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On unbounded viscosity solutions of nonlinear second order partial differential equations

神戸商船大学　石井　克幸　(Katsuyuki ISHII)
神戸商船大学　冨田　義人　(Yoshihito TOMITA)

§1. Introduction

This is a brief report of [6]. We shall consider the following nonlinear second order partial differential equations (PDEs):

$$(S) \qquad \lambda u - a_{ij}(x)u_{x_i x_j} + H(Du) - f(x) = 0 \quad \text{in} \quad \mathbb{R}^N,$$

$$(E) \qquad \begin{cases} u_t - a_{ij}(x)u_{x_i x_j} + H(Du) = 0 & \text{in} \quad (0,T)\times\mathbb{R}^N, \\ u(0,x) = \psi(x) & \text{in} \quad \mathbb{R}^N, \end{cases}$$

where λ, $T > 0$ are constants, $(a_{ij}(x))$ is a matrix of nonnegative definite, Du denotes the gradient of u with respect to $x \in \mathbb{R}^N$ and f and ψ are given functions.

When a_{ij}, H, f and ψ are smooth, $(a_{ij}(x))$ is positive definite and H grows at most quadratically as $|p| \to +\infty$, there are many papers discussing the classical solutions of the problem (S) and (E). So our interest is the case $(a_{ij}(x))$ is nonnegative and H has more general (possibly super-quadratical) growth.

As regards earlier related works, S. Aizawa - the second author [1] treated the case $a_{ij}(x) = \delta_{ij}$ and proved that, if $H \in C(\mathbb{R}^N)$ and $f \in UC(\mathbb{R}^N)$ (=the set of uniformly continuous functions in $\mathbb{R}^N$), then there exists a viscosity solution in $UC(\mathbb{R}^N)$ and the uniqueness of viscosity solutions holds in the class of continuous functions growing at most linearly as $|x| \to +\infty$. In [1] they gave the following examples showing the failure of the uniqueness of solutions growing superlinearly as $|x| \to +\infty$ even in the class of C^2 solutions.

Example 1. Let $\alpha > 1$ and dehine H by

$$H(p) = \begin{cases} -\left|\dfrac{p}{\alpha}\right|^{\alpha/(\alpha-1)} + \alpha(\alpha-1)\left|\dfrac{p}{\alpha}\right|^{(\alpha-2)/(\alpha-1)} + \dfrac{(N-1)|p|}{\left|\dfrac{p}{\alpha}\right|^{1/(\alpha-1)} - \dfrac{\alpha-2}{\alpha-1}} \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (|p| \geqq \alpha), \\ -\dfrac{|p|^2}{2\alpha(\alpha-1)} + N\alpha(\alpha-1) - \dfrac{\alpha-2}{2(\alpha-1)} \qquad (|p| \leqq \alpha). \end{cases}$$

Then the equation

$$(S_0) \qquad u - \Delta u + H(Du) = 0 \quad \text{in} \quad \mathbb{R}^N$$

has two distinct solutions:

$$u_1(x) = \begin{cases} \left(|x| + \dfrac{\alpha-2}{\alpha-1}\right)^{\alpha} & \left(|x| \geqq \dfrac{1}{\alpha-1}\right), \\ \dfrac{\alpha(\alpha-1)}{2}|x|^2 + \dfrac{\alpha-2}{2(\alpha-1)} & \left(|x| \leqq \dfrac{1}{\alpha-1}\right), \end{cases}$$

$$u_2(x) = -N\alpha(\alpha-1) + \frac{\alpha-2}{2(\alpha-1)}.$$

Example 2. Define H by

$$H(p) = \begin{cases} (1 + (N-1)|p|)\exp(1-|p|) - (|p|-1)\exp(|p|-1) & (|p| \geqq 1), \\ -\dfrac{|p|^2}{2} + N + \dfrac{1}{2} & (|p| \leqq 1). \end{cases}$$

The the equation (S_0) has two distinct solutions:

$$u_1(x) = \begin{cases} |x| \log |x| & (|x| \geqq 1), \\ \dfrac{1}{2}|x|^2 - \dfrac{1}{2} & (|x| \leqq 1), \end{cases}$$

$$u_2(x) = -N - \frac{1}{2}.$$

For general nonlinear second order elliptic PDEs, H. Ishii [5] obtained the comparison principle and existence of viscosity solutions in the class of functions having at most linear growth as $|x| \to +\infty$.

By the above examples, there arises the question whether the linear growth condition is essential to the uniqueness and existence of solutions even if we restrict the behavior of $H(p)$ as $|p| \to +\infty$ and that of $f(x)$ as $|x| \to +\infty$. From this viewpoint, M. G. Crandall - R. Newcomb - the second author [4] investigated the interaction between the growth and continuity properties of H and f and the uniqueness classes for solutions of (S) and they proved the existence of solutions of (S) in such uniqueness classes. We can consider three cases for the structure of H:

(1) H is Lipschitz continuous in $\mathbb{R}^N$.

(2) H is uniformly continuous in $\mathbb{R}^N$.

(3) $|H|$ behaves like $|p|^m$ with $m > 1$.

In the cases (1) and (2) they obtained the sharp growth conditions for the uniqueness of viscosity solutions and showed the existence of viscosity solutions in such classes. In the case (3), it is easily observed that any solution of (S) has at most m'-th order as $|x| \to +\infty$ $(m' = m/m-1)$. However, Example 1 also states that the uniqueness does not hold in the class of functions with m'-th growth. Then they proved the comparison principle of viscosity solutions of (S) in the class of locally Lipschitz continuous functions behaving like $o(|x|^{m'})$ as $|x| \to +\infty$. As to the existence of solutions, they obtained it only in the case where a_{ij} are constants. Without any continuity assumptions, S. Aizawa - the second author [2] obtained the comparison principle and existence of viscosity solutions with the growth of $(m' - \varepsilon)$-th order $(0 < \varepsilon < m')$ for general nonlinear elliptic PDEs. In this result, they also considered only the case where the coefficients of D^2u are constants.

Our main aim here is to obtain the comparison principle and existence of viscosity solutions of (S) and (E) behaving like $o(|x|^{m'})$ as $|x| \to +\infty$. To solve the question for (S) mentioned above completely, we consider the case where a_{ij} are variable and unbounded ones and do not assume any continuity for solutions.

In Section 2 we state our assumptions and recall the notion of viscosity solutions. In Section 3 we establish the comparison principle and existence of viscosity solutions of (S). Section 4 is devoted to the problem (E).

In what follows we surpress the term "viscosity" since we are mainly concerned with viscosity sub-, super- and solutions.

§2. Preliminaries

In this section we shall state our assumptions and recall the notion of solutions of (S) and (E).

We assume the following. Let $m > 1$ and let ${}^tA$ be the transposed matrix of A.

(A.1) There exist Lipschitz continuous functions $\sigma_{ij}(x)$ $(i, j = 1, \cdots N)$ such that

$$(a_{ij}(x)) = {}^t(\sigma_{ij}(x))(\sigma_{ij}(x)) \qquad (\forall x \in \mathbb{R}^N).$$

(A.2) There exists a modulus of continuity ω_H such that

$$|H(p) - H(q)| \leqq \omega_H((1 + |p|^{m-1} + |q|^{m-1})|p - q|)$$

for all p, $q \in \mathbb{R}^N$.

(A.3) There exist a modulus of continuity ω_f, and a function θ_f satisfying $\theta_f(r) \to 0$ as $r \to +\infty$ such that

$$|f(x) - f(y)| \leqq \omega_f((1 + \theta_f(|x|)|x|^{m'-1} + \theta_f(|y|)|y|^{m'-1})|x - y|)$$

for all x, $y \in \mathbb{R}^N$.

(A.4) There exist a modulus of continuity ω_ψ, and a function θ_ψ satisfying $\theta_\psi(r) \to 0$ as $r \to +\infty$ such that

$$|\psi(x) - \psi(y)| \leqq \omega_\psi((1 + \theta_\psi(|x|)|x|^{m'-1} + \theta_\psi(|y|)|y|^{m'-1})|x - y|)$$

for all x, $y \in \mathbb{R}^N$.

Remark 2.1. (1) We call a function ω on $[0, +\infty)$ a modulus of continuity when it can be represented as $\omega(r) = \inf\{M_\gamma r + \gamma \mid \gamma > 0\}$ for a set of nonnegative numbers $\{M_\gamma\}_{\gamma>0}$.
(2) (A.1) implies that $a_{ij}(x)$ have at most quadratic growth as $|x| \to +\infty$ and that there exists a constant $K_0 > 0$ such that

$$a_{ij}(y)Y_{ij} - a_{ij}(x)X_{ij} \leqq K_0\alpha|x-y|^2$$

for all $\alpha > 0$, x, $y \in \mathbb{R}^N$ and X, $Y \in \mathbb{S}^N$ satisfying

$$-3\alpha\begin{pmatrix} I & O \\ O & I \end{pmatrix} \leqq \begin{pmatrix} X & O \\ O & -Y \end{pmatrix} \leqq 3\alpha\begin{pmatrix} I & -I \\ -I & I \end{pmatrix}.$$

In the following we put $K = K_0 \vee \sup\{|a_{ij}(x))|/(1+|x|^2) \mid x \in \mathbb{R}^N,\ 1 \leqq i,j \leqq N\}$, where $a \vee b = \max\{a, b\}$.

Before recalling the notion of solutions of (S) and (E), we prepare some notations. For $u : \mathbb{R}^N \to \mathbb{R}$, we define the upper semicontinuous (u.s.c.) envelope u^* and the lower semicontinuous (l.s.c.) envelope u_* of u by

$$u^*(x) = \lim_{r\to 0} \sup\{u(y) \mid y \in \mathbb{R}^N, |y-x| < r\},\ u_*(x) = -(-u(x))^*.$$

Let $\langle \cdot, \cdot \rangle$ be the Euclidian inner product in $\mathbb{R}^N$ and let $\mathbb{S}^N$ be the set of all $N \times N$ real symmetric matrices. We denote by $J^{2,+}u(x)$, $J^{2,-}u(x)$ the super and the sub 2-jet of u at $x \in \mathbb{R}^N$, respectively:

$$J^{2,+}u(x) = \left\{(p, X) \in \mathbb{R}^N \times \mathbb{S}^N \;\middle|\; u(x+h) \leqq u(x) + \langle p, h\rangle + \frac{1}{2}\langle Xh, h\rangle + o(|h|^2) \quad \text{as } |h| \to 0\right\},$$

$$J^{2,-}u(x) = \left\{(p, X) \in \mathbb{R}^N \times \mathbb{S}^N \;\middle|\; u(x+h) \geqq u(x) + \langle p, h\rangle + \frac{1}{2}\langle Xh, h\rangle + o(|h|^2) \quad \text{as } |h| \to 0\right\}.$$

$\bar{J}^{2,+}u(x)$ and $\bar{J}^{2,-}u(x)$ are the graph closures of $J^{2,+}u(x)$ and $J^{2,-}u(x)$, respectively.

For $u : [0,T) \times \mathbb{R}^N \to \mathbb{R}$, we similarly define the u.s.c. envelope u^* and the l.s.c. envelope u_* of u by

$$u^*(t,x) = \lim_{r \to 0} \sup\{u(s,y) \mid (s,y) \in [0,T) \times \mathbb{R}^N, |s-t| + |y-x| < r\},$$

and $u_*(t,x) = -(-u(t,x))^*$. We denote by $\mathcal{P}^{2,+}u(t,x)$, $\mathcal{P}^{2,-}u(t,x)$ the parabolic super and the parabolic sub 2-jet of u at $(t,x) \in (0,T) \times \mathbb{R}^N$, respectively:

$$\mathcal{P}^{2,+}u(t,x) = \left\{(\tau,p,X) \in \mathbb{R} \times \mathbb{R}^N \times \mathbb{S}^N \,\middle|\, u(t+r,x+h) \leqq u(t,x) + \tau r + \langle p,h\rangle \right.$$
$$\left. + \frac{1}{2}\langle Xh,h\rangle + o(|r| + |h|^2) \quad \text{as } t+r \in (0,T), \text{ and } r, |h| \to 0\right\},$$
$$\mathcal{P}^{2,-}u(t,x) = \left\{(\tau,p,X) \in \mathbb{R} \times \mathbb{R}^N \times \mathbb{S}^N \,\middle|\, u(t+r,x+h) \geqq u(t,x) + \tau r + \langle p,h\rangle \right.$$
$$\left. + \frac{1}{2}\langle Xh,h\rangle + o(|r| + |h|^2) \quad \text{as } t+r \in (0,T), \text{ and } r, |h| \to 0\right\}.$$

$\bar{\mathcal{P}}^{2,+}u(t,x)$ and $\bar{\mathcal{P}}^{2,-}u(t,x)$ are the graph closures of $\mathcal{P}^{2,+}u(t,x)$ and $\mathcal{P}^{2,-}u(t,x)$, respectively.

Definition 2.2. *Let $u : \mathbb{R}^N \to \mathbb{R}$.*

(1) *We say u is a subsolution of (S) provided $u^*(x) < +\infty$ $(\forall x \in \mathbb{R}^N)$ and for all $x \in \mathbb{R}^N$ and $(p,X) \in \bar{J}^{2,+}u^*(x)$, u^* satisfies*

$$\lambda u^*(x) - a_{ij}(x)X_{ij} + H(p) - f(x) \leqq 0.$$

(2) *We say u is a supersolution of (S) provided $u_*(x) > -\infty$ $(\forall x \in \mathbb{R}^N)$ and for all $x \in \mathbb{R}^N$ and $(p,X) \in \bar{J}^{2,-}u_*(x)$, u_* satisfies*

$$\lambda u_*(x) - a_{ij}(x)X_{ij} + H(p) - f(x) \geqq 0.$$

(3) *We say u is a solution of (S) provided u is a sub- and a supersolution of (S).*

Definition 2.3. *Let $u : [0,T) \times \mathbb{R}^N \to \mathbb{R}$.*

(1) *We say u is a subsolution of (E) provided* $u^*(t,x) < +\infty$ $(\forall(t,x) \in [0,T) \times \mathbb{R}^N)$ *and for all* $(t,x) \in (0,T) \times \mathbb{R}^N$ *and* $(\tau, p, X) \in \bar{\mathcal{P}}^{2,+}u^*(t,x)$, u^* *satisfies*

$$\tau - a_{ij}(x)X_{ij} + H(p) \leqq 0.$$

(2) *We say u is a supersolution of (E) provided* $u_*(t,x) > -\infty$ $(\forall(t,x) \in [0,T) \times \mathbb{R}^N)$ *and for all* $(t,x) \in (0,T) \times \mathbb{R}^N$ *and* $(\tau, p, X) \in \bar{\mathcal{P}}^{2,-}u_*(x)$, u_* *satisfies*

$$\tau - a_{ij}(x)X_{ij} + H(p) \geqq 0.$$

(3) *We say u is a solution of (E) provided u is a sub- and a supersolution of (E).*

For the equivalent definitions to Definition 2.2 and 2.3, see [4; Section 2].

§3. The problem (S)

In this section we shall establish the comparison principle and existence of solutions of the problem (S).

The comparison principle is stated as follows.

Theorem 3.1. *Assume* (A.1) - (A.3). *Moreover assume* $\lambda \geqq \lambda_0$ *for some* $\lambda_0 = \lambda_0(N, K, m) > 0$. *Let u and v be a subsolution and a supersolution of (S), respectively. If u and v satsify*

$$\limsup_{|x| \to +\infty} \frac{u^*(x)}{|x|^{m'}} \leqq 0 \leqq \liminf_{|x| \to +\infty} \frac{v_*(x)}{|x|^{m'}}, \tag{3.1}$$

then there exists a modulus of continuity $\tilde{\omega}$ *such that*

$$u^*(x) - v_*(y) \leqq \tilde{\omega}((1 + \theta_f(|x|)|x|^{m'-1} + \theta_f(|y|)|y|^{m'-1})|x-y|)$$

for all $x, y \in \mathbb{R}^N$. *Especially,* $u^* \leqq v_*$ *in* $\mathbb{R}^N$.

Remark 3.2. It is seen by Example 1 and the fact mentioned in Section 1 that the condition (3.1) is optimal.

Outline of proof. We may assume u (resp., v) is u.s.c (resp., l.s.c.) in $\mathbb{R}^N$. By (A.2) and (A.3), for any $\gamma > 0$, there exist constants L_γ, $M_\gamma > 0$ satisfying

$$(3.2) \qquad |H(p) - H(p)| \leqq \gamma + L_\gamma(1 + |p|^{m-1} + |q|^{m-1})|p - q|$$

$$(3.3) \qquad |f(x) - f(y)| \leqq \gamma + M_\gamma(1 + \theta_f(|x|)|x|^{m'-1} + \theta_f(|y|)|y|^{m'-1})|x - y|.$$

Remarking (A.3), for any $\delta > 0$, there exists a constant $M_\delta > 0$ such that

$$|f(x) - f(y)| \leqq \gamma + M_\gamma\{1 + M_\delta + \delta(|x|^{m'-1} + |y|^{m'-1})\}|x - y|.$$

Moreover, we have, for any δ, ε, $\sigma \in (0, 1)$

$$(3.4) \qquad |f(x) - f(y)| \leqq \frac{M_\gamma^{m'}}{m'\varepsilon}\langle x - y\rangle_\sigma^{m'} + \frac{(3\varepsilon)^{m-1}}{m}\delta^m(\langle x\rangle_1^{m'} + \langle y\rangle_1^{m'}) + \gamma + \frac{(3\varepsilon)^{m-1}}{m}(1 + M_\delta)^m,$$

where $\langle x\rangle_\rho^2 = |x|^2 + \rho^2$ for $\rho > 0$.

Let $\Phi(x, y)$ be the function defined by

$$\Phi(x, y) = u(x) - v(y) - \left\{\frac{M_\gamma^{m'}}{m'\varepsilon}\langle x - y\rangle_\sigma^{m'} + \frac{(3\varepsilon)^{m-1}}{m}\delta^m(\langle x\rangle_1^{m'} + \langle y\rangle_1^{m'})\right\}.$$

and let $(\bar{x}, \bar{y}) \in \mathbb{R}^N \times \mathbb{R}^N$ be a maximum point of Φ. Using the maximum principle (cf. [3; Theorem 3.2]), for each $\mu > 0$, there exist X, $Y \in \mathbb{S}^N$ such that

$$(p, X) \in \bar{J}^{2,+}\left(u(\bar{x}) - \frac{(3\varepsilon)^{m-1}}{m}\delta^m\langle\bar{x}\rangle_1^{m'}\right),$$

$$(p, Y) \in \bar{J}^{2,-}\left(v(\bar{y}) + \frac{(3\varepsilon)^{m-1}}{m}\delta^m\langle\bar{y}\rangle_1^{m'}\right),$$

$$-\left(\frac{1}{\mu} + \|A\|\right)\begin{pmatrix} I & O \\ O & I \end{pmatrix} \leqq \begin{pmatrix} X & O \\ O & -Y \end{pmatrix} \leqq A + \mu A^2,$$

where

$$p = \frac{M_\gamma^{m'}}{\varepsilon}\langle \bar{x}-\bar{y}\rangle_\sigma^{m'-2}(\bar{x}-\bar{y}),$$

$$A = \frac{M_\gamma^{m'}}{\varepsilon}\left\{(m'-2)\langle \bar{x}-\bar{y}\rangle_\sigma^{m'-4}\begin{pmatrix}(\bar{x}-\bar{y})\otimes(\bar{x}-\bar{y}) & -(\bar{x}-\bar{y})\otimes(\bar{x}-\bar{y})\\ -(\bar{x}-\bar{y})\otimes(\bar{x}-\bar{y}) & (\bar{x}-\bar{y})\otimes(\bar{x}-\bar{y})\end{pmatrix}\right.$$
$$\left. +\langle \bar{x}-\bar{y}\rangle_\sigma^{m'-2}\begin{pmatrix} I & -I\\ -I & I\end{pmatrix}\right\}.$$

Setting $\alpha = M_\gamma^{m'}((m'-1)\vee 2)\langle \bar{x}-\bar{y}\rangle_\sigma^{m'-2}/\varepsilon$ and $\mu = 1/\alpha$, we obtain

$$-3\alpha\begin{pmatrix} I & O\\ O & I\end{pmatrix} \leqq \begin{pmatrix} X & O\\ O & -Y\end{pmatrix} \leqq 3\alpha\begin{pmatrix} I & -I\\ -I & I\end{pmatrix}. \tag{3.5}$$

By the way, since u and v are, respectively, a subsolution and a supersolution of (S), the following inequalities hold:

$$\lambda u(\bar{x}) - a_{ij}(\bar{x})\left(X + \frac{(3\varepsilon)^{m-1}}{m}\delta^m D^2\langle \bar{x}\rangle_1^{m'}\right)_{ij} + H\left(p + \frac{(3\varepsilon)^{m-1}}{m}\delta^m D\langle \bar{x}\rangle_1^{m'}\right) - f(\bar{x}) \leqq 0,$$
$$\lambda v(\bar{y}) - a_{ij}(\bar{y})\left(Y - \frac{(3\varepsilon)^{m-1}}{m}\delta^m D^2\langle \bar{y}\rangle_1^{m'}\right)_{ij} + H\left(p - \frac{(3\varepsilon)^{m-1}}{m}\delta^m D\langle \bar{y}\rangle_1^{m'}\right) - f(\bar{y}) \geqq 0.$$

Using (A.1), (3.2), (3.4), (3.5) and the above inequalities and calculating carefully, we have

$$\begin{aligned}\lambda(u(\bar{x}) - v(\bar{y})) \leqq\ & \frac{\{Km'((m'-1)\vee 1)+2\}M_\gamma^{m'}}{\varepsilon}\langle \bar{x}-\bar{y}\rangle_\sigma^{m'}\\ &+ \frac{(Km'((m'-2+N)\vee N)+2)(3\varepsilon)^{m-1}}{m}\delta^m(\langle \bar{x}\rangle_1^{m'} + \langle \bar{y}\rangle_1^{m'})\\ &+ 2\gamma + \frac{(3\varepsilon)^{m-1}}{m} + \frac{(3\varepsilon)^{m-1}}{m}(1+M_\delta)^m.\end{aligned}$$

Hence setting $\lambda_0 = \{Km'((m'-1)\vee 2)+1\}\vee\{Km'((m'-2+N)\vee N)+2\}$, if $\lambda \geqq \lambda_0$, then we conclude that

$$\Phi(x,y) \leqq \Phi(\bar{x},\bar{y}) \leqq 2\gamma + \frac{(3\varepsilon)^{m-1}}{m} + \frac{(3\varepsilon)^{m-1}}{m}(1+M_\delta)^m \qquad (\forall x, y \in \mathbb{R}^N),$$

for all σ, $\varepsilon \in (0,1)$ and $\delta \in (0,\delta_\gamma]$ $(0 < \delta_\gamma << 1)$. Thus, letting $\sigma \to 0$, we obtain

$$\begin{aligned}u(x) - v(y) \leqq\ & \frac{M_\gamma^{m'}}{m'\varepsilon}|x-y|^{m'} + \frac{(3\varepsilon)^{m-1}}{m}\delta^m(|x|^{m'} + |y|^{m'})\\ &+ \frac{(3\varepsilon)^{m-1}}{m}(1+M_\delta)^m + 2\gamma + \frac{(3\varepsilon)^{m-1}}{m} + \frac{(3\varepsilon)^{m-1}}{m}C\delta^m.\end{aligned}$$

By careful calculations, we have the result. ∎

Remark 3.3. If $\omega_G(r) = L_G r$ and $\omega_f(r) = M_f r$ for some L_G, $M_f > 0$, then $\tilde{\omega}(r) = Kr$ for some $K > 0$. Thus a solution u of (S) is locally Lipshitz continuous in $\mathbb{R}^N$ and $Du(x) = o(|x|^{m'-1})$ as $|x| \to +\infty$ except for a set of N-dim Lebesgue measure 0.

We conclude this section by proving the existence result.

Theorem 3.4. *Assume* (A.1) - (A.3). *Moreover, assume $\lambda \geqq \lambda_0$, where λ_0 is the same constant as that in Theorem 3.1. Then there exists a unique solution $u \in C(\mathbb{R}^N)$ satisfying*

$$\lim_{|x| \to +\infty} \frac{u(x)}{|x|^{m'}} = 0.$$

Outline of proof. By (A.2) we can find a constant $L > 0$ such that

$$|H(p)| \leqq L|p|^m + 1 \qquad (\forall p \in \mathbb{R}^N). \tag{3.6}$$

It follows from (A.3) that, for any $\delta > 0$, there exists a constant $M_\delta > 0$ such that

$$|f(x)| \leqq \delta \langle x \rangle_1^{m'} + M_\delta \qquad (\forall x \in \mathbb{R}^N). \tag{3.7}$$

Let $u^\delta(x) = \delta \langle x \rangle_1^{m'} + C_\delta$ for $\delta > 0$ and $x \in \mathbb{R}^N$. Then, using (3.6) and (3.7), we observe that, for all $\delta \in (0, \delta_0)$ ($\delta_0 = \delta_0(L, m)$ is small.), u^δ is a classical supersolution of (S). We put

$$\overline{u}(x) = \inf\{u^\delta(x) \mid 0 < \delta < \delta_0\}.$$

Then $\overline{u}$ is a u.s.c. supersolution of (S) and satisfies

$$\lim_{|x| \to +\infty} \frac{\overline{u}(x)}{|x|^{m'}} = \lim_{|x| \to +\infty} \frac{\overline{u}_*(x)}{|x|^{m'}} = 0.$$

Similarly we can find a subsolution $\underline{u}$ of (S) satisfying

$$\lim_{|x| \to +\infty} \frac{\underline{u}(x)}{|x|^{m'}} = \lim_{|x| \to +\infty} \frac{\underline{u}^*(x)}{|x|^{m'}} = 0.$$

Hence we can apply Perron's method and Theorem 3.1 to complete the proof. ∎

Remark 3.5. If $a_{ij}(x)$ $(i,j = 1,\cdots,N)$ are constants, then we do not need the largeness assumption for $\lambda > 0$ in Theorems 3.1 and 3.4. See [2; Section 3].

§4. The problem (E)

This section is devoted to the comparison principle and existence of solutions of the problem (E). For any $\lambda > 0$, by setting $v(t,x) = e^{-\lambda t}u(t,x)$, we can observe that the problem (E) is equivalent to the following problem:

$$(\hat{E}) \qquad \begin{cases} v_t + \lambda v - a_{ij}(x)v_{x_i x_j} + \hat{H}(Dv) = 0 & \text{in } (0,T)\times \mathbb{R}^N, \\ v(0,x) = \psi(x) & \text{in } \mathbb{R}^N, \end{cases}$$

where $\hat{H}(p) = e^{-\lambda t}H(e^{\lambda t}p)$. We note that $\hat{H}$ satisfies (A.2) by replacing ω_H with $e^{\lambda(m-1)T}\omega_H$. In what follows we consider the problem $(\hat{E})$ with $\lambda = \lambda_0$ For simplicity, we set $H = \hat{H}$ and call the problem $(\hat{E})$ the problem (E).

First we mention the comparison principle.

Theorem 4.1. *Let $T > 0$. Assume* (A.1), (A.2) *and* (A.4). *Let u and v be a subsolution and a supersolution of (E), respectively. If u and v satisfy*

$$u^*(0,x) \leqq \psi(x) \leqq v_*(0,x) \qquad (\forall x \in \mathbb{R}^N)$$
$$\limsup_{|x|\to+\infty} \frac{u^*(t,x)}{|x|^{m'}} \leqq 0 \leqq \liminf_{|x|\to+\infty} \frac{v_*(t,x)}{|x|^{m'}} \qquad \textit{uniformly in } t \in (0,T),$$

then there exists a modulus of continuity $\tilde{\omega}$ such that

$$u^*(t,x) - v_*(t,y) \leqq \tilde{\omega}((1+\theta_\psi(|x|)|x|^{m'-1} + \theta_\psi(|y|)|y|^{m'-1})|x-y|)$$

for all $t \in [0,T)$, x, $y \in \mathbb{R}^N$. Especially $u^* \leqq v_*$ in $[0,T)\times\mathbb{R}^N$.

Outline of proof. Since the strategy of the proof is quite similar to that of the proof of Theorem 3.1, we point out the differences.

We may assume u (resp., v) is u.s.c. (resp., l.s.c.) in $[0,T)\times\mathbb{R}^N$. We can estimate H and ψ similarly to (3.2) and (3.4), respectively.

For δ, ε, σ, $\eta \in (0,1)$, define the function $\Phi(t,x,y)$ on $[0,T)\times\mathbb{R}^N\times\mathbb{R}^N$ by

$$\Phi(t,x,y) = u(t,x)-v(t,y)-\left\{\frac{M_\gamma^{m'}}{m'\varepsilon}\langle x-y\rangle_\sigma^{m'} + \frac{(3\varepsilon)^{m-1}}{m}\delta^m(\langle x\rangle_1^{m'}+\langle y\rangle_1^{m'}) + \frac{\eta}{T-t}\right\},$$

Let $(\bar{t},\bar{x},\bar{y})\in(0,T)\times\mathbb{R}^N\times\mathbb{R}^N$ be a maximum point of Φ. By applying the maximum principle (cf. [3; Theorem 8.3]), there exist X, $Y\in\mathbb{S}^N$ such that

$$\begin{aligned}
&(\tau,p,X)\in\bar{\mathcal{P}}^{2,+}\left(u(\bar{x})-\frac{(3\varepsilon)^{m-1}}{m}\delta^m\langle\bar{x}\rangle_1^{m'}\right),\\
&(v,p,Y)\in\bar{\mathcal{P}}^{2,-}\left(v(\bar{y})+\frac{(3\varepsilon)^{m-1}}{m}\delta^m\langle\bar{y}\rangle_1^{m'}\right),\\
&-\left(\frac{1}{\mu}+\|A\|\right)\begin{pmatrix} I & O\\ O & I\end{pmatrix}\leqq\begin{pmatrix} X & O\\ O & -Y\end{pmatrix}\leqq A+\mu A^2,\\
&\tau-v=\frac{\eta}{(T-\bar{t})^2},
\end{aligned}$$

where p and A are the same vector and matrix, respectively, as in the proof of Theorem 3.1.

Since u and v are, a subsolution and a supersolution of (E), respectively, we have the following inequalities:

$$\tau+\lambda u(\bar{t},\bar{x})-a_{ij}(\bar{x})\left(X+\frac{(3\varepsilon)^{m-1}}{m}\delta^m D^2\langle\bar{x}\rangle_1^{m'}\right)_{ij}+H\left(p+\frac{(3\varepsilon)^{m-1}}{m}\delta^m D\langle\bar{x}\rangle_1^{m'}\right)\leqq 0,$$

$$v+\lambda v(\bar{t},\bar{y})-a_{ij}(\bar{y})\left(Y-\frac{(3\varepsilon)^{m-1}}{m}\delta^m D^2\langle\bar{y}\rangle_1^{m'}\right)_{ij}+H\left(p-\frac{(3\varepsilon)^{m-1}}{m}\delta^m D\langle\bar{y}\rangle_1^{m'}\right)\geqq 0.$$

The remainder is totally similar to that in the proof of Theorem 3.1 and hence the proof is complete. ∎

Finally we mention the existence of solutions.

Theorem 4.2. *Assume* (A.1), (A.2) *and* (A.4). *Then there exists a unique solution* $u\in C([0,T)\times\mathbb{R}^N)$ *satisfying*

$$u(0,x)=\psi(x)\quad(\forall x\in\mathbb{R}^N),\quad \lim_{|x|\to+\infty}\frac{u(t,x)}{|x|^{m'}}=0\quad\textit{uniformly in } t\in(0,T).$$

Outline of proof. By (A.4) we have the same estimate on ψ as (3.7).

Let $u^\delta(t,x) = \delta\langle x\rangle_1^{m'} + M_\delta + Ct$ for $\delta > 0$ and $(t,x) \in [0,T) \times \mathbb{R}^N)$. By the similar argument to that in the proof of Theorem 3.5 we see that u^δ is a supersolution of (E) for $\delta \in (0,\delta_0)$ and some constant $C > 0$. Hence setting

$$\overline{u}(t,x) = \inf\{u^\delta(t,x) \mid 0 < \delta < \delta_0\},$$

we conclude $\overline{u}$ is also a supersoluiton of (E). Moreover, we have

$$\overline{u}_*(0,x) \geqq \psi(x) \quad (\forall x \in \mathbb{R}^N), \quad \lim_{|x|\to+\infty} \frac{\overline{u}_*(t,x)}{|x|^{m'}} = 0 \quad \text{uniformly in } t \in [0,T).$$

In the similar way we can find a subsolution $\underline{u}$ satisfying

$$\underline{u}^*(0,x) \leqq \psi(x) \quad (\forall x \in \mathbb{R}^N), \quad \lim_{|x|\to+\infty} \frac{\underline{u}^*(x)}{|x|^{m'}} = 0 \quad \text{uniformly in } t \in [0,T).$$

Therefore, by the barrier constriction argument, Perron's method and Theorem 4.1, there exists a solution u of (E) satisfying

$$\begin{gathered} u^*(0,x) \leqq \psi(x) \qquad (\forall x \in \mathbb{R}^N), \\ \lim_{|x|\to+\infty} \frac{u_*(t,x)}{|x|^{m'}} = \lim_{|x|\to+\infty} \frac{u(t,x)}{|x|^{m'}} = \lim_{|x|\to+\infty} \frac{u^*(t,x)}{|x|^{m'}} = 0 \\ \text{uniformly in } t \in [0,T). \end{gathered}$$

Since we can show that $u_*(0,x) \geqq \psi(x)$ for all $x \in \mathbb{R}^N$, we conclude by Theorem 4.1 that u is a unique solution of (E) satisfying (4.1) and therefore $u(0,x) = \psi(x)$ for all $x \in \mathbb{R}^N$. ∎

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