\mathcal{O} \subset (0, \infty) \times \mathbb{R}^n$ is open and that there is a smooth hypersurface Γ of $\mathbb{R}^+ \times \mathbb{R}^n$, which divides $\mathcal{O}$ into two open sets $\mathcal{O}^+$ and $\mathcal{O}^-$, $\mathcal{O} = \Gamma \cup \mathcal{O}^+ \cup \mathcal{O}^-$. Then we have the following theorem.

Theorem 3.1. *Let* $y_\mathfrak{v} \in C(\mathcal{O})$ *and* $y_\mathfrak{v} = y_\mathfrak{v}^+$ *in* $\mathcal{O}^+ \cup \Gamma, y_\mathfrak{v} = y_\mathfrak{v}^-$ *in* $\mathcal{O}^- \cup \Gamma$ *where* $y_\mathfrak{v}^\pm \in C^1(\mathcal{O}^\pm \cup \Gamma)$. *Then* $y_\mathfrak{v}$ *is a viscosity solution of (H-J) in* $\mathcal{O}$ *if and only if the following conditions hold:*

    a) $y_\mathfrak{v}^+$ *and* $y_\mathfrak{v}^-$ *are classical solutions of (H-J) in* $\mathcal{O}^+$ *and* $\mathcal{O}^-$ *respectively,*

    b) If the vector $\tilde{\eta} = (H(t, x, \frac{\partial y^+}{\partial x}) - H(t, x, \frac{\partial y^+}{\partial x}), -(\frac{\partial y^+}{\partial x_1} - \frac{\partial y^-}{\partial x_1}, \ldots, \frac{\partial y^+}{\partial x_n} - \frac{\partial y^-}{\partial x_n}))$ *points into* $\mathcal{O}^+$, *then*

$$H\left(t, x, (1-\lambda)\frac{\partial y_\mathfrak{v}^+}{\partial x} + \lambda\frac{\partial y_\mathfrak{v}^-}{\partial x}\right) - (1-\lambda)H(t, x, \frac{\partial y_\mathfrak{v}^+}{\partial x}) - \lambda H(t, x, \frac{\partial y_\mathfrak{v}^-}{\partial x}) \leq 0 \ (resp. \ \geq 0),$$

where $\lambda \in [0, 1]$. *In particular, the graph of* H *lies respectively below or above the line segment joining the points* $\left(H(t, x, \frac{\partial y_\mathfrak{n}^+}{\partial x}), \frac{\partial y_\mathfrak{n}^+}{\partial x}\right)$ *and* $\left(H(t, x, \frac{\partial y_\mathfrak{n}^-}{\partial x}), \frac{\partial y_\mathfrak{n}^-}{\partial x}\right)$.

The proof of Theorem 3.1 is given in ([17],[18]) as a direct application of Theorem 1.3 in [7]. The condition b) will be referred in the sequel as the *viscosity criterion*. The hypersurface Γ in the neighbourhood of which $y_\mathfrak{v}$ has the properties specified in the above theorem is the *shock surface*. If the Hamiltonian is uniformly convex (or concave), we can automatically construct viscosity solutions from our normal forms, so that we can easily draw the pictures of shock surfaces for lower dimensional cases. In [4] Bogaevskii has shown that the potential solution of the Burgers system with vanishing viscosity is given by the minimum function of a certain family of smooth functions. It corresponds to the viscosity solution of the Hamilton-Jacobi equation when the Hamiltonian is given by $H(p_1, \ldots p_n) = \frac{1}{2}p_1^2 + \cdots + \frac{1}{2}p_n^2$. He has drawn the pictures of shocks for this case. Our pictures are same as his pictures, so we do not present these in here (see [4]).

On the other hand, Bogaevskii used Florin-Hopf-Cole method ([10],[11]) to detect the solution and it works only for the Burgers system. Here, we prove the analogous statement as the Bogaevskii's assertion in the case when the Hamiltonian $H(p_1, \ldots, p_n)$ is convex and depends only on the momentum. In this case we apply Bardi-Evans' result[2] to our situations in stead of Florin-Hopf-Cole method. The geometric solution for (P) is given by

(S) $\qquad\qquad L_{\phi,t} = \{(t, x(t, u), y(t, u), -H(p(t, u)), p(t, u)) | u \in \mathbb{R}^n\},$

where

$$\begin{cases} x(t,u) = u + t\dfrac{\partial H}{\partial p}(\dfrac{\partial \phi}{\partial x}(u))), \\[2mm] p(t,u) = \dfrac{\partial \phi}{\partial x}(u) \\[2mm] y(t,u) = t\{-H(\dfrac{\partial \phi}{\partial x}(u)) + < \dfrac{\partial \phi}{\partial x}(u), \dfrac{\partial H}{\partial p}(\dfrac{\partial \phi}{\partial x}(u)) >\} + \phi(u). \end{cases}$$

We consider a family of functions $F(t,x,p,q) = \phi(q) + < p,(x-q) > -H(p)t$, where $(t,x,p,q) \in \mathbb{R} \times \mathbb{R}^n \times (\mathbb{R}^n \times \mathbb{R}^n)$ and $<,>$ is the canonical inner product on $\mathbb{R}^n$. We have

$$\Sigma(F) = \{(t, q + \dfrac{\partial H}{\partial p}(\dfrac{\partial \phi}{\partial q}(q))t, \dfrac{\partial \phi}{\partial q}(q), q)|(t,q) \in \mathbb{R} \times \mathbb{R}^n\},$$

where $\Sigma(F)$ is the set defined to be $\frac{\partial F}{\partial p_i} = 0$ and $\frac{\partial F}{\partial q_i} = 0$. We now define a map $\Phi_F : \Sigma(F) \to J^1(\mathbb{R} \times \mathbb{R}^n, \mathbb{R})$ by $\Phi_F(t,x,p,q) = (t,x,F(t,x,p,q), \frac{\partial F}{\partial t}, \frac{\partial F}{\partial x})$. It follows that

$$\Phi_F(t, q + \dfrac{\partial H}{\partial p}(\dfrac{\partial \phi}{\partial q}(q))t, \dfrac{\partial \phi}{\partial q}(q), q) =$$
$$(t, q + \dfrac{\partial H}{\partial p}(\dfrac{\partial \phi}{\partial q}(q))t, -H(\dfrac{\partial \phi}{\partial q}(q))t + < \dfrac{\partial \phi}{\partial q}(q), \dfrac{\partial H}{\partial p}(\dfrac{\partial \phi}{\partial q}(q)) > +\phi(q), -H(\dfrac{\partial \phi}{\partial q}(q)), \dfrac{\partial \phi}{\partial q}(q)).$$

This shows that the image of the map Φ_F is equal to $L_{\phi,t}$, namely, F is a global *generating family* of $L_{\phi,t}$.

We refer the following result of Bardi-Evans [2].

Theorem 3.2. *Assume that the Hamiltonian $H(p_1, \ldots, p_n)$ is convex, then*

$$y(t,x) \equiv \inf_q \sup_p \{\phi(q) + < p,(x-q) > -H(p)t\}$$

is the unique viscosity solution of (P).

Then we have the following theorem as a corollary of the above theorem.

Theorem 3.3. *Assume that H is uniformly convex and ϕ has the minimum. Let $L_{\phi,t}$ be the geometric solution* (S) *of the Cauchy problem* (P). *Then*

$$y(t,x) \equiv \min_y \{y|(t,x,y) \in \Pi(L_{\phi,t})\}$$

is the unique viscosity solution of (P').

Proof. Consider the family of functions $F(t,x,p,q) = \phi(q) + < p,(x-q) > -H(p)t$. Since $H(p)$ is uniformly convex, we have

$$\sup_p \{\phi(q) + < p,(x-q) > -H(p)t\} = F|\Sigma_p(F),$$

where $\Sigma_p(F) = \{(t,x,p,q)|\frac{\partial F}{\partial p_i} = x_i - q_i - \frac{\partial H}{\partial p_i}(p)t = 0 \ i = 1,\ldots,n\}$. It follows that

$$\inf_q \sup_p \{\phi(q)+ <p,(x-q)> -H(p)t\} = \inf_q \{F(t,q + \frac{\partial H}{\partial p}(p)t,p,q)\}.$$

Since ϕ has the minimum , it is equal to

$$\min_q \{F(t,q + \frac{\partial H}{\partial p}(p)t,p,q)\} = \min_q \{F(t,x,p,q)|(t,x,p,q) \in \Sigma_p(F)\}.$$

On the other hand, we define functions $f_i(t,x,p,q) = \frac{\partial F}{\partial p_i} = x_i - q_i - \frac{\partial H}{\partial p_i}(p)t$ $(i = 1,\ldots,n)$. Since $H(p)$ is uniformly convex, we have $\frac{\partial f_i}{\partial p_j}(t_0,x_0,p_0,q_0) = -\frac{\partial^2 H}{\partial p_i \partial p_j}(p_0)t_0 \neq 0$, at any point (t_0,x_0,p_0,q_0), so that there exist local smooth functions $g_i(t,x,q)$ $(i = 1,\ldots,n)$ near (t_0,x_0,p_0,q_0) such that $\Sigma_p(F) = \{p_i = g_i(t,x,q)\}$. Thus we have

$$\frac{\partial F|\Sigma_p(F)}{\partial q_i} = \sum_j \frac{\partial F}{\partial p_j}(t,x,g(t,x,q),q)\frac{\partial g_j}{\partial q_i}(t,x,q) + \frac{\partial \phi}{\partial q_i}(q) - g_i(t,x,q) = \frac{\partial \phi}{\partial q_i}(q) - g_i(t,x,q),$$

so that $\Sigma(F) = \{(t,x,p,q) \in \Sigma_p(F)|\frac{\partial F|\Sigma_p(F)}{\partial q_i} = 0\}$. It follows that

$$y(t,x) \equiv \min_y \{y|(t,x,y) \in \Pi(L_{\phi,t})\}$$
$$= \min_{(p,q)} \{F(t,x,p,q)|(t,x,p,q) \in \Sigma(F)\}$$
$$= \min_q \{F(t,x,p,q)|(t,x,p,q) \in \Sigma_p(F)\}.$$

It is the unique viscosity solution for the Cauchy problem (P).

However, for general (non-convex) Hamiltonian, situations are quite different.

4. NON CONVEX HAMILTONIANS IN ONE SPACE VARIABLE

In this section we stick to the Cauchy problem of Hamilton-Jacobi equation in one space variable as follows:

(P)
$$\begin{cases} \frac{\partial y}{\partial t} + H(\frac{\partial y}{\partial x}) = 0 \\ y(0,x) = \phi(x), \end{cases}$$

where H and ϕ are C^∞-functions. Since $H(p)$ is not assumed to be uniformly convex (or concave), we cannot use Theorem 3.3, so that the situations should be quite complicated even for the one space variables case.

In this case the geometric solution is given by

$$L_{\phi,t} = \{(t,x(t,u),y(t,u),-H(p(t,u)),p(t,u))|u \in \mathbb{R}\},$$

where

$$\begin{cases} x(t,u) = u + tH'(\phi'(u)), \\ p(t,u) = \phi'(u) \\ y(t,u) = t\{-H(\phi'(u)) + \phi'(u)H'(\phi'(u))\} + \phi(u). \end{cases}$$

Before the first critical time that characteristics cross in the (t,x)—plane, W_t is the graph of the viscosity solution $y_{\mathfrak{v}}$. After the characteristics cross, W_t becomes singular. Theorem 2.2 describes the generic singularities of W_t. The first singularity appears in the form of $^1A_3$. See Figure 1a, where we show the shape of the appearing singularity.
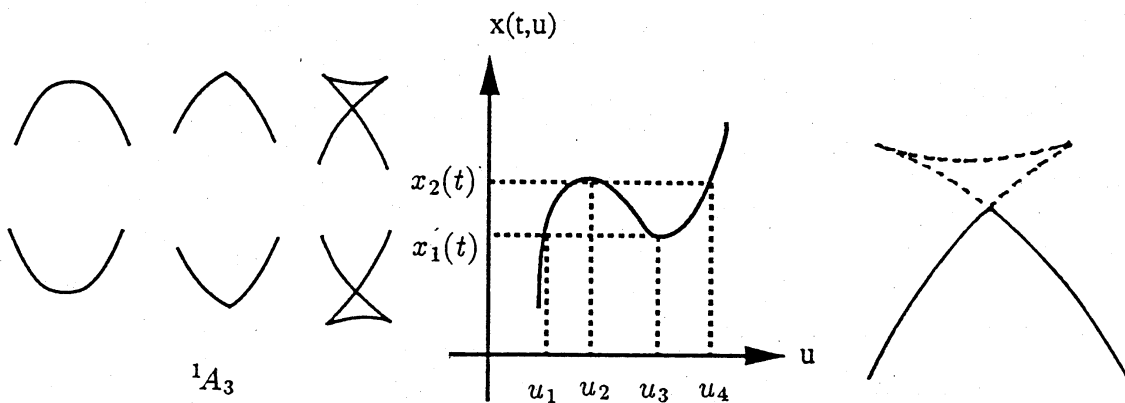
Figure 1a   Figure 1b   Figure 1c

By Theorem 2.5, these appear at the convex or the concave points of the Hamiltonian function. Away from the singularity, the viscosity solution is given by W_t. In ([17], [18]) we have constructed the unique viscosity solution past the first critical time by selecting a single-valued branch of W_t. Assume that the singularity of type $^1A_3$ appears at the point (t_0, x_0, p_0). After the critical time t_0, the wave front W_t is three-valued on an interval $(x_1(t), x_2(t))$; see Figure 1b. Let y_i, $i = 1, 2, 3$ be the three branches of W_t, where y_1 is defined on a neighborhood of $x_1(t)$ and y_2 on a neighborhood of $x_2(t)$. Then y_1, y_2 intersect at one point $\chi(t) \in (x_1(t), x_2(t))$, for $t > t_0$. We define the viscosity solution past t_0 by selecting a continuous single-valued branch of W_t as follows:

Theorem 4.1. *There exists an $\varepsilon > 0$ such that the function $y_{\mathfrak{v}}(t,x), (t,x) \in (t_0, t_0 + \varepsilon) \times (x_1(t), x_2(t))$, defined by*

$$(4.1) \qquad\qquad y_{\mathfrak{v}}(t,x) = \begin{cases} y_1(t,x), & x \le \chi(t) \\ y_3(t,x), & x \ge \chi(t), \end{cases}$$

is the viscosity solution of (P) in a neighborhood of x_0 past the time t_0.

In view of Theorem 2.5 the viscosity criterion (see Section 3) is satisfied across $\chi(t)$ while $y_{\mathfrak{v}}$ defined by (4.1) is a classical solution away from $\chi(t)$. Hence, by the uniqueness of the viscosity solution, (4.1) gives the viscosity solution of (P) past t_0.

By this construction, we have extended the viscosity solution beyond the first critical time t_0. According to Theorem 2.5 the shock is generated in a convex or concave domains of $H(p)$, so the viscosity criterion is automatically satisfied. The graph of the viscosity solution past the first critical time is depicted by a full line in Figure 1c, where we assume that H is convex in the neighborhood of the appearing singularity $^1A_3$. The shock corresponds to the intersection of the two branches and it is called a *genuine shock*. The genuine shock is defined as the intersection of two incoming characteristics (or waves) and its speed is given by the Rankine-Hugoniot condition

$$\chi'(t) = \frac{H(y^+_{v,x}(t,\chi(t))) - H(y^-_{v,x}(t,\chi(t)))}{y^+_{v,x}(t,\chi(t)) - y^-_{v,x}(t,\chi(t))},$$

where $y^\pm_{v,x} = \frac{\partial y^\pm_v}{\partial x}$ and $\chi'(t) = \frac{d\chi}{dt}(t)$. Therefore in order to follow the evolution of the shock we have to study the following questions:

a) How different branches of the multi-valued graph of W_t intersecting at one point bifurcate in time.

b) If the two branches initially defining the shock continue to cross, whether the viscosity criterion is satisfied across the intersection.

The normal forms of the generic bifurcations of different branches of W_t are given in Theorem 2.2. We depict these bifurcations in Figure 2.
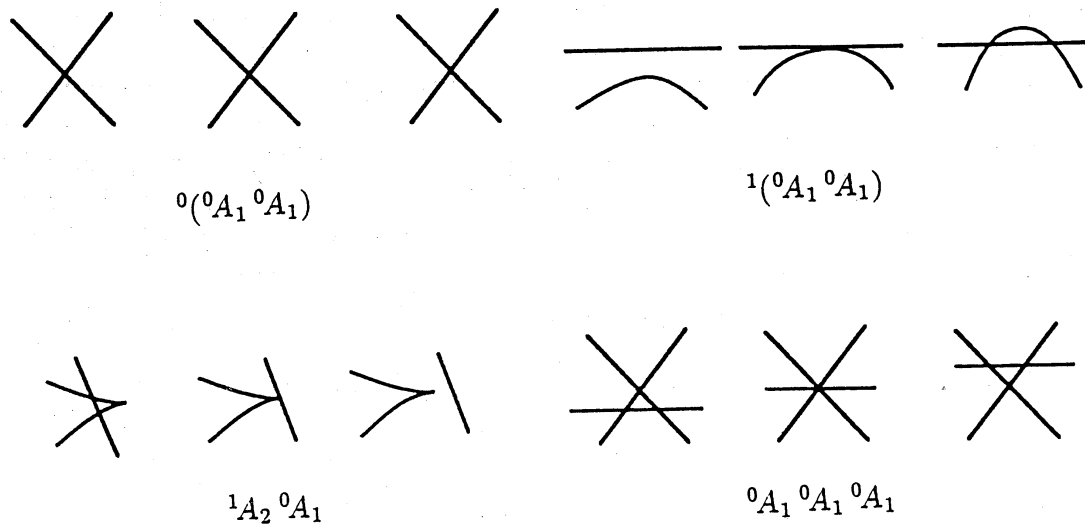

$$^0(^0A_1\,^0A_1) \qquad\qquad ^1(^0A_1\,^0A_1)$$

$$^1A_2\,^0A_1 \qquad\qquad ^0A_1\,^0A_1\,^0A_1$$

FIGURE 2

If the viscosity criterion is satisfied at the time $t_\alpha = t_0 + \varepsilon$, we can choose the correct branch of the graphs of the geometric solutions as viscosity solutions (see Figure 3).
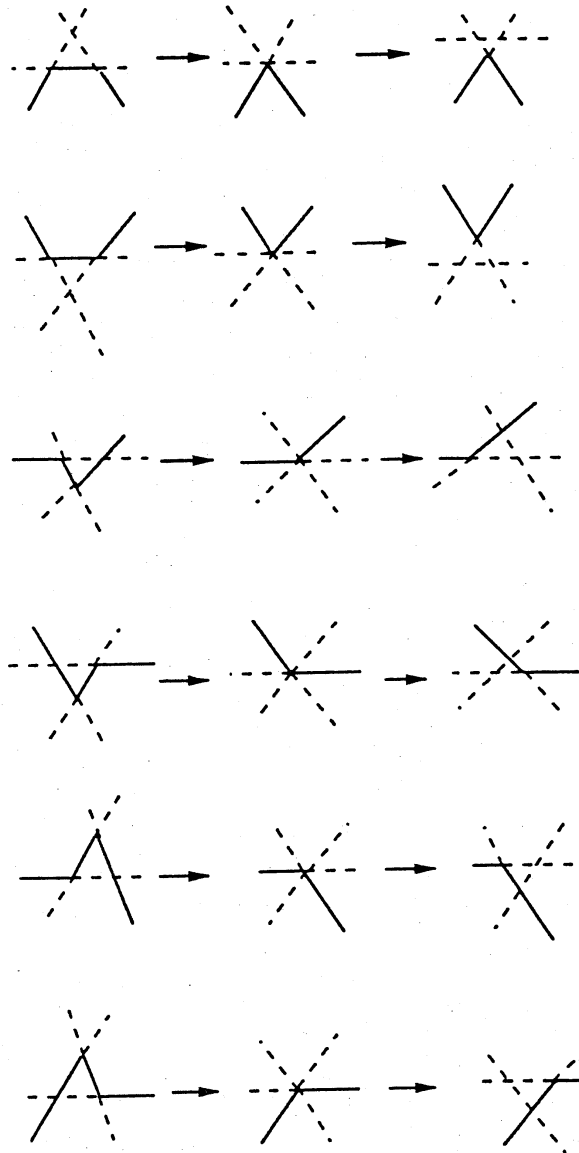

FIGURE 3

We will now investigate how the viscosity criterion can be violated across the intersection of two branches. Assume that a generated shock is defined by two intersecting branches y^- and y^+. We denote by y^- (resp. y^+) the branch representing the viscosity solution for $x < \chi(t)$

(resp. $x > \chi(t)$). If the two branches remain intersected they evolve according to $^0(^0A_1\,^0A_1)$. We denote by $\chi(t)$ the intersection of the two branches. In the case when $H(p)$ has only one inflection point Kossioris [17] studied this problem and constructed the viscosity solutions. We consider the general situation here. It is clear that for generic Hamiltonian function $H(p)$, H has only Morse type critical points and no tritangent lines. So we assume that the Hamiltonian has the above properties. By Theorem 2.2, we have the following theorem.

Theorem 4.2. *For a generic initial function ϕ, if the viscosity criterion is violated at t_α, then the only following 8 cases may occur:*

(1) *The normal form is $^0(^0A_1\,^0A_1)$ and $\overline{P^+P^-}$ is tangent to the graph of $H(p)$ at only one of the points P^+, P^- and the line is not tangent to the graph at other points between these points.*

(2) *The normal form is $^0(^0A_1\,^0A_1)$ and $\overline{P^+P^-}$ is not tangent to the graph of $H(p)$ at each point P^+, P^- and there exists only one another point between these points at where the above line is tangent to the graph.*

(3) *The normal form is $^0(^0A_1\,^0A_1)$ and $\overline{P^+P^-}$ is tangent to the graph of $H(p)$ at only one of the points P^+, P^- and there exists only one another point between these points at where the above line is tangent to the graph.*

(4) *The normal form is $^0(^0A_1\,^0A_1)$ and $\overline{P^+P^-}$ is tangent to the graph of $H(p)$ at each point of P^+, P^-.*

(5) *The normal form is $^0(^0A_1\,^0A_1)$ and $\overline{P^+P^-}$ is not tangent to the graph of $H(p)$ at each point P^+, P^- and there exists exactly two other points between these points at where the above line is tangent to the graph.*

(6) *The normal form is $^1A_2\,^0A_1$ and $\overline{P^+P^-}$ is tangent to the graph of $H(p)$ at only one of the points P^+, P^- and the line is not tangent to the graph at other points between these points.*

(7) *The normal form is $^0A_1\,^0A_1\,^0A_1$ and $\overline{P^+P^-}$ is tangent to the graph of $H(p)$ at only one of the points P^+, P^- and is not tangent to the graph at other points between these points.*

(8) *The normal form is $^0A_1\,^0A_1\,^0A_1$ and $\overline{P^+P^-}$ is not tangent to the graph of $H(p)$ at each point P^+, P^- and there exists only one another point between these points at where the above line is tangent to the graph.*

Here, $P^+ = (y_x^+(t_\alpha,\chi(t_\alpha)), H(y_x^+(t_\alpha,\chi(t_\alpha))))$, $P^- = (y_x^-(t_\alpha,\chi(t_\alpha)), H(y_x^-(t_\alpha,\chi(t_\alpha))))$ and $\overline{P^+P^-}$ denotes the line through P^+, P^- in the $(p, H(p))$-plane.

Proof. By Theorem 2.2, we may assume that the first singularities appear in the form of $^1A_3$. After that the singularities of the graph of the geometric solution bifurcate in the forms of $^0(^0A_1\,^0A_1)$, $^1(^0A_1\,^0A_1)$, $^1A_2\,^0A_1$ or $^0A_1\,^0A_1\,^0A_1$. Since the characteristics in $J^1(\mathbb{R} \times \mathbb{R}, \mathbb{R})$ never cross, we can get rid of the case $^1(^0A_1\,^0A_1)$.

We already mentioned that the viscosity criterion is satisfied past the first critical time t_0, so that it is satisfied until the time t_α when $\overline{P^+P^-}$ is tangent to the graph of $H(p)$. By the assumptions on the Hamiltonian $H(p)$, we may consider the case that $\overline{P^+P^-}$ is at most a double tangent line for each normal form. We now distinguish each normal form. We denote that $p^+ = y_x^+(t_\alpha,\chi(t_\alpha)) = \phi'(u_+)$ and $p^- = y_x^-(t_\alpha,\chi(t_\alpha)) = \phi'(u_-)$.

(A) $^0(^0A_1\,^0A_1)$: In this case each branch of the graph of geometric solution is a non-singular curve. We remark that $y^\pm(t,\chi(t)) = t\{-H(\phi'(u_\pm)) + \phi'(u_\pm)H'(\phi(u_\pm))\} + \phi(u_\pm)$.

Since the normal form $^0(^0A_1\,^0A_1)$ has trivial bifurcations along the time parameter, the condition $y^+(t,\chi(t)) = y^-(t,\chi(t))$ defines a codimension 0 submanifold in the corresponding jet space, so that we may ignore this condition. We now consider the following conditions which correspond to all possible cases:

(a) $\pm H'(\phi'(u_+)) = \frac{H(\phi'(u_+))-H(\phi'(u_-))}{\phi'(u_+)-\phi'(u_-)}$, which defines a submanifold in $_2J^1(\mathbb{R},\mathbb{R})$ of codimension 1. Of course, we have to consider the case that $\pm H'(\phi'(u_-)) = \frac{H(\phi'(u_+))-H(\phi'(u_-))}{\phi'(u_+)-\phi'(u_-)}$, however this case is essentially contained in the above, so that we may ignore such non-essentially different cases in the following arguments.

(b) There exists u_0 with $u_0 \neq u_\pm$ such that $\pm H'(\phi'(u_0)) = \frac{H(\phi'(u_+))-H(\phi'(u_-))}{\phi'(u_+)-\phi'(u_-)}$. This condition defines a submanifold in $_2J^1(\mathbb{R},\mathbb{R})$ of codimension 1.

(c) There exists u_0 with $u_0 \neq u_\pm$ such that

$$\pm H'(\phi'(u_+)) = \pm H'(\phi'(u_0)) = \frac{H(\phi'(u_+)) - H(\phi'(u_-))}{\phi'(u_+) - \phi'(u_-)} = \frac{H(\phi'(u_+)) - H(\phi'(u_0))}{\phi'(u_+) - \phi'(u_0)}.$$

This condition defines a submanifold in $_3J^1(\mathbb{R},\mathbb{R})$ of codimension 3.

(d) $\pm H'(\phi'(u_+)) = \pm H'(\phi'(u_-)) = \frac{H(\phi'(u_+))-H(\phi'(u_-))}{\phi'(u_+)-\phi'(u_-)}$, which defines a submanifold in $_2J^1(\mathbb{R},\mathbb{R})$ of codimension 2.

(e) There exist u_0, u_1 which are different from $u_\pm$ such that

$$\pm H'(\phi'(u_0)) = \pm H'(\phi'(u_1)) = \frac{H(\phi'(u_+)) - H(\phi'(u_-))}{\phi'(u_+) - \phi'(u_-)}$$

$$= \frac{H(\phi'(u_+)) - H(\phi'(u_0))}{\phi'(u_+) - \phi'(u_0)} = \frac{H(\phi'(u_+)) - H(\phi'(u_1))}{\phi'(u_+) - \phi'(u_1)}.$$

This condition defines a submanifold of $_4J^1(\mathbb{R},\mathbb{R})$ of codimension 4. Here, $_rJ^1(\mathbb{R},\mathbb{R})$ is a multi-1-jet space of function germ $\mathbb{R} \to \mathbb{R}$. Each submanifold in $_rJ^1(\mathbb{R},\mathbb{R})$ has at most codimension r, so that we can not avoid such conditions by the multi-jet transversality theorem.

(B) $^1A_2\,^0A_1$: In this case the normal form $^1A_2\,^0A_1$ bifurcates at the time t_α, so that we should consider the condition $y^+(t_\alpha,\chi(t_\alpha)) = y^-(t_\alpha,\chi(t_\alpha))$ for fixed t_α. It defines a submanifold in $_2J^1(\mathbb{R},\mathbb{R})$ of codimension 1. By the same arguments as the above, we can avoid the conditions (c), (d) and (e). So we may consider the condition (a) or (b). We now show that the condition (a) holds for the normal form $^1A_2\,^0A_1$. On the (t,x)-plane, we denote $(t,\chi(t))$ the genuine shocks for $t \leq t_\alpha$. Suppose that the point u_- corresponds to the cusp point at the time t_α. Then there exists a smooth function $u(t)$ such that $\chi(t) = u(t) + tH'(\phi'(u(t)))$ for $t \leq t_\alpha$ and $u(t_\alpha) = u_-$, where we choose one of the branches of the graph of the geometric solution corresponding to u_-. It follows that we have

$$\chi'(t) = u'(t)(1 + H''(\phi'(u(t))\phi''(t)) + H'(\phi'(u(t))).$$

Since the graph of the geometric solution has a singularity at t_α, we have $\frac{\partial x}{\partial u}(t_\alpha, u_-) = 1 + t_\alpha H''(\phi'(u_-))\phi''(u_-) = 0$. So we have $\chi'(t_\alpha) = \lim_{t \to t_\alpha} \chi'(t) = H'(\phi'(u_-))$.

On the other hand, by the Rankine-Hugoniot condition we have

$$\chi'(t) = \frac{H(y_{v,x}^+(t,\chi(t))) - H(y_{v,x}^-(t,\chi(t)))}{y_{v,x}^+(t,\chi(t)) - y_{v,x}^-(t,\chi(t))},$$

for $t \le t_\alpha$. Since $\lim_{t \to t_\alpha} y_{v,x}^\pm(t,\chi(t))) = \phi'(u_\pm)$, we have $\chi'(t_\alpha) = \frac{H(\phi'(u_+)) - H(\phi'(u_-))}{\phi'(u_+) - \phi'(u_-)}$, so that we have $H'(\phi'(u_-)) = \frac{H(\phi'(u_+)) - H(\phi'(u_-))}{\phi'(u_+) - \phi'(u_-)}$. This condition corresponds to the case (a) and we may get rid of the case (b).

(C) ${}^0A_1\,{}^0A_1\,{}^0A_1$: In this case the normal form also bifurcate at the point t_α, so that we can get rid of the case (c), (d) and (e) by the similar reasons as those of the case (B). Since each branch of the normal form is non-singular, the remaining two cases may occur in generic. This completes the proof.

We can solve local Riemann problems and construct viscosity solutions for each case in the above theorem. However, we only consider the cases (1) and (6) in this paper. We will give the detailed considerations for all cases in elsewhere.

Case (1). We assume that the graph of the viscosity solution at the time $t \le t_\alpha$ is depicted as in Figure 4a.
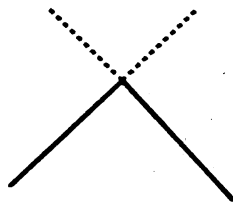

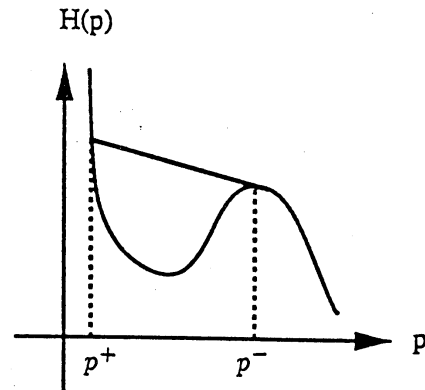Figure 4a   Figure 4b

For convenience, we assume that $\overline{P^+P^-}$ is tangent to the graph of $H(p)$ at the point $(y_x^-(t_\alpha,\chi(t_\alpha)), H(y_x^-(t_\alpha,\chi(t_\alpha))))$ and $H''((y_x^-(t_\alpha,\chi(t_\alpha)))) < 0$ (see Figure 4b). For the case $H''((y_x^-(t_\alpha,\chi(t_\alpha)))) = 0$, detailed arguments will be appeared in elswhere. As we already mentioned that the genuine shocks satisfies the Rankin-Hugoniot condition. So we should construct new characteristics which satisfies both of the Rankin-Hugoniot condition and the viscosity criterion. In this case we have

$$H'(y_x^-(t_\alpha,\chi(t_\alpha))) = \frac{H(y_x^+(t_\alpha,\chi(t_\alpha))) - H(y_x^-(t_\alpha,\chi(t_\alpha)))}{y_x^+(t_\alpha,\chi(t_\alpha)) - y_x^-(t_\alpha,\chi(t_\alpha))} = \chi'(t_\alpha).$$

We now distinguish two cases as follows:
a) If

$$H'(y_x^-(t,\chi(t))) \ge \frac{H(y_x^+(t,\chi(t))) - H(y_x^-(t,\chi(t)))}{y_x^+(t,\chi(t)) - y_x^-(t,\chi(t))}$$

for $t_\alpha \leq t < t_\alpha + \varepsilon$ for sufficiently small $\varepsilon > 0$, then we can easily show that the viscosity criterion is satisfied for $t < t_\alpha + \varepsilon$. So we can choose single valued continuous branches of the geometric solution as the viscosity solution.

b) If

$$H'(y_x^-(t,\chi(t))) < \frac{H(y_x^+(t,\chi(t))) - H(y_x^-(t,\chi(t)))}{y_x^+(t,\chi(t)) - y_x^-(t,\chi(t))}$$

for $t_\alpha \leq t < t_\alpha + \varepsilon$ for sufficiently small $\varepsilon > 0$, then we can easily show that the viscosity criterion is violated for $t_\alpha < t < t_\alpha + \varepsilon$, so that a new way to build the solution is required (cf., Figure 5).
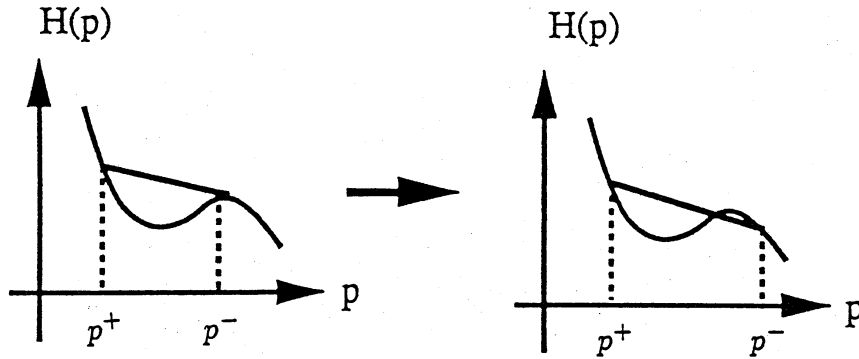

FIGURE 5

In this case we can use the techniques in [12] to construct the contact discontinuity shock curve and then obtain new characteristics. Lets consider the relation $H'(q) = \frac{H(p)-H(q)}{p-q}$ around (q_0, p_0) with $q_0 \neq p_0$, $H'(q_0) = \frac{H(p_0)-H(q_0)}{p_0-q_0}$ and $H''(q_0) \neq 0$. By the implicit function theorem, there exists a smooth function ψ around p_0 such that the above relation is equivalent to $q = \psi(p)$. We will first construct the contact discontinuity as the solution of the following initial value problem.

$$\begin{cases} \chi_c'(t) = H'(\psi(y_x^+(t,\chi_c(t)))), \\ \chi_c(t_\alpha) = \chi(t_\alpha). \end{cases}$$

The characteristic which is started at a point $(\tau, \chi_c(\tau))$ should be satisfied the following:

$$\begin{cases} x'(t) = H'(p(t)), \\ p'(t) = 0 \\ y'(t) = -H(p(t)) + p(t)H'(p(t)), \end{cases}$$

with initial condition $x(\tau) = \chi_c(\tau)$, $y(\tau) = y^+(\tau, \chi_c(\tau))$ and $p(\tau) = \psi(y_x^+(\tau, \chi_c(\tau)))$. So the solution is exactly given as follows:

$$\begin{cases} \tilde{x}(t) = \chi_c(\tau) + (t-\tau)H'(\psi(y_x^+(\tau, \chi_c(\tau)))), \\ \tilde{p}(t) = \psi(y_x^+(\tau, \chi_c(\tau))) \\ \tilde{y}(t) = y^+(\tau, \chi_c(\tau)) \\ \qquad + (t-\tau)\{-H(\psi(y_x^+(\tau, \chi_c(\tau)))) + \psi(y_x^+(\tau, \chi_c(\tau)))H'(\psi(y_x^+(\tau, \chi_c(\tau))))\}. \end{cases}$$

By definition of the contact discontinuity, we have

$$\chi_c''(t) = H''(\psi(\phi(u_+(t))\frac{\partial\psi}{\partial p}(\phi'(u_+(t))\phi''(u_+(t))u_+'(t),$$

where $\chi_c(t) = u_+(t) + tH'(\phi(u_+(t)))$. Since $\frac{\partial\psi}{\partial p} = \frac{H'(p)-H'(q)}{H''(q)(p-q)}$, we have

$$\chi_c''(t) = \frac{H'(\phi'(u_+(t))) - H'(\psi(\phi'(u_+(t))))}{\phi'(u_+(t)) - \psi(\phi'(u_+(t)))}\phi''(u_+(t))u_+'(t).$$

We also have

$$\chi'(t) = u_+'(t)\{1 + tH''(\phi'(u_+(t))\phi''(u_+(t))\} + H'(\phi'(u_+(t))).$$

It follows that

$$\chi_c''(t) = -\frac{(H'(\phi'(u_+(t))) - H'(\psi(\phi'(u_+(t)))))^2}{\phi'(u_+(t)) - \psi(\phi'(u_+(t)))}\frac{\phi''(u_+(t))}{1 + tH''(\phi'(u_+(t))\phi''(u_+(t))}.$$

Since

$$\frac{\partial x}{\partial u}(t, u_+(t)) = 1 + tH''(\phi'(u_+(t)))\phi''(u_+(t)),$$

we may assume that $1 + tH''(\phi'(u_+(t)))\phi''(u_+(t)) > 0$. So $\chi_c(t)$ is convex if and only if $\phi''(u_+(t)) > 0$. We suppose that $\phi''(u_+(t)) \leq 0$ and denote $\chi_c(t) = u_+(t) + tH'(\phi(u_+(t)) = u_-(t) + tH'(\phi(u_-(t)))$, where $u_-(t)$ (resp. $u_+(t)$) is the point corresponding to the characteristic from the right (resp. left) side of $(t, \chi_c(t))$. We distinguish two cases as follows:

b-1) If $\phi''(u_-(t)) > 0$, then ϕ' is monotone. Since $u_-'(t) < 0$, $\phi'(u_-(t))$ moves to the left direction, so that the viscosity criterion is satisfied across χ.

b-2) If $\phi''(u_-(t)) < 0$ and the viscosity criterion is violated across χ for $t > t_\alpha$, then $1 + tH''(\phi'(u_-(t)))\phi''(u_-(t)) > 0$ near t_α. Differentiate the equality $\chi_c(t) = u_-(t) + tH'(\phi(u_-(t))$ with respect to t, then we have

$$\chi'(t) - H'(\phi'(u_-(t))) = \{1 + tH''(\phi'(u_-(t)))\phi''(u_-(t))\}u_-'(t).$$

Since

$$\chi'(t) = \frac{H(\phi'(u_+(t))) - H(\phi'(u_-(t)))}{\phi'(u_+(t)) - \phi'(u_-(t))} > H'(\phi'(u_-(t))),$$

we have $u_-'(t) > 0$, so that $u_-(t)$ is increase, which is a contradiction.

Hence, if the viscosity criterion is violated for $t > t_\alpha$, the contact discontinuity curve χ is convex and the viscosity solution can be constructed.

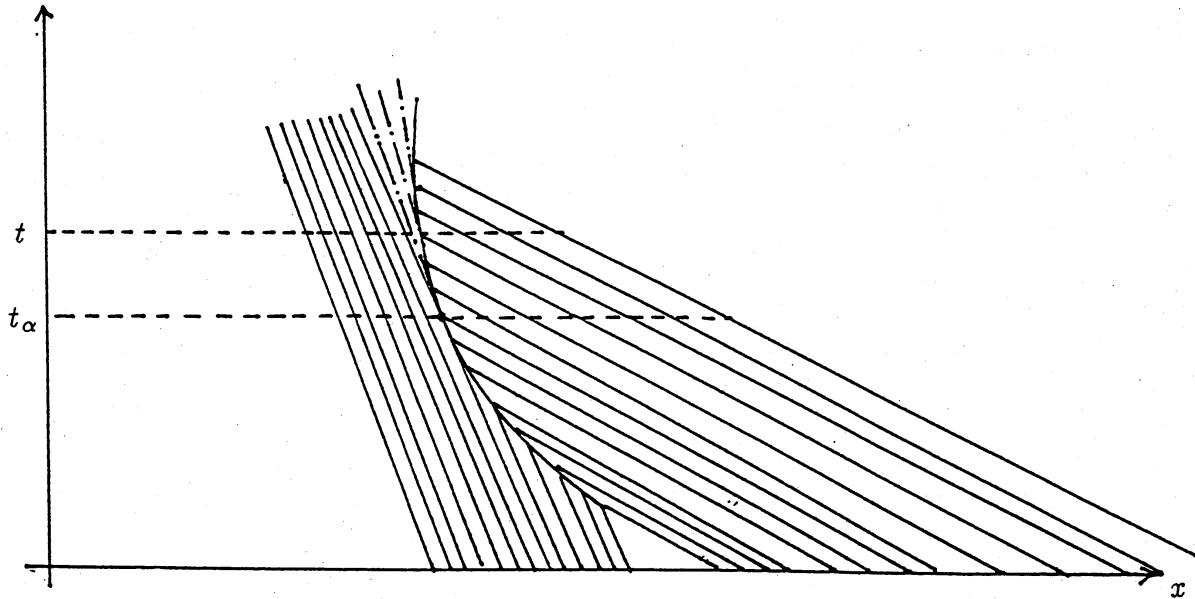We draw the picture which is illustrating the situations as follows :

FIGURE 6

Then we can draw the picture of the graph of the viscosity solution for $t > t_\alpha$ and the shock curve around t_α.
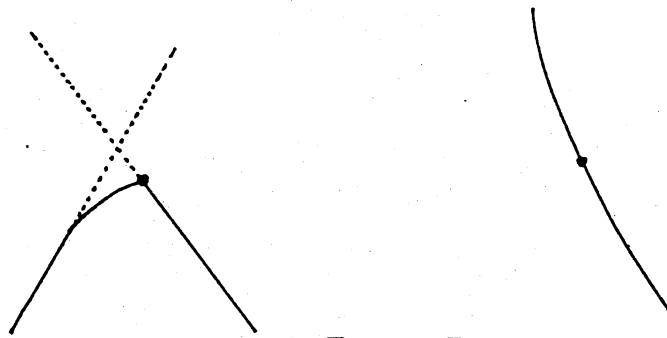

FIGURE 7

Case (6). The bifurcations of the graphs of the geometric solution at the time t_α is depicted as follows:
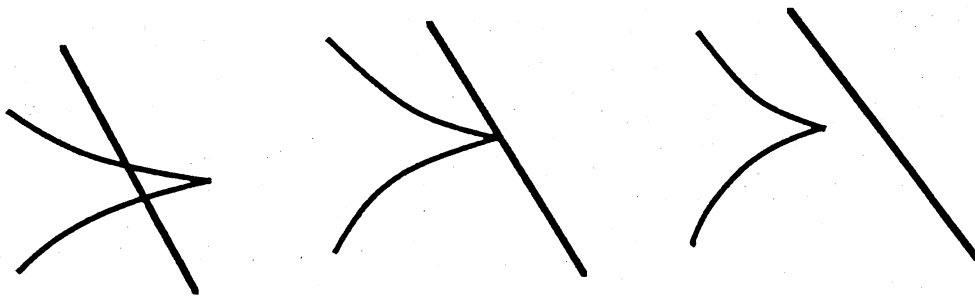

FIGURE 8

We use the same notation as the case (1). Since $u_-(t_\alpha)$ corresponds to the cusp point, we have $1 + t_\alpha H''(\phi(u_-(t_\alpha)))\phi''(u_-(t_\alpha)) = 0$. Let $(t, \sigma(t))$ be the locus of the cusps, where we denote $\sigma(t) = \sigma_(t) + tH'(\phi'(\sigma_-(t)))$ as the family of characteristics come from the left side, so that we have $1 + tH''(\phi'(\sigma_-(t)))\phi''(\sigma_-(t)) = 0$ and $\sigma'_-(t) < 0$. It follows that $H''(\phi'(\sigma_-(t)))\phi''(\sigma_-(t)) < 0$ and $\sigma'(t) = H'(\phi'(\sigma'_-(t)))$. Differentiating the equation again, we get $\sigma''(t) = H''(\phi'(\sigma_-(t)))\phi''(\sigma_-(t))\sigma'_-(t) > 0$. Therefore $(t, \sigma(t))$ is strictly convex. We denote $\chi(t) = u_-(t) + tH'(\phi'(u_-(t))) = u_+(t) + tH'(\phi'(u_+(t)))$ for $t \le t_\alpha$, then we have $u_-(t_\alpha) = \sigma_-(t_\alpha)$, so that $\sigma'(t_\alpha) = H'(\phi'(\sigma_-(t_\alpha))) = H'(\phi(u_-(t_\alpha))) = \chi'(t_\alpha)$ by the proof of Theorem 4.2. We also construct the contact discontinuity $(t, \chi_c(t))$ exactly the same as that of in the case (1). We need examine the following two subcases.

a) Assume that $\sigma(t) \ge \chi_c(t)$ for $t \ge t_\alpha$. Since both χ_c and σ are convex near t_α, we have that $\sigma''(t_\alpha) > \chi''(t_\alpha)$.

On the other hand, we have

$$y(t, \sigma_\pm(t)) = t\{-H(\phi'(\sigma_\pm(t))) + \phi'(\sigma_\pm(t))H'(\phi'(\sigma_\pm(t)))\} + \phi(\sigma_\pm(t)),$$

$$\frac{dy}{dt}(t, \sigma_+(t)) = -H(\phi'(\sigma_+(t))) + \phi'(\sigma_+(t))H'(\phi'(\sigma_+(t)))$$
$$+\phi'(\sigma_+(t))\sigma'_+(t)\{1 + tH''(\phi'(\sigma_+(t)))\phi''(\sigma_+(t))\}$$

and

$$\frac{dy}{dt}(t, \sigma_-(t)) = -H(\phi'(\sigma_-(t))) + \phi'(\sigma_-(t))H'(\phi'(\sigma_-(t))).$$

Let $A(t) = y(t, \sigma_+(t)) - y(t, \sigma_-(t))$ for $t \ge t_\alpha$.

Differentiating the equality $\sigma_-(t) + tH'(\phi'(\sigma_-(t))) = \sigma_+(t) + tH'(\phi'(\sigma_+(t)))$ with respect to t, we get

$$\sigma'_+(t)\{1 + tH''(\phi'(\sigma_+(t)))\phi''(\sigma_+(t))\} = H'(\phi'(\sigma_-(t))) - H'(\phi'(\sigma_+(t))).$$

It follows that

$$A'(t) = H'(\phi(\sigma_-(t)))\{\phi'(\sigma_+(t)) - \phi'(\sigma_-(t))\} - (H(\phi'(\sigma_+(t))) - H(\phi'(\sigma_-(t)))).$$

Furthermore, we have

$$A''(t) = (H'(\phi'(\sigma_-(t)) - H'(\phi'(\sigma_+(t)))\phi''(\sigma_+(t))\sigma'_+(t)$$
$$+H''(\phi'(\sigma_-(t)))\phi''(\sigma_-(t))\{\phi'(\sigma_+(t)) - \phi'(\sigma_-(t))\}\sigma'_-(t).$$

Since $\sigma'_+(t) = \frac{H'(\phi'(\sigma_-(t)) - H'(\phi'(\sigma_+(t)))}{1 + tH''(\phi'(\sigma_+(t))\phi''(\sigma_+(t)))}$ and $\sigma''(t) = H''(\phi'(\sigma_-(t)))\phi''(\sigma_-(t))\sigma'_-(t)$, we have

$$A''(t) = \frac{(H'(\phi'(\sigma_-(t)) - H'(\phi'(\sigma_+(t))))^2}{1 + tH''(\phi'(\sigma_+(t))\phi''(\sigma_+(t))}\phi''(\sigma_+(t)) + \sigma''(t)\{\phi'(\sigma_+(t)) - \phi'(\sigma_-(t))\}.$$

On the other hand, as we already calculated in the case (1) that

$$\chi''_c(t) = -\frac{(H'(\phi'(u_+(t))) - H'(\psi(\phi'(u_+(t)))))^2}{\phi'(u_+(t)) - \psi(\phi'(u_+(t)))}\frac{\phi''(u_+(t))}{1 + tH''(\phi'(u_+(t)))\phi''(u_+(t))}.$$

At the point $t = t_\alpha$, we have $u_\pm(t_\alpha) = \sigma_\pm(t_\alpha)$ and $\psi(\phi'(\sigma_+(t_\alpha))) = \phi'(\sigma_-(t_\alpha))$, so that

$$\chi_c''(t_\alpha) = -\frac{(H'(\phi'(u_+(t_\alpha))) - H'(\psi(\phi'(u_+(t_\alpha)))))^2}{\phi'(u_+(t_\alpha)) - \psi(\phi'(u_+(t_\alpha)))} \frac{\phi''(u_+(t_\alpha))}{1 + t_\alpha H''(\phi'(u_+(t_\alpha))\phi''(u_+(t_\alpha))}.$$

Thus, we have

$$A''(t_\alpha) = (\sigma''(t_\alpha) - \chi_c''(t_\alpha lpha))(\phi'(\sigma_+(t_\alpha) - \phi'(\sigma_-(t_\alpha))).$$

Since $\sigma''(t_\alpha) > \chi_c''(t_\alpha)$ and $\phi'(\sigma_+(t_\alpha) < \phi'(\sigma_-(t_\alpha))$, we have $A''(t_\alpha) < 0$. This means that $A'(t) < 0$ near t_α, so $y(t, \sigma_+(t)) < y(t, \sigma_-(t))$.

We also consider

$$y_+(t) = t\{-H(\phi'(u_+(t))) + \phi'(u_+(t))H'(\phi'(u_+(t)))\} + \phi(u_+(t))$$

and

$$y_\alpha(t) = t\{-H(\phi'(u_\alpha)) + \phi'(u_\alpha)H'(\phi'(u_\alpha))\} + \phi(u_\alpha),$$

where $u_\alpha = \sigma_-(t_\alpha)$ and $x(t, u_\alpha) = u_\alpha + tH'(\phi'(u_\alpha)) = x(t, u_+(t)) = u_+(t) + tH'(\phi'(u_+(t)))$. Differentiating the last equality, we get

$$H'(\phi'(u_\alpha)) = u_+'(t) + H'(\phi'(u_+(t))) + tH''(\phi'(u_+(t)))\phi''(u_+(t))u_+'(t).$$

Then

$$\begin{aligned}
y_+'(t) = &- H(\phi'(u_+(t)) + \phi'(u_+(t))H'(\phi'(u_+(t))) \\
&+ tH''(\phi'(u_+(t)))\phi'(u_+(t))\phi''(u_+(t))u_+'(t) + \phi'(u_+(t))u_+'(t) \\
= &- H(\phi'(u_+(t))) + \phi'(u_+(t))H'(\phi'(u_+(t))).
\end{aligned}$$

So we obtain

$$\frac{d}{dt}(y_\alpha(t) - y_+(t)) = H(\phi'(u_+(t))) - H(\phi'(u_\alpha)) - H'(\phi'(u_\alpha))(\phi'(u_+(t)) - \phi'(u_\alpha)).$$

Since $\phi'(u_+(t)) < \phi'(u_\alpha)$ and $\phi'(u_\alpha)$ is in the convex region of $H(p)$, we have

$$H'(\phi'(u_\alpha)) < \frac{H(\phi'(u_+(t))) - H(\phi'(u_\alpha))}{\phi'(u_+(t)) - \phi'(u_\alpha)},$$

so that we have $\frac{d}{dt}(y_\alpha(t) - y_+(t)) < 0$. This means that $y_\alpha(t) < y_+(t)$ for $t > t_\alpha$.

These situation describes that the two branches of the multi-valued graph have intersection for $t > t - \alpha$. This contradicts to the assumption that the singularity is [1]
b) Here we assume that $\sigma(t) < \chi_c(t)$ for $t > t_\alpha$. In this subcase two shocks bifurcates from the point (t_α, x_α).
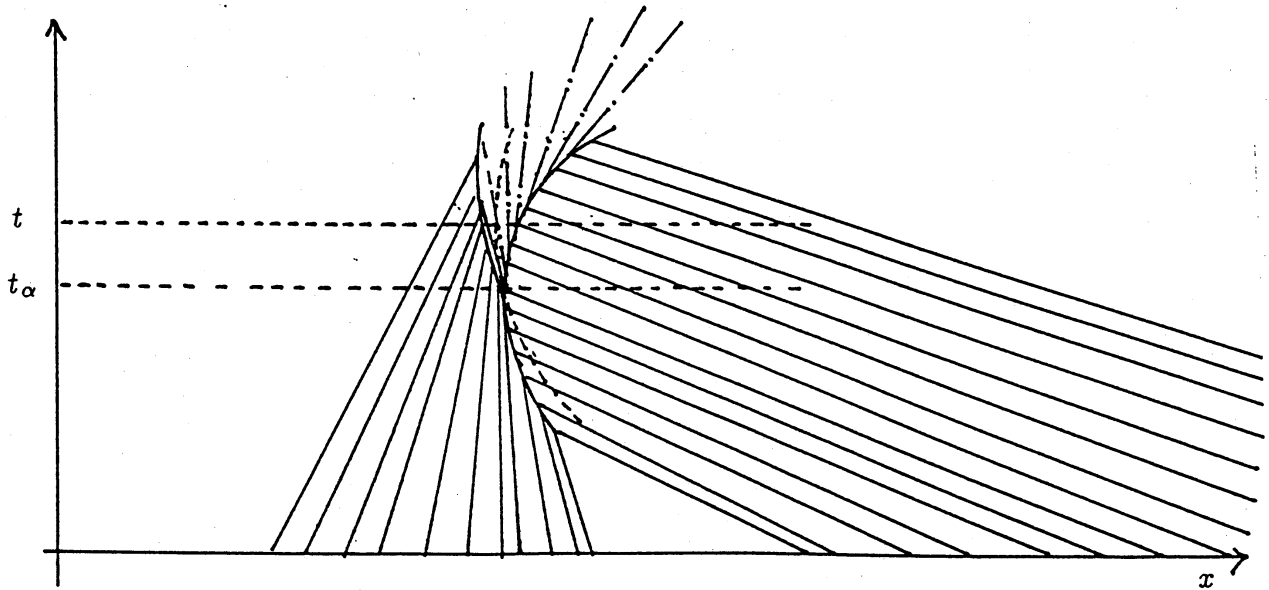
See Figure 9.

FIGURE 9

The left one is a new shock given by the intersection of the original characteristic from the left side and the new characteristic from the contact discontinuity (i.e., *the rarefaction waves*).

By definition, we have

$$\tilde{y}_\alpha(t) = (t - t_\alpha)\{-H(\psi(y_x^+(t_\alpha, \chi_c(t_\alpha)))) + \psi(y_x^+(t_\alpha, \chi_c(t_alpha)))H''(\psi(y_x^+(t_\alpha, \chi_c(t_\alpha))))\}$$
$$+y^+(t_\alpha, \chi(t_\alpha))$$

and

$$y_-(t) = t\{-H(\phi'(u_-(t))) + \phi'(u_-(t))H'(\phi(u_(t)))\} + \phi(u_(t)),$$

where $u_\alpha + tH'(\phi'(u_alpha))) = x(t, u_-(t)) = u_-(t) + tH'(\phi'(t))$.

Differentiating the last equality, we get

$$u'_-(t) + H'(\phi'(u_-(t))) + tH''(\phi'(u_-(t)))\phi''(u_-(t))u'_-(t) = H'(\phi'(u_\alpha)).$$

Then

$$y'_-(t) = - H(\phi'(u_-(t))) + \phi'(u_-(t))H'(\phi(u_(t)))$$
$$+tH''(\phi'(u_-(t)))\phi'(u_(t))\phi''(u_-(t))u'_-(t) + \phi'(u_-(t))u'_-(t)$$
$$= - H(\phi'(u_-(t))) + \phi'(u_-(t))H'(\phi(u_\alpha)$$

So we obtain

$$\frac{d}{dt}(\tilde{y}_\alpha(t) - y_-(t)) = H(\phi'(u_-(t))) - H(\phi'(u_\alpha)) - H'(\phi'(u_alpha))(\phi'(u_-(t)) - \phi'(u_\alpha)).$$

Since $u_-(t) < u_\alpha$ and both of $\phi'(u_(t))$, $\phi'(u_\alpha)$ are in the convex region of $H(p)$, we have $\phi'(u_(t)) > \phi'(u_\alpha))$ and

$$H'(\phi'(u_\alpha)) > \frac{H(\phi'(u_-(t))) - H(\phi'(u_\alpha))}{\phi'(u_-(t)) - \phi'(u_\alpha)},$$

so that we obtain $\frac{d}{dt}(\tilde{y}_\alpha(t) - y_-(t)) < 0$. Since $\tilde{y}_\alpha(t_\alpha) = y_-(t_\alpha)$, the last inequality means that $\tilde{y}_\alpha(t) < y_-(t)$ for $t > t_\alpha$.

We also consider

$$\begin{aligned}
\tilde{y}_\alpha(t, \sigma(t)) =& (t - \tau(t))\{-H(\psi(y_x^+(\tau(t), \chi_c(\tau(t))))) \\
&+ \psi(y_x^+(\tau(t), \chi_c(\tau(t))))H''(\psi(y_x^+(\tau(t), \chi_c(\tau(t)))))\} \\
&+ y_x^+(\tau(t), \chi_c(\tau(t)))
\end{aligned}$$

and

$$y_-(t, \sigma(t)) = t\{-H(\phi'(\sigma_-(t))) + \phi'(\sigma_-(t))H'(\phi(\sigma_-(t))\} + \phi(\sigma_-(t)),$$

where

$$y^+(\tau(t), \chi_c(\tau(t))) = \tau(t)\{-H(\phi'(\tau(t))) + \phi'(\tau(t))H'(\phi(\tau(t))\} + \phi(\tau(t)),$$
$$y_x^+(\tau(t), \chi_c(\tau(t))) = \phi'(u_+(\tau(t)))$$

and

$$\sigma(t) = \sigma_-(t)) + tH'(\phi'(\sigma_-(t)) = \chi_c(\tau(t)) + (t - \tau(t))H'(\psi(\phi'(u_+(t))).$$

Since $\chi_c(\tau) = H'(\psi(\phi'(u_+(\tau))))$ and $1 + tH''(\phi'(\sigma_-(t)))\phi''(\sigma_-(t)) = 0$, differentiating the above equality, we get

$$\begin{aligned}
H'(\phi'(\sigma_-(t))) =& H'(\psi(\phi'(u_+(\tau(t)))) \\
&+ (t - \tau(t))H''(\psi(\phi(u_+(\tau(t))))\frac{d\psi}{dp}\phi'(u_+(\tau(t)))u_+'(\tau(t))\tau'(t).
\end{aligned}$$

Then

$$\begin{aligned}
\tilde{y}'(t, \sigma(t)) =& \tau'(t)\{-H(\phi'(u_+(\tau(t)))) + \phi'(u_+(\tau(t)))H'(\phi'(u_+(\tau(t))))\} \\
&+ \tau(t)\phi'(u_+\tau(t))H''(\phi'(u_+(\tau(t))))\phi''(u_+(\tau(t)))u_+'(\tau(t))\tau'(t) \\
&+ \phi'(u_+(\tau(t)))u_+'(\tau(t))\tau'(t) \\
&+ (1 - \tau'(t))\{-H(\psi(\phi'(u_+(\tau(t))))) + \psi(\phi'(u_+(\tau(t))))H'(\psi(\phi'(u_+(\tau(t)))))\} \\
&+ (t - \tau(t))\psi(\phi'(u_+(\tau(t))))H''(\psi(\phi'(u_+(\tau(t)))))\frac{d\psi}{dp}\phi''(u_+(\tau(t)))\tau'(t) \\
=& \tau'(t)\{H(\psi(\phi'(u_+(\tau(t))))) - H(\phi'(u_+(\tau(t)))) \\
&+ (\phi'(u_+(\tau(t))) - \psi(\phi'(u_+(\tau(t)))))H'(\psi(\phi'(u_+(\tau(t)))))\} \\
&+ \psi(\phi'(u_+(\tau(t))))H'(\phi'(\sigma_-(t)) - H(\psi(\phi'(u_+(\tau(t))))).
\end{aligned}$$

By definition, we have

$$H'(\psi(\phi'(u_+(\tau(t)))) = \frac{H(\phi'(u_+(\tau(t)))) - H(\psi(\phi'(u_+(\tau(t)))))}{\phi'(u_+(\tau(t))) - \psi(\phi'(u_+(\tau(t))))},$$

so that

$$\tilde{y}'(t, \sigma(t)) = \psi(\phi'(u_+(\tau(t))))H'(\phi'(\sigma(t))) - H(\psi(\phi'(u_+(\tau(t))))).$$

Thus we have

$$\frac{d}{dt}(\tilde{y}(t, \sigma(t)) - y_-(t, \sigma(t))) = H'(\phi'(\sigma_t t)) - H(\psi(\phi'(u_+(\tau(t)))))$$

$$- H'(\phi'(\sigma(t)))(\phi'(\sigma_-(t)) - \psi(\phi'(u_+(\tau(t))))).$$

Since both $\phi'(\sigma_-(t))$ and $\psi(\phi'(u_+(\tau(t))))$ belong to the concave region of $H(p)$, $\phi'(\sigma_-(t)) < \psi(\phi'(u_+(\tau(t))))$ for $t > t_\alpha$. Therefore

$$\frac{H'(\phi'(\sigma_t t)) - H(\psi(\phi'(u_+(\tau(t)))))}{\phi'(\sigma_-(t)) - \psi(\phi'(u_+(\tau(t))))} < H'(\phi'(\sigma(t))),$$

hence we have $\frac{d}{dt}(\tilde{y}(t, \sigma(t)) - y_-(t, \sigma(t))) > 0$. Since $\tilde{y}(t_\alpha, \sigma(t_\alpha)) = y_-(t_\alpha, \sigma(t_\alpha))$, the above inequality means that $\tilde{y}(t, \sigma(t)) > y_-(t, \sigma(t))$ for $t > t_\alpha$. It follows that, there exists a unique $(t, \chi_r(t))$ with $x(t, u_\alpha) < \chi_r(t) < \sigma(t)$, such that $\tilde{y}(t, \chi_r(t)) = y_-(t, \chi_r(t))$.

Then we can draw the picture of the graph of the viscosity solution for $t > t_\alpha$ and the bifurcation of the shock curves around t_α. (cf., Figure 10)
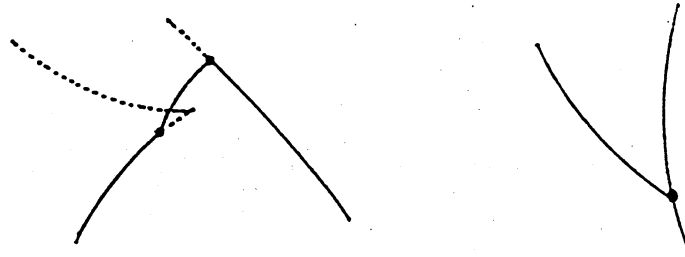

FIGURE 10

For other cases, the detailed discussions will appear in elsewhere.

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