数理解析研究所講究録
913 巻 1995 年 90-107

The curve shortening equation and its generalizations

Masayoshi Tsutsumi 堤 正義

Department of Applied Physics, Waseda University
早稲田大学理工学部

1 Introduction

We consider evolution equations for closed curves $\mathcal{C}(t),\ t>0$ in $\mathbb{R}^2$ moving with curvature dependent speed.

Let $\mathcal{C}(t):\sigma\mapsto(x(\sigma,t),y(\sigma,t)):S^1\to\mathbb{R}^2$ be a smooth closed curve, which is governed by

$$\frac{\partial\mathcal{C}}{\partial t}=g(\kappa)\mathcal{N} \tag{1}$$

where κ is the curvature of $\mathcal{C}$, $\mathcal{N}$ is its unit inward normal vector and g is a given nondecreasing function on $\mathbb{R}$. Let s be the arclength parameter:

$$ds=v(\sigma,t)\,d\sigma,\qquad v(\sigma,t)=\sqrt{(x_\sigma)^2+(y_\sigma)^2}.$$

We can rewrite (1) as the system

$$x_t=-g(\kappa)\frac{y_\sigma}{v}=\frac{g(\kappa)}{\kappa}x_{ss},\quad y_t=g(\kappa)\frac{x_\sigma}{v}=\frac{g(\kappa)}{\kappa}y_{ss}$$

where

$$\kappa=\frac{y_{\sigma\sigma}x_\sigma-x_{\sigma\sigma}y_\sigma}{v(\sigma,t)^3}=y_{ss}x_s-x_{ss}y_s$$

and

$$\frac{\partial}{\partial s}=\frac{1}{v}\frac{\partial}{\partial\sigma}.$$

When $g(\kappa)=\kappa$, (1) is called the curve shortening equation. In this case, the behaviour of the curves governed by (1) has been studied by many authors, see [7], [20], [23], [14].

Although the curve shortening equation originally arises in the pure mathematical field, there are many applications. For example, it is related to physical phenomena such as crystal growth and flame propagation (see [29] and its reference). As to the engineering, it is used for shape representation and recognition ([26], [27], [28]).

Our aim of this lecture is to present some extensions of previous results of the curve shortening equation to eq.(1) and to investigate global behaviour of solutions of (1) for smooth immersed convex closed curves in the plane.

It will be shown that the convex curves determined by eq. (1) do not necessarily shrink, but may expand to infinity under some assumptions on g. We also consider the case when the initial curve contains line segments and investigate whether or not line segments disappear instantaneously. We will show that they do not disappear instantaneously in some cases.

The organization of this talk is as follows:

In Section 2 we consider evolution equations for the curvature κ and the arclength density of the curves. We will establish the short time existence results of smooth (and weak) solutions of eq. (1) by solving the equations of the curvature and the arclength density.

Section 3 is devoted to show that the curves governed by eq. (1) may shrink to a round point in a finite time, or expand roundly to infinity in a finite (or infinite) time.

In Section 4, we rewrite evolution equations for the curvature in terms of the Gauss map to get nonlinear degenerate parabolic equations of non divergent form. In the following subsections we investigate regional blow up of solutions of the nonlinear degenerate parabolic equation and the behaviour of them near blow-up time. We will prove that the solutions may blow up regionally, but not on the whole interval. It means that the corresponding immersed convex closed curve becomes singular before shrinking to a point In these subsection our proof is inspired by Angenent [7] as well as Friedman-McLeod [19], but is much simpler and more precise than them.

Precise statements and most of proofs will be appeared elsewhere.

2 The evolution equation of curvature

The evolution of the curve $\mathcal{C}(t)$ can be expressed in terms of the curvature $\kappa(\sigma, t)$ and the arclength density $v(\sigma, t)$ in the following manner (see Gage and Hamilton [20]):

$$\begin{aligned} \frac{\partial \kappa}{\partial t} &= \frac{\partial^2}{\partial s^2} g(\kappa) + g(\kappa)\kappa^2 \qquad \sigma \in S^1, \quad t > 0, && (2)\\ \frac{\partial v}{\partial t} &= -g(\kappa)\kappa\, v \qquad \sigma \in S^1, \quad t > 0. && (3)\end{aligned}$$

with prescribed initial data

$$\begin{aligned} \kappa(\sigma, 0) &= \kappa_0(\sigma), \qquad \sigma \in S^1, && (4)\\ v(\sigma, 0) &= v_0(\sigma), \qquad \sigma \in S^1. && (5)\end{aligned}$$

If $\kappa(\sigma, t)$ and $v(\sigma, t)$ solve the initial value problem for (2)-(5), the curve the curve is completely determined up to a translation and a rotation. The inversion is given by

$$\mathcal{C}(t)(\sigma) = \mathcal{C}(0)(\sigma) + \int_0^\sigma \mathcal{T}(\alpha, t)\, v(\alpha, t)\, d\alpha, \qquad \sigma \in S^1, \quad t > 0 \tag{6}$$

where $\mathcal{T}(\sigma,t)$ can be obtained by solving

$$\frac{\partial \mathcal{T}}{\partial t} - \frac{1}{v^2\,\kappa}\frac{\partial g(\kappa)}{\partial \sigma}\frac{\partial \mathcal{T}}{\partial \sigma} = 0, \qquad \sigma \in S^1, \quad t > 0 \tag{7}$$

with

$$\mathcal{T}(\sigma,0) = \mathcal{T}_0(\sigma), \qquad \sigma \in S^1. \tag{8}$$

Note that $\mathcal{T}(\sigma,t)$ is given by

$$\mathcal{T}(\sigma,t) = \mathcal{T}_0(\gamma(t\,;\sigma)), \qquad t \geq 0 \tag{9}$$

where $\gamma(t\,;\sigma)$ is a solution of the characteristic equation

$$\begin{aligned} \frac{d\gamma}{dt} &= \frac{1}{v^2(\gamma,t)\,\kappa(\gamma,t)}\frac{\partial g(\kappa(\gamma,t)}{\partial \gamma}, \qquad t > 0 && (10)\\ \gamma(0\,;\sigma) &= \sigma \in S^1. && (11) \end{aligned}$$

From (3) and (5) we have

$$v(\sigma,t) = v_0(\sigma)e^{-\int_0^t g(\kappa(\sigma,\tau))\kappa(\sigma,\tau)d\tau} \tag{12}$$

Eliminating $v(\sigma,t)$ from (2)-(3), we have a nonlinear degenerate integro-differential equation of the form:

$$\kappa_t = A(\sigma,t,\kappa)\,\kappa_{\sigma\sigma} + B(\sigma,t,\kappa,\kappa_\sigma), \qquad \sigma \in S^1, \quad t > 0 \tag{13}$$

with

$$\kappa(\sigma,0) = \kappa_0(\sigma), \qquad \sigma \in S^1. \tag{14}$$

Here

$$A(\sigma,t,\kappa) = v_0(\sigma)^{-2}e^{2\int_0^t g(\kappa(\sigma,\tau))\kappa(\sigma,\tau)d\tau}g'(\kappa(\sigma,t))$$

and

$$\begin{aligned} B(\sigma,t,\kappa,\kappa_\sigma) &= v_0(\sigma)^{-2}e^{2\int_0^t g(\kappa(\sigma,\tau))\kappa(\sigma,\tau)d\tau}\Big\{g''(\kappa)\,\kappa_\sigma^2 - v_0^{-1}v_0'g'(\kappa)\,\kappa_\sigma \\ &\quad + \int_0^t (g'(\kappa(\sigma,\tau))\kappa(\sigma,\tau) + g(\kappa(\sigma,\tau)))\,\kappa_\sigma(\sigma,\tau)\,d\tau\, g'(\kappa)\kappa_\sigma\Big\} \\ &\quad + g(\kappa)\kappa^2. \end{aligned}$$

If $g'(\kappa) \geq \lambda > 0$ $(\forall\kappa \in \mathbb{R})$, eq. (13) is uniformly parabolic. We consider a successive approximation and apply the classical result of Ladyzhenskaya-Solonnikov-Ural'ceva [25] to obtain the short time existence of classical solutions to the initial value problem (13)-(14) provided that $g(\kappa)$, v_0 and κ_0 are smooth. In order to treat degenerate and non smooth cases, we approximate g and κ_0 by $g_\varepsilon \in C^1(\mathbb{R})$ with $g_\varepsilon'(\kappa) \geq \lambda_\varepsilon > 0\,(\forall\kappa \in \mathbb{R})$ and $\kappa_{0\varepsilon} \in C^\infty(S^1)$, respectively, to get the approximate solutions $\kappa_\varepsilon(\sigma,t)$ in a short time which can be taken independent of ε. Under suitable assumptions on g and κ_0, we may obtain a priori estimate $\|\kappa_\varepsilon\|_{L^\infty}$, by using some modifications of usual energy estimates, and then, get a priori estimates of higher order derivatives. Then, taking the limit of $\kappa_\varepsilon(\sigma,t)$ as $\varepsilon \to 0$, we obtain the short time existence for (13)-(14).

We also note that the equation of κ shows that if $\kappa(\sigma,0)$ is positive (nonnegative), then $\kappa(\sigma,t)$ is positive (nonnegative) for all $t \geq 0$. This fact is useful in the subsequent sections.

3 Fundamental properties of solutions

Evolution of the geometric quantities such as the length of the curve, the area which the curve encloses and the rotation number of the curve is fundamental to the study of evolution of the curve. We denote the length, the area and the rotation number of $\mathcal{C}(t)$ by $L(t)$, $A(t)$ and $R(t)$, respectively. They are given by

$$L(t) = \int_{\mathcal{C}(t)} ds = \int_0^1 v(\sigma, t)\, d\sigma,$$

$$A(t) = -\frac{1}{2} \int_{\mathcal{C}(t)} \langle \mathcal{C}(t), \mathcal{N} \rangle ds,$$

and

$$R(t) = \int_{\mathcal{C}(t)} \kappa\, ds = \int_0^1 \kappa(\sigma, t) v(\sigma, t)\, d\sigma.$$

Note that $A(t)$ means the area which $\mathcal{C}(t)$ encloses only if $\mathcal{C}(t)$ is embedded.

We have

Lemma 1

$$\frac{d}{dt} L(t) = -\int_{\mathcal{C}(t)} g(\kappa)\kappa\, ds,$$

$$\frac{d}{dt} A(t) = -\int_{\mathcal{C}(t)} g(\kappa)\, ds,$$

and

$$\frac{d}{dt} R(t) = 0.$$

When $g(\kappa) = \kappa$, we have

$$\frac{d}{dt} A(t) = -\int_{\mathcal{C}(t)} \kappa\, ds = -R(t) = -R(0).$$

Hence,

$$A(t) = A(0) - R(0)t$$

and

$$\lim_{t \nearrow A(0)/R(0)} A(t) = 0.$$

In this case, Gage-Hamilton [20] and Grayson [23] showed that the smooth embedded closed curve $\mathcal{C}(t)$ shrinks to a point, becoming round in the limit. We can extend the above mentioned results to more general cases.

Theorem 2 *Suppose that* $R_0 = R(0) = \int_0^1 \kappa(\sigma, 0) v(\sigma, 0)\, d\sigma \neq 0$ *and that* $G(\kappa) = g(\kappa)\kappa$ *is nonnegative and convex. Suppose that* $\int_0^L \frac{1}{g(R_0/\ell)}\, d\ell < +\infty$ *for all* $L > 0$*. Then,*

$$\lim_{t \nearrow T} L(t) = 0$$

where

$$T = \int_0^{L(0)} \frac{1}{g(R_0/\ell)R_0} d\ell$$

provided that the curve $\mathcal{C}(t)$ is smooth on $[0, T)$.

If we restrict ourselves to convex closed curves, we can find $g(\kappa)$ for which the curve $\mathcal{C}(t)$ may expand to infinity.

Theorem 3 *Suppose that $\mathcal{C}(t)$ is smooth convex and closed. Suppose that $R_0 = \int_0^1 \kappa(\sigma, 0)$ $v(\sigma, 0)\, d\sigma \neq 0$ and $G(\kappa) = g(\kappa)\kappa$ is a nonpositive convex function of $\kappa \geq 0$ such that $G(\kappa) \leq$ $C|\kappa|^{\alpha+1}$ where C is a positive constant and $-1 \leq \alpha < 0$. Suppose that $\int_L^{+\infty} \frac{1}{g(R_0/\ell)} d\ell = -\infty$ for all $L > 0$. Then,*

$$\lim_{t \nearrow \infty} L(t) = +\infty$$

provided that the curve $\mathcal{C}(t)$ is smooth on $[0, \infty)$.

Theorem 4 *Suppose that $\mathcal{C}(t)$ is smooth convex and closed. Suppose that $R_0 = \int_0^1 \kappa(\sigma, 0)$ $v(\sigma, 0) d\sigma \neq 0$ and that $G(\kappa) = g(\kappa)\kappa$ is nonpositive and concave. Suppose that $\int_L^{+\infty} \frac{1}{g(R_0/\ell)} d\ell$ $> -\infty$ for all $L > 0$. Taking $T = -\int_{L(0)}^{\infty} \frac{1}{g((R_0/\ell)R_0} d\ell$, one has*

$$\lim_{t \nearrow T} L(t) = +\infty$$

provided that the curve $\mathcal{C}(t)$ is smooth on $[0, T)$.

We can prove Theorems 2-4 by using the evolution of $L(t)$, $A(t)$ and $R(t)$, and Jensen's inequality.

Remark 5 Let $g(\kappa) = \text{sign}\alpha\, |\kappa|^{\alpha-1}\kappa$, $\alpha \in \mathbb{R}$. Then, if $\alpha > 0$, then the curve $\mathcal{C}(t)$ shrinks to a point in a finite time if it does not develop singularity before shrinking to a point. If $-1 \leq \alpha < 0$, the convex curve may expand to infinity as $t \to \infty$ if it exists for all $t \geq 0$. If $\alpha < -1$, the convex curve may blow up in a finite time.

The following proposition concerns the evolution of the total curvature of the curves.

Proposition 6 *Suppose that $g(\kappa)$ is a nondecreasing differentiable function on $\mathbb{R}\backslash 0$ and the curve $\mathcal{C}(t)$ evolved by (1) is smooth on $[0, T)$. Then, we have*

$$\sup_{t \in [0,T)} \int_{\mathcal{C}(t)} |\kappa| ds \leq C,$$

where C is a positive constant.

Corollary 7 *If $\mathcal{C}(0)$ is oval, then $\mathcal{C}(t)$ is oval for all $t > 0$.*

Corollary 8 *For any $\eta > 0$, there exists a constant B such that $\kappa(s, t) \leq B$ except on subarcs of length less than or equal to η.*

4 Immersed closed curves and degenerate parabolic equations

Let $\mathcal{C}: S^1 \to \mathbb{R}^2$ be a convex immersed curve, and for each $\sigma \in S^1$ we denote the unit tangent to $\mathcal{C}$ by $\mathcal{T}(\sigma) \in S^1$. Since the curve is convex, $\mathcal{T}: S^1 \to S^1$ is locally one-to-one, and $\mathcal{T}$ is a covering. The degree ν of the covering is the index of the curve. Let $\mathrm{T}_\nu = \mathbb{R}/2\nu\pi\mathbb{Z}$. For any $\sigma \in S^1$ we can write $\mathcal{T}(\sigma) = (\cos\theta(\sigma), \sin\theta(\sigma))$ for some $\theta(\sigma) \in \mathrm{T}_\nu$, where $\theta(\sigma)$ can be chosen so that it depends continuously on $\sigma \in S^1$. Then, $\theta: S^1 \to T_\nu$ is bijective and $\mathcal{C} \circ \theta^{-1}: \mathrm{T}_\nu \to \mathbb{R}^2$ is a parametrization of the curve for which the tangent is $(\cos\theta, \sin\theta)$.

If the curvature κ of the curve is known as a function of the angle θ, then the curve is completely determined, up to a translation. In this case the inversion is given by

$$\mathcal{C}(\theta_0) = \mathcal{C}(0) + \int_{\theta=0}^{\theta_0} \mathcal{T}\, ds = \mathcal{C}(0) + \int_0^{\theta_0} \frac{e^{i\theta}}{\kappa(\theta)}\, d\theta, \tag{15}$$

where $\mathbb{R}^2$ is identified with $\mathbb{C}$. Hence, any positive function $\kappa \in C(\mathrm{T}_\nu)$ satisfying the closedness condition

$$\int_0^{2\nu\pi} \frac{e^{i\theta}}{\kappa(\theta)}\, d\theta = 0 \tag{16}$$

defines a C^2 convex immersed closed curve. When κ has zeros on a finite number of points, if $\kappa^{-1} \in L^1(\mathrm{T}_\nu)$ and satisfies (16), the inversion formula (15) still defines an immersed closed curve. But it is obvious that many closed curves containing line segments may have the same curvature κ in this case.

It can be shown that eq. (1) for convex immersed curves is equivalent to the following partial differential equation for the curvature κ, as a function of (θ, t),

$$\frac{\partial \kappa}{\partial t} = \kappa^2 \left[\frac{\partial^2 g(\kappa)}{\partial \theta^2} + g(\kappa)\right] \qquad (\theta \in \mathrm{T}_\nu, 0 \le t < T). \tag{17}$$

If $\kappa(\theta, t)$ satisfies (16) at $t = 0$, then it satisfies (16) for all $t \ge 0$ so long as eq. (17) has a sense. Indeed,

$$\begin{aligned}\frac{d}{dt}\int_0^{2\nu\pi} \frac{e^{i\theta}}{\kappa(\theta,t)}\, d\theta &= -\int_0^{2\nu\pi} \frac{\kappa_t(\theta,t)e^{i\theta}}{\kappa^2(\theta,t)}\, d\theta \\ &= -\int_0^{2\nu\pi} \left[\frac{\partial\theta^2 g(\kappa)}{\partial\theta^2} + g(\kappa)\right] e^{i\theta}\, d\theta = 0.\end{aligned}$$

In the sequel we always assume that $g(\kappa) = \mathrm{sign}\alpha\, |\kappa|^{\alpha-1}\kappa$, $\alpha \in \mathbb{R}$. Then, putting $u = \beta g(\kappa)$, $\beta = |\alpha|^{\alpha/(\alpha+1)}$, (17) becomes

$$u_t = u^\delta(u_{\theta\theta} + u), \qquad (\theta \in \mathrm{T}_\nu, 0 \le t < T). \tag{18}$$

where $\delta = \dfrac{\alpha+1}{\alpha} \neq 1$. If $\alpha = -1$, eq. (18) turns to be the linear heat equation

$$u_t = u_{\theta\theta} + u.$$

If we assume that $g(\kappa) = e^{-1/\kappa}$. Then, putting $u = g(\kappa)$, we have

$$u_t = u\,(u_{\theta\theta} + u), \qquad (\theta \in \mathrm{T}_\nu, 0 \le t < T).$$

Hence we can consider eq.(18) for all $\delta \in \mathbb{R}$. As is suggested in the case $\delta = 2$ (see [7]), selfsimilar solutions of the form

$$u(\theta, t) = U(\theta)[\delta(T-t)]^{-1/\delta}$$

may play an important role in studying the asymptotic behaviour of solutions of (18). Here $U(\theta)$ satisfies

$$U_{\theta\theta} + U = U^{1-\delta} \quad \theta \in \mathrm{T}_\nu$$

We see that $U(\theta) \equiv 1$ is a solution. It is interesting to find other solutions if exist. When $\delta = 2$, Abresch and Langer [1] classified all positive solutions of this equation. Epstein and Weinstein [14] investigated their stability. When $\delta = 1$, the equation turns to be $U_{\theta\theta} + U = 1$. Hence we have a continuous family of selfsimilar solutions

$$u(\theta, t) = (1 + A\cos\theta + B\sin\theta)(T-t)^{-1}$$

where A and B are positive constants satisfying $\sqrt{A^2 + B^2} < 1$. Note that the corresponding curvature κ to the selfsimilar solution does not satisfy the closedness condition (16) unfortunately unless $A = B = 0$.

4.1 blow-up of solutions.

We consider the initial value problem

$$\begin{aligned} u_t &= u^\delta(u_{\theta\theta} + u), & \theta \in \mathrm{T}_\nu, \quad 0 \le t < T, & \qquad (19)\\ u(\theta, 0) &= u_0(\theta), & \theta \in \mathrm{T}_\nu & \qquad (20) \end{aligned}$$

where u_0 is a continuous nonnegative function on T_ν.

In this and the following subsections we only investigate the case $\delta > 0$.

Viscosity solutions of (19), (20) can be obtained by the method of vanishing viscosity (see [2]). That is, we approximate the solution u of (19), (20) by u_ε satisfying

$$\begin{aligned} u_{\varepsilon t} &= u_\varepsilon^\delta(u_{\varepsilon\theta\theta} + u_\varepsilon), & \theta \in \mathrm{T}_\nu, \quad 0 \le t < T, & \qquad (21)\\ u_\varepsilon(\theta, 0) &= (u_0(\theta) + \varepsilon), & \theta \in \mathrm{T}_\nu, & \qquad (22) \end{aligned}$$

for $\varepsilon > 0$ and take the limit as $\varepsilon \to 0$.

Selfsimilar solutions suggest that when $\delta > 0$ solutions of (19), (20) may blow up in a finite time.

Theorem 9 *Suppose that $u_0^{1-\delta} \in L^1(\mathrm{T}_\nu)$ when $\delta \neq 1$, and $\log u_0 \in L^1(\mathrm{T}_\nu)$ when $\delta = 1$. Then, there exists a finite $T > 0$ such that any viscosity solution u blows up to infinity as $t \uparrow T$ where*

$$T \le \begin{cases} \dfrac{1}{\delta}\left(\dfrac{1}{2\nu\pi}\displaystyle\int_0^{2\nu\pi} u_0(\theta)^{1-\delta}\,d\theta\right)^{\frac{\delta}{\delta-1}} & \text{if } \delta \neq 1,\\ \exp\left(-\dfrac{1}{\nu}\displaystyle\int_0^{2\nu\pi} \log\, u_0(\theta)\,d\theta\right) & \text{if } \delta = 1. \end{cases}$$

Proof. We only consider the case $\delta > 1$. We may prove the others in the same manner. Since u_ε is strictly positive, we can rewrite (21) into

$$-\frac{1}{\delta-1}(u_\varepsilon^{1-\delta})_t = u_{\varepsilon\theta\theta} + u_\varepsilon. \tag{23}$$

Integrating (23) from 0 to $2\nu\pi$, we have

$$\frac{1}{\delta-1}\frac{d}{dt}\int_0^{2\nu\pi} u_\varepsilon^{1-\delta}\,d\theta = -\int_0^{2\nu\pi} u_\varepsilon\,d\theta.$$

Since

$$2\nu\pi = \int_0^{2\nu\pi} d\theta \le \left(\int_0^{2\nu\pi} u_\varepsilon^{1-\delta}d\theta\right)^{1/\delta}\left(\int_0^{2\nu\pi} u_\varepsilon\,d\theta\right)^{(\delta-1)/\delta}, \tag{24}$$

we have

$$\frac{d}{dt}\int_0^{2\nu\pi} u_\varepsilon^{1-\delta}\,d\theta \le -(\delta-1)(2\nu\pi)^{\delta/(\delta-1)}\left(\int_0^{2\nu\pi} u_\varepsilon^{1-\delta}\,d\theta\right)^{-\frac{1}{\delta-1}}. \tag{25}$$

From (25), letting ε tend to 0, we have

$$\int_0^{2\nu\pi} u^{1-\delta}\,d\theta \le \left\{\left(\int_0^{2\nu\pi} u_0^{1-\delta}\,d\theta\right)^{\frac{\delta}{\delta-1}} - \delta\,(2\nu\pi)^{\frac{\delta}{\delta-1}}\,t\right\}^{\frac{\delta-1}{\delta}}, \tag{26}$$

which implies that

$$\sup_{0\le\theta\le 2\nu\pi} u(\theta,t) \ge \left\{\left(\frac{1}{2\nu\pi}\int_0^{2\nu\pi} u_0^{1-\delta}\,d\theta\right)^{\frac{\delta}{\delta-1}} - \delta\,t\right\}^{-\frac{1}{\delta}}. \tag{27}$$

This completes the proof of the case $\delta > 1$.

4.2 Regional blow-up

From now on we always assume that

(A1) u_0 is a nonnegative continuous function satisfying $u_0^{1-\delta} \in L^1(\mathrm{T}_\nu)$ for $\delta \ne 1$, and $\log u_0 \in L^1(\mathrm{T}_\nu)$ for $\delta = 1$.

We define the blow-up set to be the set S of points θ in T_ν for which there is a sequence of the form (θ_n, t_n) with θ_n converging to θ, t_n converging to the blow-up time T, and $u(\theta_n, t_n)$ tending to infinity. Let S_* be the set of monotone blow up points $\theta \in \mathrm{T}_\nu$ for which there is a sequence $\{t_n\}$ such that t_n converges to T and $u(\theta, t_n)$ tends to infinity. Note that by definition $S_* \subset S$.

Let M_t be the set of local maximum points of solutions $u(\theta,t)$ at each fixed $t \in [0,T)$. Define the set M by

$$\begin{aligned} M = \{\,\theta \mid \exists\{(\theta(t_n), t_n)\} \in \bigcup_{t\in[0,T)} M_t \times t \\ \textit{such that}\quad t_n \to T,\, \theta(t_n) \to \theta \quad \textit{and} \quad u(\theta(t_n), t_n) \to \infty\,\} \end{aligned}$$

Theorem 9 imply that S is nonempty, and so is M. Moreover we have

Theorem 10 *For any $\theta \in M$, $(\theta - \frac{\pi}{2}, \theta + \frac{\pi}{2}) \subset S_*$.*

In order to prove Theorem 10, we introduce the set D defined by

$$D = \left\{ (\theta, t) \mid u(\theta, t) > \max_{0 \leq \theta \leq 2\nu\pi} \left[(u_0(\theta) + \varepsilon_0)^2 + (u_0'(\theta))^2 \right]^{1/2} \right\}$$

where $\varepsilon_0 > 0$.

Proposition 11 *Let $0 < \varepsilon < \varepsilon_0$ and (θ_0, t_0) be any fixed point in D. Let $\varphi_\varepsilon \in (-\frac{\pi}{2}, \frac{\pi}{2})$ be determined by $\tan \varphi_\varepsilon = u_{\varepsilon\theta}(\theta_0, t_0)/u_\varepsilon(\theta_0, t_0)$. For any $\theta \in (\theta_0 + \varphi_\varepsilon - \frac{\pi}{2}, \theta_0 + \varphi_\varepsilon + \frac{\pi}{2})$,*

$$u_\varepsilon(\theta, t_0) \geq u_\varepsilon(\theta_0, t_0) \cos(\theta - \theta_0) + u_{\varepsilon\theta}(\theta_0, t_0) \sin(\theta - \theta_0). \tag{28}$$

Moreover we have

$$\frac{\partial u}{\partial t}(\theta, t) \geq 0 \qquad \forall (\theta, t) \in D.$$

This proposition can be proved by using Strumian comparison theorem (see Angenent [4] and is essentially due to Angenent [7].

Proof of Theorem 10. Let $\theta \in M$. Then, there exists a sequence $\{\theta(t_n), t_n\}$ such that $t_n \to T$, $\theta(t_n) \to \theta$ and $u(\theta(t_n), t_n) \to \infty$, Hence, there exists a $N_* \in \mathbb{N}$ such that $(\theta(t_n), t_n) \subset D$ for any $n \geq N_*$. Then, by virtue of Proposition 11 we have for $n \geq N_*$

$$u_\varepsilon(\theta, t_n) \geq u_\varepsilon(\theta(t_n), t_n) \cos(\theta - \theta(t_n)) + u_{\varepsilon\theta}(\theta(t_n), t_n) \sin(\theta - \theta(t_n))$$

for any $\theta \in (\theta(t_n) + \varphi_\varepsilon - \frac{\pi}{2}, \theta(t_n) + \varphi_\varepsilon + \frac{\pi}{2})$. Then, letting $\varepsilon \downarrow 0$ and noting that $u_{\varepsilon\theta}(\theta(t_n), t_n) \to u_\theta(\theta(t_n).t_n) = 0$, we have

$$u(\theta, t_n) \geq u(\theta(t_n), t_n) \cos(\theta - \theta(t_n)). \tag{29}$$

Hence, from (29) we see that for any $\theta \in M$,

$$(\theta - \frac{\pi}{2}, \theta + \frac{\pi}{2}) \subset S_*.$$

This completes the proof.

Remark 12 Under the zero-Dirichlet condition, we have analogous assertions which are proved by different methods to ours (see Friedman-Mcleod [19], Fukuda-Anada-Tsutsumi [3]).

Our interest is whether the blow-up set is exactly $\bigcup_{\theta \in M} \left[\theta - \frac{\pi}{2}, \theta + \frac{\pi}{2} \right]$ or not (see Friedman-Mcleod [19], Angenent [7]).

First we show that

Proposition 13 *Suppose that* $0 < \delta < 2$. *We have*

$$u(\theta, t) \leq C(T-t)^{-1/\delta} \qquad \theta \in \mathrm{T}_\nu \quad 0 \leq t < T. \tag{30}$$

Here and in the sequel by C we denote various positive constants changeable from line to line.

Proof can be done by considering

$$J(t) = \frac{1}{2-\delta} \int_0^{2\nu\pi} u^{2-\delta}\, d\theta.$$

Indeed, differentiating under integration, integrating by parts and making use of (19), we have a differential inequality

$$(J'(t))^2 \leq \frac{2-\delta}{2} J''(t) J(t) \tag{31}$$

from which we can deduce the assertion.

Remark 14 In case of $\delta \geq 2$, we will show later in the next subsection that for $\theta \in M, u(\theta, t)$ blows up faster than $(T-t)^{-1/\delta}$.

Using Proposition 13, we have

Proposition 15 *Suppose that* $0 < \delta < 2$. *We have*

$$\bigcup_{\theta \in M} \left[\theta - \frac{\pi}{2}, \theta + \frac{\pi}{2}\right] \subset S_*$$

When $\delta \geq 2$, we get a more exact assertion.

Theorem 16 *Let* $\delta \geq 2$. *Suppose that* $S \neq [0, 2\nu\pi]$. *Then,*

$$S = \bigcup_{\theta \in M} \left[\theta - \frac{\pi}{2}, \theta + \frac{\pi}{2}\right].$$

.

Outline of the proof. Proof is rather long and is divided into several lemmas. Among the lemmas, we will give the proof of Lemma 21 which is the most essential part of the arguments.

First we can assume that S consists of exactly one interval or there exist $\theta_1, \theta_2 \in M$ such that $\theta_1 + \frac{\pi}{2} < \theta_2 - \frac{\pi}{2}$. and $(\theta_1 + \frac{\pi}{2}, \theta_2 - \frac{\pi}{2}) \cap S^c \neq \phi$. Since the former case can be treated in the same manner, below we only consider the latter case. In order to establish Theorem 16, it suffices to prove that $\theta_1 + \frac{\pi}{2} + \varepsilon \in S^c$ and $\theta_2 - \frac{\pi}{2} - \varepsilon \in S^c$ for any sufficiently small $\varepsilon >$.

Let $\alpha_j \geq \frac{\pi}{2}$ $(j = 1, 2)$ be such that $[\theta_1, \theta_1 + \alpha_1] \subset S$, $[\theta_2 - \alpha_2, \theta_2] \subset S$, $\theta_1 + \alpha_1 + \varepsilon \notin S^c$, and $\theta_2 - \alpha_1 - \varepsilon \notin S^c$ for any sufficiently small $\varepsilon > 0$.

Lemma 17 *For any sufficiently small $\varepsilon >$, there is a $t_\varepsilon < T$ such that for all $t \in (t_\varepsilon, T)$ one has*

$$u_\theta(\theta, t) < 0 \quad (> 0)$$

on the interval $(\theta_1 + \alpha_1 - \frac{\pi}{2}, \theta_1 + \alpha_1 - \varepsilon)$ *(or on the interval* $(\theta_2 - \alpha_1 + \epsilon, \theta_2 - \alpha + \frac{\pi}{2})$ *).*

Lemma 18 *Let $z(\theta) \in C^2(\theta_1, \theta_2)$ satisfy*

$$\begin{cases} z_{\theta\theta} + z = \dfrac{1}{\delta g^{\delta-1}} & (\,\theta_1 < \theta < \theta_2\,), \\ z(\theta) > 0, & (\,\theta_1 < \theta < \theta_2\,), \\ \lim_{\theta \uparrow \theta_2} z(\theta) = 0. \end{cases}$$

Then, $\delta < 2$.

Lemma 19 *For $\theta_0 \in M$*

$$u(\theta_0, t) > C(T-t)^{-1/\delta} \qquad 0 < t < T. \tag{32}$$

Lemma 20 *Suppose that $\delta > 1$. For any $\theta_0 \in M$, we have*

$$\min_{\theta_0 \le \theta \le \theta_0 + \frac{\pi}{2}} u(\theta, t) \le C(T-t)^{-1/\delta}. \tag{33}$$

Lemma 21 *Suppose that $\alpha > \frac{\pi}{2}$. Under the same assumptions as in* Lemma 17, *there exists a sequence $t_n \uparrow T$ such that*

$$u(\theta, t_n) = o(u(\theta_1, t_n)) \qquad \forall\, \theta \in [\theta_1 + \frac{\pi}{2}, \theta_1 + \alpha]. \tag{34}$$

Proof of Lemma 21. If the assertion is not true, then there is a point $\bar\theta \in [\theta_1 + \frac{\pi}{2}, \theta_1 + \alpha]$ and $\rho > 0$ such that $u(\bar\theta, t) > \rho u(\theta_1, t)$ for all $0 < t < T$. Then, by virtue of Lemma 17 we get

$$u(\theta, t) > \rho u(\theta_1, t) \qquad \text{if } \theta_1 \le \theta \le \bar\theta. \tag{35}$$

Then, Lemma 19 and Lemma 20 yield that there exist two positive constants M_1, M_2 such that

$$M_1\,(T-t)^{-1/\delta} \le u(\theta, t) \qquad \text{if } \theta_1 \le \theta \le \bar\theta, \quad 0 < t < T \tag{36}$$

and

$$u(\theta, t) \le M_2\,(T-t)^{-1/\delta} \qquad \text{for any } \theta \in \mathrm{T}_\nu. \tag{37}$$

Introducing the new time variable

$$s = -\log\,(1 - \frac{t}{T}) : \qquad [0, T) \to (0, \infty),$$

we define

$$Z(\theta, s) = (T-t)^{1/\delta} u(\theta, t).$$

Then $Z(\theta, s)$ solves the initial-periodic boundary value problem

$$
\text{(RP)} \qquad \begin{cases} Z_s = Z^\delta(Z_{\theta\theta} + Z) - \dfrac{Z}{\delta} & \theta \in \mathrm{T}_\nu,\ s > 0, \\ Z(\theta, 0) = T^{1/\delta} u_0(\theta) & \theta \in \mathrm{T}_\nu. \end{cases} \qquad \begin{matrix} (38) \\ (39) \end{matrix}
$$

Take any sequence $t_n \uparrow \infty$ and define

$$
z_n(\theta) = \int_{t_n}^{t_n+1} Z(\theta, s)\, ds.
$$

Then, by virtue of (37) and $Z_s(\theta, s) \geq 0$ for sufficiently large s and $\theta \in S$, we see that $\{z_n\}$ forms a Cauchy sequence in $H^1(\mathrm{T}_\nu) \subset C(\mathrm{T}_\nu)$ and $z_n \to z$ as well as $Z(\cdot, t_n) \to z$ when $n \uparrow \infty$. The limit function z is not identically zero because of (36) and satisfies

$$
\text{(LP)} \qquad \begin{cases} z_{\theta\theta} + z - \dfrac{1}{\delta z^{\delta-1}} = 0, & \theta \in [\theta_1, \theta^* + \kappa), \\ z(\theta^* + \kappa) = 0, \end{cases} \qquad \begin{matrix} (40) \\ (41) \end{matrix}
$$

where $[\theta_1, \theta^*] \supset [\theta_1, \bar{\theta}]$ is the blow-up interval and $\theta^* + \kappa \in S^c$ for an arbitrary $\kappa > 0$. This contradicts the assertion of Lemma 18. □

Proof of Theorem 16 completed. Suppose that $\alpha > \dfrac{\pi}{2}$. Let $\theta(t) \in M_t$ be such that $\theta(t) \to \theta_1$ as $t \uparrow T$. Let κ be an arbitrary positive small number and $\tilde{\theta}$ be such that $\theta_1 + \dfrac{\pi}{2} + \kappa < \tilde{\theta} < \min(\theta_1 + \alpha, \pi)$. We have

$$
\begin{aligned}
&\frac{d}{dt} \int_{\theta(t)}^{\tilde{\theta}} u^{1-\delta}(\theta, t) \cos(\theta - \theta(t))\, d\theta \\
&\qquad = (1 - \delta)(u_\theta(\tilde{\theta}, t) \cos(\tilde{\theta} - \theta(t)) + u(\tilde{\theta}, t) \sin(\tilde{\theta} - \theta(t)) \qquad (42)
\end{aligned}
$$

Let t_n be a sequence such that $t_n \uparrow T$. For sufficiently large n and $\tilde{\theta}$ be sufficiently near $\theta_1 + \dfrac{\pi}{2}$, taking $t \geq t_n$, the left hand side of (42) is nonpositive since $u_\theta(\tilde{\theta}, t) \leq 0$. Hence, by Theorem 10 and Lemma 21 we see that for $t_n < t$

$$
\begin{aligned}
\int_{\theta(t)}^{\tilde{\theta}} u^{1-\delta}(\theta, t) \cos(\theta - \theta(t))\, d\theta &\leq \int_{\theta(t_n)}^{\tilde{\theta}} u^{1-\delta}(\theta, t_n) \cos(\theta - \theta(t_n))\, d\theta \\
&\leq \int_{\theta(t_n)}^{\theta(t_n) + \frac{\pi}{2} - \kappa} u^{1-\delta}(\theta, t_n) \cos(\theta - \theta(t_n))\, d\theta \\
&\qquad + \int_{\theta(t_n) + \frac{\pi}{2} + \kappa}^{\tilde{\theta}} u^{1-\delta}(\theta, t_n) \cos(\theta - \theta(t_n))\, d\theta \\
&\qquad + 2\kappa\, u^{1-\delta}(\theta(t_n) + \frac{\pi}{2} - \kappa, t_n) \\
&\leq -C < 0
\end{aligned}
$$

for sufficiently large n. Then $u(\theta, t)$ remains bounded on a interval contained in $\theta(t) + \dfrac{\pi}{2}, \tilde{\theta}$ as $t \uparrow T$. This is a contradiction. □

4.3 Behaviour near the blow-up time

Under the assumption (A1) we can investigate the behaviour of the solutions near the blow-up time.

Theorem 22 *Suppose that $\delta \geq 2$ and u_0 satisfies* (A1). *Suppose that $S \neq [0, 2\nu\pi]$. Let u blow up at T and $\theta_0 \in M$. Then,*

$$\frac{u(\theta,t)}{u(\theta_0,t)} \longrightarrow \cos(\theta-\theta_0) \quad for \quad \theta \in (\theta_0 - \frac{\pi}{2}, \theta_0 + \frac{\pi}{2}) \tag{43}$$

uniformly as $t \longrightarrow T$. Moreover, if $\theta \notin \bigcup_{\theta \in M}[\theta - \frac{\pi}{2}, \theta + \frac{\pi}{2}]$, then

$$\frac{u(\theta,t)}{u(\theta_0,t)} \longrightarrow 0. \tag{44}$$

Theorem 23 *Suppose that $\delta \geq 2$ and u_0 satisfies* (A1). *Suppose that $S = [0, 2\nu\pi]$. Let u blow up at T. Then,*

$$u(\theta,t) \leq C(T-t)^{-1/\delta} \qquad \forall \theta \in \mathrm{T}_\nu \tag{45}$$

and

$$(T-t)^{1/\delta} u(\theta,t) \longrightarrow z(\theta) \qquad \forall \theta \in \mathrm{T}_\nu \tag{46}$$

uniformly as $t \longrightarrow T$, where $z(\theta)$ is a positive solution of the boundary value problem

$$z_{\theta\theta} + z - \frac{1}{\delta z^{\delta-1}} = 0, \qquad \theta \in \mathrm{T}_\nu. \tag{47}$$

Remark 24 *Let $\delta \geq 2$. Then,*

$$S = [0, 2\nu\pi] \quad \text{if and only if} \quad u(\theta,t) \leq C(T-t)^{-/1\delta} \qquad \forall \theta \in \mathrm{T}_\nu.$$

Theorem 25 *Suppose that $1 < \delta < 2$, and u_0 satisfies* (A1). *Let u blow up at T. Then,*

$$(T-t)^{1/\delta} u(\theta,t) \longrightarrow \begin{cases} z(\theta) & for \;\; \theta \in S, \\ 0 & for \;\; [0, 2\nu\pi] \backslash S \end{cases} \tag{48}$$

uniformly as $t \longrightarrow T$, where $z(\theta)$ is the unique positive solution of the boundary value problem

$$z_{\theta\theta} + z - \frac{1}{\delta z^{\delta-1}} = 0, \qquad \theta \in S, \tag{49}$$

$$z(\theta) = 0, \qquad \theta \in \partial S. \tag{50}$$

If $S = [0, 2\nu\pi]$, $z(\theta)$is periodic with period $2\nu\pi$.

We only outline the proof of Theorem 22.

We prepare two lemmas.

Lemma 26 *Let $\theta_0 \in M$. Under the same assumptions as in Theorem 22, we have*

$$(T-t)^{1/\delta}u(\theta_0,t) \longrightarrow \infty \qquad as\ t \longrightarrow T. \tag{51}$$

Proof. Assume on the contrary that there exists a sequence $t_n \uparrow T$ such that

$$(T-t_n)^{1/\delta}u(\theta_0,t_n) \le C \tag{52}$$

for some positive constant C. Then, we see that $[\theta_0 - \frac{\pi}{2}, \theta_0 + \frac{\pi}{2}] \subset S_*$. Putting

$$F(\theta,t) = \frac{1}{(T-t)\,u(\theta_0,t)} \int_t^T u(\theta,s)\,ds,$$

we have

$$F_{\theta\theta} + F = \frac{1}{(T-t)\,u(\theta_0,t)} \int_t^T \frac{u_s(\theta,s)}{u^\delta(\theta,s)} ds \equiv f(\theta,t). \tag{53}$$

Note that $[\theta_0 - \frac{\pi}{2}, \theta_0 + \frac{\pi}{2}] \times [t_0, T) \subset D$. for some $t_0 \in [0,T)$. In the same manner as in the proof of Lemma 17, we can assume that for any $\kappa > 0$ $t \ge t_0$

$$u_\theta(\theta_0 + \frac{\pi}{2} + \kappa, t) \le 0 \quad \text{and} \quad u_\theta(\theta_0 - \frac{\pi}{2} - \kappa, t) \ge 0.$$

For the sake of simplicity we write for a while $\theta_0 - \frac{\pi}{2} - \kappa$ and $\theta_0 + \frac{\pi}{2} + \kappa$ by θ_1 and θ_2, respectively . Multiplying (53) by F and integrating from θ_1 to θ_2, we have

$$-\int_{\theta_1}^{\theta_2} F_\theta^2(\theta,t)\,d\theta + \int_{\theta_1}^{\theta_2} F^2(\theta,t)\,d\theta \ge \int_{\theta_1}^{\theta_2} f(\theta,t)\,F(\theta,t)\,d\theta \ge 0 \quad \forall t \ge t_0.$$

from which we deduce that

$$\int_{\theta_1}^{\theta_2} F_\theta^2(\theta,t)\,d\theta \le C \qquad \forall t \ge t_0$$

since $0 \le F \le 1$. Hence $\{F(\theta,t_n)\}_{n\in\mathbb{N}}$ forms a bounded sequence in $H^1(I)$ where $I = (\theta_1, \theta_2)$. We can now extract a subsequence $\{t_{n_k}\}$ converging to T such that

$$F(\theta, t_{n_k}) \longrightarrow \zeta(\theta) \qquad \text{weakly in } H^1(I) \text{ and strongly in } C(\overline{I})$$

for some $\zeta(\theta) \in H^1(I) \subset C(\overline{I})$.
Since $[\theta_0 - \frac{\pi}{2} - \kappa, \theta_0 - \frac{\pi}{2}) \cup (\theta_0 + \frac{\pi}{2}, \theta_0 + \frac{\pi}{2} + \kappa] \subset S^c$ and ζ is continuous, we see that $\zeta(\theta_0 \pm \frac{\pi}{2}) = 0$.

Multiplying (53) by $\cos(\theta - \theta_0)$ and integrating over $(\theta_0, \theta_0 + \frac{\pi}{2})$, we get

$$F(\theta_0 + \frac{\pi}{2}, t) = \int_{\theta_0}^{\theta_0 + \frac{\pi}{2}} f(\theta,t)\cos(\theta - \theta_0)\,d\theta.$$

For $\theta_0 \leq \theta < \theta_0 + \frac{\pi}{2}$, we get

$$\int_t^T \frac{u_s(\theta,s)}{u^\delta(\theta,s)}\,ds = \frac{1}{\delta-1}u^{1-\delta}(\theta,t) \geq \frac{1}{\delta-1}u^{1-\delta}(\theta_0,t).$$

Hence, in view of (52), we see that

$$f(\theta,t_n) \geq C \qquad \forall\,\theta \in [\theta_0,\theta_0+\frac{\pi}{2}).$$

Therefore we have

$$F(\theta_0+\frac{\pi}{2},t_n) \geq C$$

which leads to a contradiction. □

Lemma 27 *For any $\theta_0 \in M$*

$$\frac{1}{u(\theta_0,t)}\int_{\theta_0}^{\theta_0+\frac{\pi}{2}} u^{-\delta}u_t\cos(\theta-\theta_0)\,d\theta \longrightarrow 0 \qquad \textit{as } t \longrightarrow T. \tag{54}$$

Proof. We have

$$\frac{1}{u(\theta_0,t)}\int_{\theta_0}^{\theta_0+\frac{\pi}{2}} u^{-\delta}u_t\cos(\theta-\theta_0),d\theta = \frac{u(\theta_0+\frac{\pi}{2},t)}{u(\theta_0,t)} = \frac{C}{u(\theta_0,t)(T-t)^{1/\delta}}.$$

Then, the assertion follows from Lemma 26. □

Proof of Theorem 22. Suppose that $\theta_0 \in M$. For $\theta \in (\theta_0,\theta_0+\frac{\pi}{2})$, we have

$$\begin{aligned}
&\frac{1}{u(\theta_0,t)}\int_{\theta_0}^{\theta} u^{-\delta}u_t\cos(\theta-\theta_0)\,d\theta\\
&\quad\geq \frac{\cos(\theta-\theta_0)}{u(\theta_0,t)}\int_{\theta_0}^{\theta} u^{-\delta}u_t\,d\theta\\
&\quad\geq \frac{\cos(\theta-\theta_0)}{u(\theta_0,t)}\int_{\theta_0}^{\theta} u^{-\delta}u_t\sin(\theta-\theta_0)\,d\theta \geq 0
\end{aligned}$$

provided that t is sufficiently close to T. Hence,

$$\frac{1}{u(\theta_0,t)}\int_{\theta_0}^{\theta} u^{-\delta}u_t\sin(\theta_0-\theta)\,d\theta \longrightarrow 0 \qquad \text{as } t \longrightarrow T. \tag{55}$$

Making use of the equation (1.1), we obtain

$$\frac{1}{u(\theta_0,t)}\int_{\theta_0}^{\theta} u^{-\delta}u_t\sin(\theta_0-\theta)\,d\theta = \frac{u(\theta,t)}{u(\theta_0,t)} - \cos(\theta-\theta_0).$$

Then, from (55) we conclude the first assertion. □

References

[1] U. Abresch and J. Langer, *The normalized curve shortening flow and homothetic solutions*, J. Diff. Geometry **23** (1986), 175–196.

[2] K. Anada, I. Fukuda and M. Tsutsumi, *Local existence and uniqueness of viscosity solutions for* $u_t = u\Delta u - \gamma|\nabla u|^2 + f(t,u)$, preprint.

[3] K. Anada, I. Fukuda and M. Tsutsumi, *Regional blow-up of solutions to the initial-boundary value problem for* $u_t = u^\delta(u_{xx} + \mu u)$, preprint

[4] S. Angenent, *The zero set of a solution of a parabolic equation*, J. reine und angew. Math., **390** (1988), 79–96.

[5] S. Angenent, *Parabolic equations for curves on surfaces Part I. Curves with p-integrable curvature*, Ann. of Math. **132** (1990), 451–483.

[6] S. Angenent, *Parabolic equations for curves on surfaces Part II. Intersections, blow-up and generalized solutions*, Ann. of Math. **133** (1991), 171–215.

[7] S. Angenent, *On the formation of singularities in the curve shortening flow*, J. Diff. Geometry, **33** (1991), 601–633.

[8] D. G. Aronson, *Regularity properties of flows through porous media; the interface*, Arch. Rat. Mech. Anal. **37** (1970), 1–10.

[9] D. G. Aronson, *The porous medium equation*, in Some Problems in Nonlinear Diffusion (A. Fasano and M. Primiceris, eds), Lecture Notes in Math. **1224**, Springer-Verlag 1986.

[10] M. Bertsch, R. Dal Passo and M. Ughi, *Nonuniqueness of solutions of a degenerate parabolic equation*, Annali di Matematica pura ed applicate (IV), **161** (1992), 57–81.

[11] M. Bertsch, R. Dal Passo and M. Ughi, *Discontinuous "viscosity" solutions of a degenerate parabolic equation*, Trans. Amer. Math. Soc., **320** (1990), 779–798.

[12] M. Bertsch and M. Ughi, *Positivity properties of viscosity solutions of a degenerate parabolic equation*, Nonlinear Analysis T.M.A., **14** (1990), 571–592.

[13] C. Dohmen, Y. Giga and N. Mizoguchi, *Existence of selfsimilar shrinking curves for anisotropic curvature flow equations*, preprint.

[14] C. Epstein and M. Weintein, *A stable manifold theorem for the curve shortening equation*, Comm. Pure Appl. Math. **40** (1987), 119–139.

[15] H. Fujita, *On some nonexistence and nonuniqueness theorems for nonlinear parabolic equations*, in Proc. Symp. in Pure Math. **18** American Mathematical Society, Providence, Rhode Island, (1970), 105–113.

[16] I. Fukuda, H. Ishii and M. Tsutsumi, *Uniqueness of solutions to the Cauchy problem for* $u_t - u\Delta u + \gamma|\nabla u|^2 = 0$, Differential and Integral Eq., **6** (1993), 1231–1252.

[17] I. Fukuda and M. Tsutsumi, *The initial-boundary value problem for a nonlinear degenerate parabolic equation*, Nonlinear Analysis T.M.A., **16** (1991), 997–1009.

[18] A. Friedman and J. B. McLeod, *Blow-up of solutions of semilinear heat equations*, Indiana Univ. Math. **34** (1985), 425–447.

[19] A. Friedman and J. B. McLeod, *Blow-up of solutions of nonlinear degenerate parabolic equations*, Arch. Rational Mech. Anal. **96** (1987), 55–80.

[20] M. E. Gage and R. S. Hamilton, *The heat equation shrinking convex plane curves,* J. Diff. Geometry, **23** (1986), 69–96.

[21] M. E. Gage, *On the size of the blow-up set for a quasilinear parabolic equation*, Contemporary Mathematics, **127** (1992), 41–58.

[22] V. A. Galaktionov, *Asymptotic behaviour of unbounded solutions of the nonlinear parabolic equation* $u_t = (u^\sigma u_x)_x + u^{\sigma+1}$, Differential'nye Uravneniya, **21** (1985), 1126-1134 (in Russian).

[23] M. Grayson, *The heat equation shrinks embedded plane curves to round points,* J. Diff. Geometry, **26** (1987), 285—314.

[24] S. Kaplan, *On the growth of solutions of quasilinear parabolic equations*, Comm. Pure Appl. Math. **16** (1963), 305–330.

[25] O. A. Ladyzhenskaya, V. A. Solonnikov and N. N. Ural'ceva, *Linear and quasilinear equations of parabolic type*, Transl. Math. Monographs, **23**, Amer. Math. Soc., Providence, R.I. (1968)

[26] F. Mokhtarian and A. Mackworth, *Scale-based description and recognition of planar curves and two-dimensional shapes*, IEEEE Trans. Pattern Anal. Machine Intell, **PAMI-8** (1986), 34–43

[27] F. Mokhtarian and A. Mackworth, *Fingerprint theorem for curvature and torsion zero-crossings*, Proc. IEEEE Trans. CVPR (San Diego,CA), (1989), 269—275.

[28] F. Mokhtarian and A. Mackworth, *A Theory of multiscale, Curvature-based shape representation for planar curves*, IEEEE Trans. Pattern Anal. Machine Intell, **14** (1992), 789–805

[29] S. Osher and J. A. Sethian, *Fronts propagating with curvature-dependent speed: algorithms based on Hamilton-Jacobi formulations*, J. Comput. Physics **79** (1988) 12–49.

[30] Y-W. Qi, *The global feature of unbounded solutions to a nonlinear parabolic equation*, J. Diff. Eqs., **99** (1992), 139–152.

[31] F. B. Weissler, *Single point blow up of semilinear initial value problems*, J. Diff. Eqs. **55** (1984), 202–224.

[32] M. Wiegner, *Blow-up for solutionss of some degenerate parabolic equations*, Differential and Integral Eq., **7** (1994), 1641–1647.