数理解析研究所講究録
914 巻 1995 年 103-111



# Bénard-Marangoni convection with a deformable surface

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## 1 Introduction

We consider a model of Bénard-Marangoni convection using the Boussinesq equations for the velocity, pressure and temperature :

$$\frac{1}{\mathrm{Pr}}(u_t + u\cdot\nabla u) + \nabla p \;=\; \Delta u - \rho(T)\nabla z\,, \quad \nabla\cdot u \;=\; 0\,, \quad T_t + u\cdot\nabla T \;=\; \Delta T$$

in the strip $\{-1 < z < \eta(x,t), -\infty < x < \infty\}$, where $\rho(T) = \mathrm{G} - \mathrm{Ra}T$ is assumed for the density of the fluid.

We consider the boundary condition $u = 0$ and $T = 1$ on the bottom . The top surface $\eta(x,t)$ is deformable and has the kinematic boundary condition

$$\eta_t \;=\; u_3 - u_h\partial_x\eta|_{z=\eta(x,t)}\;,$$

and the stress balance equation is satisfied on it :

$$\Big((p - p_{\mathrm{air}})I - (\nabla u + {}^t\nabla u)\Big)\cdot n \;=\; \sigma H n - (t\cdot\nabla)\sigma t.$$

Here $n$ and $t$ are the normal and tangential unit vectors of the surface and $H$ is the curvature of the surface . The surface stress $\sigma$ is assumed to be given by

$$\sigma \;\equiv\; \mathrm{W} - \mathrm{Ma}T + \mathrm{Vi}\,(t\cdot\nabla)(u\cdot t).$$

We also have the boundary condition of temperature $n \cdot \nabla T + \mathrm{Bi}\, T = -1$ on the upper surface .

These equations have a stationary solution

$$\eta = 0, \quad u = 0, \quad T = \tilde{T}(z) \equiv -z, \quad p = \tilde{p}(z) \equiv -\frac{\mathrm{Ra}}{2} z^2 - \mathrm{G} z + p_{\mathrm{air}}$$

representing the purely heat conducting state.

We will consider the stability of this stationary state. We assume that all functions are periodic in $x$ with period $L$. Perturbation $(u, p, \theta, \eta)$ to the heat conducting state satisfies a nonlinear system, whose linearization is

$$\frac{1}{\mathrm{Pr}} u_t + \nabla p - \Delta u - \mathrm{Ra}\, \theta \nabla z = F \,, \quad \nabla \cdot u = 0 \,, \quad \theta_t - \Delta \theta - u_3 = F_0 \quad \text{in } \Omega, \tag{1}$$

$$\eta_t - u_3|_{S_F} = 0 \,,$$

$$p n - (\nabla u + \nabla u^t) \cdot n - (-\mathrm{W} \Delta_h + \mathrm{G}) \eta n - (\mathrm{Ma}\, \nabla_h (\theta - \eta) - \mathrm{Vi}\, \Delta_h u_h) t = f \,,$$

$$\theta_z + \mathrm{Bi}\, (\theta - \eta) = f_0 \quad \text{on } S_F, \tag{2}$$

$$u = 0 \,, \; \theta = 0 \quad \text{on } S_B. \tag{3}$$

Here $\Omega = \{-1 < z < 0\}$ is the domain occupied by the fluid at the heat conducting state and $S_F = \{z = 0\}$ and $S_B = \{z = -1\}$ are its boundaries.

We use Sobolev spaces $H^r(\Omega)$ and $H^r(S_F)$ and denote their norm by $\|\cdot\|_r$ and $\|\cdot\|_{r,S_F}$ respectively, and we use the function spaces.

$$\begin{aligned}
K^r(\Omega \times (0,\infty)) &\equiv H^0(0,\infty; H^r(\Omega)) \cap H^{r/2}(0,\infty; H^0(\Omega)) \\
K^r_{-\gamma}(\Omega \times (0,\infty)) &\equiv \{f : e^{\gamma t} f \in K^r(\Omega \times (0,\infty))\} \,, \\
K^{r,\frac{1}{2}}(S_F \times (0,\infty)) &\equiv H^0(0,\infty; H^{r+\frac{1}{2}}(S_F)) \cap H^{r/2}(0,\infty; H^{\frac{1}{2}}(S_F)) \\
K^{r,\frac{1}{2}}_{-\gamma}(S_F \times (0,\infty)) &\equiv \{f : e^{\gamma t} f \in K^{r,\frac{1}{2}}(S_F \times (0,\infty))\} \,.
\end{aligned}$$

## 2 Existence for nonlinear problems

We have the following for the Laplace transform of the solution of the linearized system .

**Proposition 1** *Assume* $r \geq 2$. *For small constants* Ra *and* Ma, *there is a positive constant* $\gamma$ *such that for non-zero* $\lambda$ *in* $\{Re\ \lambda > -\gamma\}$ *and data* $F$, $F_0 \in H^{r-2}$, $f$, $f_0 \in H^{r-2+\frac{1}{2}}(S_F)$, *there is a unique solution* $u, \theta \in H^r, \nabla p \in H^{r-2}, \eta \in H^{r+\frac{1}{2}}(S_F)$ *and this solution satisfy*

$$\|u,\theta\|_r + |\lambda|^{\frac{r}{2}}|u,\theta| + \|\nabla p\|_{r-2} + |\lambda|^{\frac{r-2}{2}}|\nabla p| + \|\eta\|_{r+\frac{1}{2},S_F} + |\lambda|^{\frac{r+\frac{1}{2}}{2}}|\eta|_{S_F}$$
$$\leq C\left(\|F,F_0\|_{r-2} + |\lambda|^{\frac{r-2}{2}}|F,F_0|\right) + C\left(\|f,f_0\|_{r-\frac{3}{2},S_F} + |\lambda|^{\frac{r-2}{2}}\|f,f_0\|_{\frac{1}{2},S_F}\right).$$

*Here* $C$ *does not depend on* $\lambda$ . *When* Vi *is positive,* $u_h|_{S_F} \in H^{r+\frac{1}{2}}(S_F)$ *and also* $\|u_h\|_{r+\frac{1}{2},S_F} + |\lambda|^{\frac{r-2}{2}}\|u_h\|_{2+\frac{1}{2},S_F}$ *can be estimated by the right hand side above.*

The nonlinear system has $F, F_0, f, f_0$ in (1)(2) which are quadratic or higher order terms of the unknowns and their derivatives. We have the following for small Ra and Ma.

**Theorem 1** *(See [4].) Assume* $\frac{5}{2} < r < 3$.
*(1) When* Vi $> 0$, *for small initial conditions* $\tilde{u}_0, \tilde{\theta}_0 \in H^{r-1}(\Omega)$, $\eta_0, \tilde{u}_h|_{S_F} \in H^{r-\frac{1}{2}}(S_F)$ *which satisfy conditions* $\nabla\cdot\tilde{u}_0 = 0$, $\tilde{u}_0, \tilde{\theta}_0\big|_{S_B} = 0$ *and* $\int \eta_0 dx = 0$, *there exists a global in time solution* $u, \theta \in K^r_{-\gamma}, p \in K^{r-2}_{-\gamma}, \eta, u_h|_{S_F} \in K^{r+\frac{1}{2}}_{-\gamma}(S_F)$.
*(2) When* Vi $= 0$, *for small initial conditions* $\tilde{u}_0, \tilde{\theta}_0 \in H^{r-1}(\Omega)$, $\eta_0 \in H^{r-\frac{1}{2}}(S_F)$ *which satisfy conditions* $\nabla\cdot\tilde{u}_0 = 0$, $\tilde{u}_0, \tilde{\theta}_0\big|_{S_B} = 0$ *and* $\int \eta_0 dx = 0$, *there exists a global in time solution* $u, \theta \in K^r_{-\gamma}, p \in K^{r-2}_{-\gamma}, \eta \in K^{r+\frac{1}{2}}_{-\gamma}(S_F)$.

**Remark** *The solution constructed in the theorem decays exponentially. Thus, the results say that the purely heat conducting state is stable for small* Ra *and* Ma.

## 3 Eigenvalue problems

Here we increase Rayleigh number and Marangoni number in the system (1)-(3) to investigate the instability of the purely heat conducting state. Rewrite the system using the stream function $\psi$ for the linearized perturbed flow.

$$\psi = 0, \quad \psi_z = 0, \quad \theta = 0 \qquad \text{on} \quad z = 0. \tag{4}$$

$$\Delta \psi_t + \mathrm{Pr}\,\mathrm{Ra}\,\theta_x = \mathrm{Pr}\,\Delta^2 \psi, \quad \theta_t + \psi_x = \Delta\theta \qquad \text{in} \quad 0 < z < 1. \tag{5}$$

$$\begin{aligned} \eta_t + \psi_x &= 0, \\ \psi_{zz} - \psi_{xx} + \mathrm{Ma}\,(\theta_x - \eta_x) + \mathrm{Vi}\,\psi_{zxx} &= 0, \\ -\frac{1}{\mathrm{Pr}}\psi_{zt} + 3\psi_{xxz} + \psi_{zzz} + \mathrm{W}\eta_{xxx} - \mathrm{G}\,\eta_x &= 0, \\ \theta_z + \mathrm{Bi}\,(\theta - \eta) &= 0 \qquad \text{on} \quad z = 1. \end{aligned} \tag{6}$$

We can consider $\psi$, $\theta$ and $\eta$ of the form

$$\psi = \varphi(z)\exp(\, inx + \lambda t\,),$$
$$\theta = \theta(z)\exp(\, inx + \lambda t\,), \qquad \eta = \eta \exp(\, inx + \lambda t\,).$$

Thus the instability problem (4)-(6) is reduced to the eigenvalue problem of the ODE for $\varphi$, $\theta$ and $\eta$ :

$$\varphi(0) = 0, \quad \varphi'(0) = 0, \quad \theta(0) = 0 \qquad \text{on} \quad z = 0. \tag{7}$$

$$\begin{aligned} \mathrm{Pr}\,(\varphi'''' - 2n^2\varphi'' + n^4\varphi) &= \mathrm{Pr}\,\mathrm{Ra}\,in\theta + \lambda(\varphi'' - n^2\varphi), \\ \theta'' - n^2\theta &= in\varphi + \lambda\theta \qquad \text{in} \quad 0 < z < 1. \end{aligned} \tag{8}$$

$$\begin{aligned} \lambda\eta + in\varphi(1) &= 0, \\ \varphi''(1) - \mathrm{Vi}\,n^2\varphi'(1) + n^2\varphi(1) + \mathrm{Ma}\,in\,(\theta(1) - \eta) &= 0, \end{aligned} \tag{9}$$

$$\varphi'''(1) - \frac{1}{\mathrm{Pr}}\lambda\,\varphi'(1) - 3n^2\varphi'(1) - (\mathrm{W}n^2 + \mathrm{G})\,i\,n\,\eta \;=\; 0\,,$$
$$\theta'(1) \;+\; \mathrm{Bi}\,(\theta(1) - \eta) \;=\; 0 \qquad \text{on} \qquad z \;=\; 1\,.$$

By this formulation, the original problem of stability is reduced to investigate the behavior of the real part of the eigenvalue $\lambda$ when the parameters Ra, Ma and $n$ vary. The problem is to find the critical Rayleigh number

$$\mathrm{Ra} \;=\; \mathrm{Rc} \qquad \text{at which} \qquad \lambda \;=\; \pm\mathrm{i}\,\omega \quad (\,\omega \in \mathbf{R}\,)$$

for certain periodicity in $x$ , namely $n$ fixed , and further to show

$$\left.\frac{\partial\,\mathrm{Re}\,\lambda}{\partial\,\mathrm{Ra}}\right|_{\mathrm{Ra}=\mathrm{Rc}} \;>\; 0\,.$$

By this motion of eigenvalue and by the fact that the original evolution problem for the linearized system forms a sectorial operator, we see that a sufficient condition given in the paper of Crandall and Rabinowitz [1],[2] for the occurence of the stationary bifurcation or the Hopf bifurcation for the infinite dimensional system holds . Hence, we see that *The heat conducting state becomes unstable for* Ra $>$ Rc *and the stationary bifurcation or the Hopf bifurcation occurs at* Ra $=$ Rc *corresponding to* $\omega = 0$ *or* $\omega \neq 0$ *respectively.*

In order to justify the above argument about the instability and the bifurcation we use the method given in [6] to prove the existence of the eigenvalue and the critical Rayleigh number in a small neighbourhood of the computed eigenvalue and critical Rayleigh number based on the Newton method.
(i) the shooting method and the Newton method to obtain an approximate eigenvalue, eigen function and critical Rayleigh number of the problem (7) -(9) by numerical computation using the fourth order Taylor finite difference scheme for the fundamental solutions

(ii) the interval analysis by a computer software for the bound of round-off errors in the computation of the fundamental solutions ,
(iii) the theory of pseudo trajectory to estimate the difference of the fundamental solutions and the computed fundamental solutions and
(iv) a theorem of criterion to guarantee the existence of the pure imaginary eigenvalue at the critical Rayleigh number in the neighbourhood of the computed eigenvalue and critical Rayleigh number based on the simplified Newton method.

Example 1. We take $\mathrm{G} = 400$ , $\mathrm{W} = 0$ , $\mathrm{Pr} = 1$ , $\mathrm{Vi} = 0$ , $\mathrm{Bi} = 0$ and $\mathrm{Ma} = 0$ .

$$\lambda_0 = 0 \quad \text{for} \quad \mathrm{R}_0 = 1108.10829\,10299 \quad \text{and} \quad n = 1$$
$$\overline{\frac{\partial \lambda}{\partial \mathcal{R}}}\bigg|_{\mathcal{R}=\mathcal{R}_0} = 0.00601\,95610 \ .$$
$$\lambda_0 = 0 \quad \text{for} \quad \mathrm{R}_0 = 670.28924\,90412 \quad \text{and} \quad n = 2$$
$$\overline{\frac{\partial \lambda}{\partial \mathcal{R}}}\bigg|_{\mathcal{R}=\mathcal{R}_0} = 0.01176\,43040 \ .$$
$$\lambda_0 = 0 \quad \text{for} \quad \mathrm{R}_0 = 782.78265\,39432 \quad \text{and} \quad n = 3$$
$$\overline{\frac{\partial \lambda}{\partial \mathcal{R}}}\bigg|_{\mathcal{R}=\mathcal{R}_0} = 0.01623\,62555 \ .$$
$$\lambda_0 = 0 \quad \text{for} \quad \mathrm{R}_0 = 1131.04272\,17360 \quad \text{and} \quad n = 4$$
$$\overline{\frac{\partial \lambda}{\partial \mathcal{R}}}\bigg|_{\mathcal{R}=\mathcal{R}_0} = 0.01723\,00682 \ .$$

For this gravity G we see the stationary bifurcation.

Example 2. We take $\mathrm{G} = 100$ , $\mathrm{W} = 0$ , $\mathrm{Pr} = 1$ , $\mathrm{Vi} = 0$ , $\mathrm{Bi} = 0$ and $\mathrm{Ma} = 0$ .

$$\lambda_0 = \mathrm{i} \times 2.91543\,59477 \quad \text{for} \quad \mathrm{R}_0 = 447.81500\,11036 \quad \text{and} \quad n = 0.5$$
$$\overline{\frac{\partial \lambda}{\partial \mathcal{R}}}\bigg|_{\mathcal{R}=\mathcal{R}_0} = 0.00323\,04147 \ - \ i \times 0.00182\,89195 \ .$$

$$\lambda_0 = \mathrm{i} \times 4.55206\,09938 \quad \text{for} \quad \mathrm{R}_0 = 391.30728\,48837 \quad \text{and} \quad n = 1$$
$$\left.\frac{\partial\lambda}{\partial\mathcal{R}}\right|_{\mathcal{R}=\mathcal{R}_0} = 0.00739\,43320 - i \times 0.00639\,79827 .$$
$$\lambda_0 = \mathrm{i} \times 5.15597\,17779 \quad \text{for} \quad \mathrm{R}_0 = 424.67690\,19853 \quad \text{and} \quad n = 2$$
$$\left.\frac{\partial\lambda}{\partial\mathcal{R}}\right|_{\mathcal{R}=\mathcal{R}_0} = 0.01092\,75779 - i \times 0.01304\,26310 .$$
$$\lambda_0 = \mathrm{i} \times 5.83570\,00744 \quad \text{for} \quad \mathrm{R}_0 = 514.01005\,34704 \quad \text{and} \quad n = 3$$
$$\left.\frac{\partial\lambda}{\partial\mathcal{R}}\right|_{\mathcal{R}=\mathcal{R}_0} = 0.01003\,54801 - i \times 0.01216\,26611 .$$
$$\lambda_0 = \mathrm{i} \times 6.52511\,06375 \quad \text{for} \quad \mathrm{R}_0 = 749.27424\,80405 \quad \text{and} \quad n = 4$$
$$\left.\frac{\partial\lambda}{\partial\mathcal{R}}\right|_{\mathcal{R}=\mathcal{R}_0} = 0.00818\,05053 - i \times 0.00902\,53196 .$$

For this gravity G we see the Hopf bifurcation.

Example 3. We give another interesting example taking $\mathrm{G} = 100$ , $\mathrm{W} = 0$ , $\mathrm{Pr} = 1$ , $\mathrm{Vi} = 0$ , $\mathrm{Bi} = 0$ and $\mathrm{Ma} \approx -43.73$ , and $n = \pm 1$ .

$$\lambda_0 = \pm\mathrm{i} \times 0.29905\,33540 \quad \text{for} \quad \mathrm{R}_0 = 712.52096\,87507 , \quad \mathrm{M}_0 = -43.735$$
$$\left.\frac{\partial\lambda}{\partial\mathcal{R}}\right|_{\mathcal{R}=\mathcal{R}_0} = -0.00047\,64522 \mp i \times 0.20119\,02855 .$$
$$\lambda_0 = 0 \quad \text{for} \quad \mathrm{R}_0 = 713.26319\,60036 , \quad \mathrm{M}_0 = -43.735$$
$$\left.\frac{\partial\lambda}{\partial\mathcal{R}}\right|_{\mathcal{R}=\mathcal{R}_0} = 162.83867\,58322 .$$
$$\lambda_0 = 0 \quad \text{for} \quad \mathrm{R}_0 = 713.27868\,68247 , \quad \mathrm{M}_0 = -43.73$$
$$\left.\frac{\partial\lambda}{\partial\mathcal{R}}\right|_{\mathcal{R}=\mathcal{R}_0} = -352.57294\,77357 .$$

Thus it suggests an existence of the double zero eigenvalue of the determinant at $\mathrm{Ra} \approx 713$ , $\mathrm{Ma} \approx -43.73$ . Detailed arguements will be given elsewhere.

Example 4. We give another example taking $\mathrm{G} = 100$ , $\mathrm{W} = 0$ , $\mathrm{Pr} = 1$ , $\mathrm{Vi} = 0$ , $\mathrm{Bi} = 0$ and $\mathrm{Ma} \approx 8$ , and $n =$ 1 or 2 .

$$\lambda_0 = \mathrm{i} \times 4.43652\,66171 \quad \text{for} \quad \mathrm{R}_0 = 374.05568\,57561\,, \quad \mathrm{Ma} = 8.0 \quad \text{and} \quad n = 1$$
$$\left.\frac{\overline{\partial \lambda}}{\partial \mathcal{R}}\right|_{\mathcal{R}=\mathcal{R}_0} = 0.00814\,67572 - i \times 0.00575\,47700\,.$$
$$\lambda_0 = \mathrm{i} \times 4.43483\,51773 \quad \text{for} \quad \mathrm{R}_0 = 373.85774\,26405\,, \quad \mathrm{Ma} = 8.1 \quad \text{and} \quad n = 1$$
$$\left.\frac{\overline{\partial \lambda}}{\partial \mathcal{R}}\right|_{\mathcal{R}=\mathcal{R}_0} = 0.00815\,64085 - i \times 0.00574\,64556\,.$$
$$\lambda_0 = \mathrm{i} \times 5.75871\,35816 \quad \text{for} \quad \mathrm{R}_0 = 374.21235\,43595\,, \quad \mathrm{Ma} = 8.0 \quad \text{and} \quad n = 2$$
$$\left.\frac{\overline{\partial \lambda}}{\partial \mathcal{R}}\right|_{\mathcal{R}=\mathcal{R}_0} = 0.01251\,04768 - i \times 0.00987\,39172\,.$$
$$\lambda_0 = \mathrm{i} \times 5.76349\,04450 \quad \text{for} \quad \mathrm{R}_0 = 373.67749\,50581\,, \quad \mathrm{Ma} = 8.1 \quad \text{and} \quad n = 2$$
$$\left.\frac{\overline{\partial \lambda}}{\partial \mathcal{R}}\right|_{\mathcal{R}=\mathcal{R}_0} = 0.01252\,95986 - i \times 0.00984\,38143\,.$$

It suggests the neutral curves $\lambda = i\omega_1$ for $n = 1$ and $\lambda = i\omega_2$ for $n = 2$ intersect at $\mathrm{Ra} \approx 374$ and $\mathrm{Ma} \approx 8.0$ . Detailed arguments will be given elsewhere.

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